Mathematical Inequalities & Applications
Volume 16, Number 3 (2013), 679–695 doi:10.7153/mia-16-51

SOME NEW SCALES OF REFINED HARDY TYPE INEQUALITIES VIA FUNCTIONS RELATED TO SUPERQUADRACITY

S. ABRAMOVICH AND L.-E. PERSSON

(Communicated by S. Varošanec)

Abstract. For the Hardy type inequalities the "breaking point" (= the point where the inequality reverses) is $p=1$. Recently, J. Oguntoase and L. E. Persson proved a refined Hardy type inequality with a breaking point at $p=2$. In this paper we prove a new scale of refined Hardy type inequality which can have a breaking point at any $p \geqslant 2$. The technique is to first make some further investigations for superquadratic and superterzatic functions of independent interest, among which, a new Jensen type inequality is proved.

1. Introduction

Hardy's famous inequality reads: If f is nonnegative and is p-integrable over $(0,\infty)$, then:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)\,dx, \quad p>1. \tag{1.1}$$

This inequality was stated by G. H. Hardy in 1920 (see [4]) and finally proved by him in 1925 (see [5]). The first weighted version of (1.1) was proved in 1928 also by G. H. Hardy (see [6]) and it reads: If f is nonnegative and measurable on $(0,\infty)$, then

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^\alpha dx \leqslant \left(\frac{p}{p-1-\alpha}\right)^p \int_0^\infty f^p(x)\,x^\alpha dx, \tag{1.2}$$

whenever $p>1$ and $\alpha<p-1$. But it has been recently pointed out in [14] that these two inequalities are in fact equivalent, since the substitutions $f(x)=g(x^{1-\frac{1}{p}})x^{-\frac{1}{p}}$ and $f(x)=g(x^{\frac{p-\alpha-1}{p}})x^{-\frac{\alpha+1}{p}}$, respectively, carry over both inequalities to the inequality

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(y)\,dy\right)^p \frac{dx}{x} \leqslant 1\int_0^\infty g^p(x)\,\frac{dx}{x}. \tag{1.3}$$

Since (1.3) follows directly from Jensen's inequality and reversing the order of integration, we get a very simple proof of the weighted Hardy's inequality (1.2) even with

Mathematics subject classification (2010): 26D15.
Keywords and phrases: Inequalities, refined Hardy type inequalities, scales, superquadratic functions, superterzatic functions, Jensen type inequalities.

equality for $p=1$ and holding in reverse direction for $0<p<1$. All constants above are obviously sharp, and by discussing in the same way, when the interval $(0,\infty)$ is replaced by an interval $(0,b)$, $0<b\leqslant\infty$, we obtain the following version of (1.2), still with sharp constant:

$$\int_0^b\left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^\alpha dx\leqslant\left(\frac{p}{p-1-\alpha}\right)^p\int_0^b f^p(x)\,x^\alpha\left(1-\left(\frac{x}{b}\right)^{\frac{p-\alpha-1}{p}}\right)dx, \tag{1.4}$$

where $p\geqslant 1$ and $\alpha<p-1$.

For this and more information of this type see [14]. Concerning Hardy type inequalities with general weights we refer to [8], [9] and [10].

Even if all constants above are sharp we can improve all inequalities above by making so called “refinements” i.e., inserting some additional strictly positive terms on the left hand-side of the inequalities.

Here we will mention some of these results.

An early result of this type is the following one by C. O. Imoru from 1977 [7]:

$$\int_0^b\left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^\alpha dx+\frac{p}{p-1-\alpha}b^{1-p-\alpha}\left(\int_0^b f(y)\,dy\right)^p \leqslant\left(\frac{p}{p-1-\alpha}\right)^p\int_0^b f^p(x)\,x^\alpha dx,$$

where $p\geqslant 1$, $\alpha<p-1$ and $0<b<\infty$. This result was further generalized (and also previous results by D. T. Shum) in the paper [12]. In the paper [11] (cf. also [13]) the same authors made a refinement of a completely different type, namely the following: Let $p\geqslant 1$, $\alpha<p-1$ and $0<b\leqslant\infty$. If $p\geqslant 2$, then

$$\begin{aligned}&\int_0^b\left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^\alpha dx\\ &\quad+\frac{p-1-\alpha}{p}\int_0^b\int_t^b\left|\frac{p}{p-\alpha-1}\left(\frac{t}{x}\right)^{1-\frac{p-\alpha-1}{p}}f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right|^p\\ &\quad\times x^{\alpha-\frac{p-\alpha-1}{p}}\,dx\,t^{\frac{p-\alpha-1}{p}-1}dt\\ &\leqslant\left(\frac{p}{p-\alpha-1}\right)^p\int_0^b f^p(x)\,x^\alpha\left(1-\left(\frac{x}{b}\right)^{\frac{p-\alpha-1}{p}}\right)dx\end{aligned} \tag{1.5}$$

If $1<p\leqslant 2$, then (1.5) holds in the reverse direction. In particular, for $p=2$ we have equality in (1.5). This means that the natural “breaking point” is $p=2$ in this refined Hardy inequality. In all other Hardy type inequalities discussed above and elsewhere (see e.g. the books and the references there) the corresponding natural breaking point is $p=1$. The crucial point in the proof of the result in [11] was to use the concept of superquadratic functions and a corresponding refined Jensen type inequality by Abramovich & al (see [2]). In [3] it was proved that the refinement with breaking point $p=2$ is not unique and that (1.5) can be replaced by another inequality with breaking

point at $p=2$. Moreover, another new refined Hardy type inequality was proved there with breaking point $p=3$. This was obtained by using the concept of superterzatic functions, which was introduced and studied in the paper [1]. The main purpose of this paper is to derive a whole scale of refined Hardy type inequalities which can have a breaking point at any point $p=\alpha,\ \alpha>2$.

In Section 2 of this paper we define all concepts mentioned above and other preliminaries. In Section 3 we prove some equivalent ways to define superterzatic functions including a new Jensen type inequality (see Theorem 1 and Theorem 2). In Section 4 we study the case when $K(x):=x^{\gamma}\varphi(x),\ \gamma\in\mathbb{R}_{+}$, where φ is superquadratic or convex. In this section we compare some bounds in the crucial inequality (4.1), which is important in our Section 5 but also of independent interest. Our new refined Hardy type inequalities are presented and proved in Section 5 (see especially Theorem 3) but now with other natural breaking points. We derive a refinement of Hardy's inequality in a similar way as the refinement which was achieved via superquadratic and subquadratic functions in [11]. In this way we can obtain a whole scale of refined Hardy-type inequalities with natural breaking points $p=p_0\geqslant 2$.

2. Preliminaries

First we define the crucial concept of superquadratic and subquadratic functions (see [2]).

DEFINITION 1. Let $\varphi:[0,b)\to\mathbb{R}$. The function φ is superquadratic if for all $x\in[0,b)$ there exists $C_{\varphi}(x)\in\mathbb{R}$ such that

$$\varphi(y)-\varphi(x)\geqslant C_{\varphi}(x)(y-x)+\varphi(|y-x|) \tag{2.1}$$

for all $y\in[0,b)$.

The function φ is subquadratic if $-\varphi$ is superquadratic and the reverse inequality of (2.1) holds.

REMARK 1. Inequality (2.1) holds for all $\varphi(x)=x^{p},\ x\geqslant 0,\ p\geqslant 2$ and reduces to equality for $\varphi(x)=x^{2}$. The reverse of (2.1) holds for all $\varphi(x)=x^{p},\ x\geqslant 0,\ 0\leqslant p\leqslant 2$.

The following result is useful (see [2, Lemma 2.1]):

LEMMA 1. *Let φ be a superquadratic function with $C_{\varphi}(x)$ as in (2.1).*
(i) Then $\varphi(0)\leqslant 0$.
(ii) If $\varphi(0)=\varphi'(0)=0$, then $C_{\varphi}(x)=\varphi'(x)$ whenever φ is differentiable at $x>0$.
(iii) If $\varphi\geqslant 0$, then φ is convex and $\varphi(0)=\varphi'(0)=0$.

The following refined Jensen type inequality was proved in [2]:
The inequality

$$\varphi\left(\int_{\Omega}f(x)\,d\mu(x)\right)\leqslant\int_{\Omega}\varphi(f(x))\,d\mu(x)-\int_{\Omega}\varphi\left(\left|f(x)-\int_{\Omega}f(x)\,d\mu(x)\right|\right)d\mu \tag{2.2}$$

holds for all probability measure spaces (Ω,μ) of μ-integrable nonnegative functions f if and only if φ is superquadratic. Moreover, (2.2) holds in the reverse direction if and only if φ is subquadratic.

EXAMPLE 1. For the case $\varphi(x)=x^p$ (2.2) implies that the inequality

$$\left(\int_\Omega f(x)\,d\mu(x)\right)^p \leqslant \int_\Omega (f(x))^p\,d\mu(x) - \int_\Omega \left| f(x) - \int_\Omega f(x)\,d\mu(x)\right|^p d\mu(x) \quad (2.3)$$

holds for $p\geqslant 2$ and (2.3) holds in the reversed direction if $0<p\leqslant 2$.

Now, following [1] superterzatic and subterzatic functions are defined as follows:

DEFINITION 2. A function $g:[0,b)\to\mathbb{R}$ is called superterzatic provided that for all $x\in[0,b)$ there exists a constant $C(x)\in\mathbb{R}$ such that the inequality

$$\begin{aligned}
&\sum_{i=1}^N \alpha_i g(x_i) - g(x)\\
&\qquad \geqslant \sum_{i=1}^N \alpha_i x_i\left[(x_i-x)\,C(x) + |x_i-x|^{-1} g(|x_i-x|)\right]\\
&\qquad = \sum_{i=1}^N \alpha_i (x_i-x)^2 C(x) + \sum_{i=1}^N \alpha_i x_i |x_i-x|^{-1} g(|x_i-x|) \qquad (2.4)
\end{aligned}$$

holds for all $x_i:[0,b)$ and $\alpha_i\geqslant 0$, $i=1,\dots,N$, such that $\sum_{i=1}^N \alpha_i=1$, where $x=\sum_{i=1}^N \alpha_i x_i$.

The function g is called subterzatic if $-g$ is superterzatic and the reverse inequality in (2.4) holds.

Also, according to [1, Theorem 1, Case A], we have:

LEMMA 2. *Let $\varphi:[0,b)\to\mathbb{R}$ be a superquadratic function, and let $g:[0,b)\to\mathbb{R}$ and be defined by $g(x)=x\varphi(x)$. Then g is superterzatic. If φ is subquadratic, then g is subterzatic. Moreover $C(x)=C_\varphi(x)$, where $C(x)$ is as in (2.4) and $C_\varphi(x)$ is as in (2.1).*

The name "Superterzatic Function" is given to g because (2.4) holds for $g(x)=x^p$, $p\geqslant 3$, $x\geqslant 0$, with equality for $p=3$. (2.4) holds in the reversed direction for $1<p<3$.

In the next section we characterize in paricular superterzatic and subterzatic functions via a new Jensen type inequality in a similar way to the characterization by (2.2) of superquadratic functions and the characterization of subquadratic functions by the reverse inequality of (2.2) (See Theorem 2).

3. Characterization of the superterzatic functions – a new Jensen type inequality

In Theorem 1 and in Theorem 2 we present equivalent definitions to the one in [1] for superterzacity, which are important for our further investigations but also of independent interest.

THEOREM 1. *The function $g : [0,b) \to \mathbb{R}$ is superterzatic if and only if for all $x \in [0,b)$, there exist constants $C(x) \in \mathbb{R}$ and $D(x) \in R$ such that*

$$g(y) - g(x) \geqslant C(x)y(y-x) + D(x)(y-x) + y|y-x|^{-1} g(|y-x|) \tag{3.1}$$

for all $y \in [0,b)$.

If $g(x) = x\varphi(x)$ where φ is superquadratic, then $C(x) = C_\varphi(x)$, where $C_\varphi(x)$ is as appears in (2.1).

If φ is differentiable, superquadratic, $g(x) = x\varphi(x)$ and $\varphi(0) = \varphi'(0) = 0$, then $C(x) = \varphi'(x)$ and $D(x) = \varphi(x)$.

g is subterzatic if and only if the reverse of inequality (3.1) holds.

Proof. First we show that if (2.4) holds, then (3.1) holds.

Inequality (2.4) for $n = 2$ reads

$$\begin{aligned}\alpha g(y_1) &+ \beta g(y_2) - g(x) \\ &\geqslant C(x)\alpha\beta(y_2-y_1)^2 + \alpha y_1 (\beta|y_2-y_1|)^{-1} g(\beta|y_2-y_1)| \\ &\quad + \beta y_2 (\alpha|y_2-y_1|)^{-1} g(\alpha|y_2-y_1)|,\end{aligned} \tag{3.2}$$

where $0 \leqslant \alpha \leqslant 1$, $\alpha + \beta = 1$.

Let us assume that

$$0 \leqslant y_1 < x < y_2 < b$$

and choose

$$\alpha = \frac{y_2 - x}{y_2 - y_1}, \qquad \beta = \frac{x - y_1}{y_2 - y_1}.$$

Then from (3.2), after some manipulations, we get that

$$\begin{aligned}&\frac{g(y_2)}{y_2-x} - \frac{g(x)}{y_2-x} - C(x)y_2 - \frac{y_2}{(y_2-x)}\frac{g(y_2-x)}{(y_2-x)} \\ &\qquad \geqslant \frac{g(y_1)}{y_1-x} - \frac{g(x)}{y_1-x} - C(x)y_1 - \frac{y_1}{(y_1-x)}\frac{g(x-y_1)}{(x-y_1)}.\end{aligned} \tag{3.3}$$

By fixing $y_1 \in (0,x)$ we obtain a lower bound, which shows that

$$D(x) = \inf_{y_2 > x} \frac{g(y_2) - g(x) - C(x)y_2(y_2-x) - y_2(y_2-x)^{-1}g(y_2-x)}{y_2 - x}$$

exists. Now we take $y_2 = y$ to see that

$$g(y) - g(x) - C(x)y(y-x) - y(y-x)^{-1}g(y-x) \geqslant D(x)(y-x)$$

for all $y > x$, and take $y_1 = y$ to get that

$$g(y) - g(x) - C(x)y(y-x) - y(x-y)^{-1}g(x-y) \geqslant D(x)(y-x)$$

for all $y < x$. Thus (3.1) holds.

Conversely, we get (2.4) from (3.1) by replacing in (3.1) y by x_i, where $x_i \in [0,b)$ $i = 1,...,N$, and by choosing $x = \sum_{i=1}^{N} \alpha_i x_i$, multiplying each of the N inequalities by α_i and taking the sum of these N inequalities, where $0 \leqslant \alpha_i \leqslant 1$, $i = 1,...,N$ and $\sum_{i=1}^{N} \alpha_i = 1$.

In the case that $g(x) = x\varphi(x)$ where φ is superquadratic it follows from [1] and [2] that $C(x) = C_\varphi(x)$ and from the differentiability and superquadracity of φ, and since $\varphi(0) = \varphi'(0) = 0$, it follows that $C(x) = \varphi'(x)$. Dividing (3.1) by $y - x$ and then letting $y \to x$ in the case $y > x$ and then in the case $y < x$, we get that $D = \varphi$.

The result for a subterzatic function g is obtained by dealing with the superterzatic function $-g$. The proof is complete. □

In our next characterization, (3.4) may be regarded as a new refined Jensen type inequality yielding for superterzatic functions.

THEOREM 2. *The function g is superterzatic if and only if*

$$\begin{aligned}
&\int_\Omega g(f(s))d\mu(s) - g\left(\int_\Omega f(s)d\mu(s)\right) \\
&\quad \geqslant C\left(\int_\Omega f(s)d\mu(s)\right)\int_\Omega f(s)\left(f(s) - \int_\Omega f(t)d\mu(t)\right)d\mu(s) \\
&\qquad + \int_\Omega f(s)\left(\left|f(s) - \int_\Omega f(t)d\mu(t)\right|\right)^{-1} g\left(\left|f(s) - \int_\Omega f(t)d\mu(t)\right|\right)d\mu(s) \quad (3.4)
\end{aligned}$$

holds for all probability measures μ and all nonnegative μ-integrable functions f on the measure space (Ω,μ).

If $g(x) = x^3$, $x \geqslant 0$, then (3.4) reduces to equality.

The reverse of inequality (3.4) holds if g is subterzatic.

Proof. According to Theorem 1, if g is superterzatic, then (3.1) holds.

Fix a probability measure μ and a nonnegative, μ-integrable function f. Set $x = \int_\Omega f d\mu$, and let $C(x)$ and $D(x)$ be the constants in (3.1). Then, by integrating we find that

$$\begin{aligned}
&\int_\Omega \Big(g(f(s)) - g(x) - C(x) f(s)(f(s) - x) \\
&\qquad - f(s)|f(s) - x|^{-1} g(|f(s) - x|)\Big) d\mu(s) \\
&\qquad \geqslant D(x)\int_\Omega (f(s) - x)\, d\mu(s) = 0,
\end{aligned}$$

which by rearranging yields (3.4). So we have proved that if g is superterzatic, then (3.4) holds.

Now we show that from (3.4) we get (3.1). Suppose that $0 < y_1 < x < y_2$ and let μ be the probability measure on $[0,1]$ with $\mu(0) = \frac{x-y_1}{y_2-y_1}$ and $\mu(1) = \frac{y_2-x}{y_2-y_1}$. With $f(0) = y_2$, $f(1) = y_1$ we have that $\int_\Omega f d\mu = x$ so inequality (3.4) becomes (3.3) and from this inequality we continue as in the proof of Theorem 1 and we get that (3.1) holds and therefore that g is superterzatic.The remaining part of the proof follows similarly by using Theorem 1, Lemma 2 and Definition 2. The proof is complete. □

By using the results above and Theorem 1 for the function $g(x) = x^p$, $x \geqslant 0$, $p \geqslant 3$ and $1 < p \leqslant 3$ we get the following inequalities:

COROLLARY 1. *Let $p \geqslant 3$. If $x = \sum_{i=1}^{n} \alpha_i x_i$, $\alpha_i \geqslant 0$, $\sum_{i=1}^{n} \alpha_i = 1$, $x_i \in [0,\infty)$, then the following inequality holds:*

$$\begin{aligned}\sum \alpha_i x_i^p - x^p &\geqslant \sum \alpha_i x_i \left[(x_i - x)(p-1) x^{p-2} + |x_i - x|^{p-1} \right] \\ &= \sum_{i=1}^{n} \alpha_i (x_i - x)^2 (p-1) x^{p-2} + \sum_{i=1}^{n} \alpha_i x_i |x_i - x|^{p-1}. \end{aligned} \tag{3.5}$$

If $x,\ y \geqslant 0$, then

$$y^p - x^p \geqslant (p-1) x^{p-2} y (y-x) + y |y-x|^{p-1} + x^{p-1}(y-x), \tag{3.6}$$

and

$$\begin{aligned}&\int_\Omega (f(s))^p d\mu(s) - \left(\int_\Omega f(s) d\mu(s) \right)^p \\ &\geqslant (p-1) \left(\int_\Omega f(s) d\mu(s) \right)^{p-2} \int_\Omega f(s) \left(f(s) - \int_\Omega f(t) d\mu(t) \right) d\mu(s) \\ &\quad + \int_\Omega f(s) \left| f(s) - \int_\Omega f(t) d\mu(t) \right|^{p-1} d\mu(s). \end{aligned} \tag{3.7}$$

holds for all probability measure spaces (Ω,μ) of nonnegative μ-integrable functions f.

If $1 \leqslant p \leqslant 3$, then the reverse of these inequalities hold.

Inequalities (3.5), (3.6) and (3.7) reduce to equalities for $p = 3$, where f is any nonnegative μ-integrable function on the probability measure space (Ω,μ).

4. The case $K(x) = x^\gamma \varphi(x)$, where φ is superquadratic or convex

In order to get more refinements of Hardy's inequalities similarly to those in [11], we prove in this section some inequalities that hold when the given function $K(x)$ satisfies $K(x) = x^\gamma \varphi(x)$, $\gamma \in \mathbb{R}_+$, where φ is a superquadratic function. These inequalities include and generalize the results in Section 2 related to superquadratic function $\varphi : [0,b) \to R$ and in Section 3 to superterzatic functions $g : [0,b) \to R$ that satisfy $g(x) = x\varphi(x)$.

LEMMA 3. *Let $K(x) = x^\gamma \varphi(x)$, $\gamma \in \mathbb{R}_+$, where $\varphi(x)$ is superquadratic on $[0,b)$. Then*

$$K(y) - K(x) \geqslant \varphi(x)(y^\gamma - x^\gamma) + C_\varphi(x) y^\gamma (y-x) + y^\gamma \varphi(|y-x|), \tag{4.1}$$

holds for $x \in [0,b)$, $y \in [0,b)$. *Moreover,*

$$\begin{aligned}&\sum_{i=1}^{N}\alpha_i K(y_i) - K\left(\sum_{i=1}^{N}\alpha_i y_i\right)\\ &\quad\geqslant \varphi\left(\sum_{j=1}^{N}\alpha_j y_j\right)\left(\sum_{i=1}^{N}\alpha_i y_i^{\gamma} - \left(\sum_{j=1}^{N}\alpha_j y_j\right)^{\gamma}\right)\\ &\qquad + C_{\varphi}\left(\sum_{j=1}^{N}\alpha_j y_j\right)\sum_{i=1}^{N}\alpha_i y_i^{\gamma}\left(y_i - \sum_{j=1}^{N}\alpha_j y_j\right) + \sum_{i=1}^{N}\alpha_i y_i^{\gamma}\varphi\left(\left|y_i - \sum_{j=1}^{N}\alpha_j y_j\right|\right)\end{aligned} \tag{4.2}$$

holds for $x_i \in [0,b)$, $y_i \in [0,b)$, $0 \leqslant \alpha_i \leqslant 1$, $i = 1,\ldots,n$, *and* $\sum_{i=1}^{N}\alpha_i = 1$; *and*

$$\begin{aligned}&\int_{\Omega} K(f(s))\,d\mu(s) - K\left(\int_{\Omega} f(s)\,d\mu(s)\right)\\ &\geqslant \int_{\Omega}\left[\varphi(x)\left(f^{\gamma}(s) - x^{\gamma}\right) + C_{\varphi}(x) f^{\gamma}(s)(f(s) - x) + f^{\gamma}(s)\varphi(|f(s) - x|)\right] d\mu(s).\end{aligned} \tag{4.3}$$

holds, where f is any nonnegative μ*-integrable function on the probability measure space* (Ω,μ) *and* $x = \int_{\Omega} f(s)\,d\mu(s)$.

If φ *is subquadratic, then the reverse inequality of (4.1), (4.2), and (4.3) hold, in particular*

$$\begin{aligned}&\int_{\Omega} K(f(s))\,d\mu(s) \; - K\left(\int_{\Omega} f(s)\,d\mu(s)\right)\\ &\leqslant \int_{\Omega}\left[\varphi(x)\left(f^{\gamma}(s) - x^{\gamma}\right) + C_{\varphi}(x) f^{\gamma}(s)(f(s) - x) + f^{\gamma}(s)\varphi(|f(s) - x|)\right] d\mu(s).\end{aligned} \tag{4.4}$$

Inequalities (4.1), (4.2) and (4.3) are satisfied in particular by $K(x) = x^p$, $p \geqslant \gamma + 2$. *For* $\gamma < p \leqslant \gamma + 2$ *the reverse inequalities hold. They reduce to equalities for* $p = \gamma + 2$.

Proof. Multiplying (2.1) by y^{γ}, by simple manipulations we get that $K(x) = x^{\gamma}\varphi(x)$ satisfies (4.1) when φ is superquadratic.

By fixing in (4.1) a probability measure μ and a nonnegative integrable function f, setting $x = \int_{\Omega} f\,d\mu$ and $C_{\varphi}(x)$ is as in the definition of superquadracity, we obtain for $K(x) = x^{\gamma}\varphi(x)$, $\gamma \in \mathbb{R}_+$, where $\varphi(x)$ is superquadratic, that (4.3) holds.

(4.2) is the discrete case of (4.3) and is obtained from (4.1) in the same way as (2.4) was derived from (3.1).

Similarly, since $-\varphi$ is superquadratic, inequality (4.4) and the reverse inequalities of (4.1) and (4.2) are obtained for subquadratic functions.

Since $\varphi(x) = x^p$, is superquadratic for $p \geqslant 2$, $x > 0$, and subquadratic for $0 < p \leqslant 2$, $x > 0$, we find that inequalities (4.1), (4.2) and (4.3) hold when $p \geqslant \gamma + 2$ and the reverse of inequalities (4.1), (4.2) and (4.3) hold when $\gamma < p \leqslant \gamma + 2$. As a result, we in particular find that (4.1), (4.2) and (4.3) reduce to equalities for $p = \gamma + 2$. The proof is complete. □

In an analogous way we can start with a convex function ψ satisfying $\psi(y) - \psi(x) \geqslant C_\psi(x)(y-x)$, which reduces to equality for $\psi(x) = x$ and by the same procedure we obtain that for the function $T(x) = x^\gamma \psi(x)$, $\gamma \in \mathbb{R}_+$, where $\psi(x)$ is convex, it yields that

$$T(y) - T(x) \geqslant \psi(x)(y^\gamma - x^\gamma) + C_\psi(x) y^\gamma (y-x). \tag{4.5}$$

In particular, inequality (4.5) holds for $T(x) = x^p$, $p \geqslant \gamma + 1$ with equality for $T(x) = x^{\gamma+1}$.

In Section 5 (see Theorem 3), we prove a new Hardy type inequaliy related to the functions $\varphi(x) = x^m$, $x \geqslant 0$, $m \geqslant 1$, which are superquadratic when $m \geqslant 2$ and subquadratic when $1 < m \leqslant 2$. These functions are evidently differentiable, nonnegative, convex, increasing and satisfy $\varphi(0) = \varphi'(0) = 0$. But in order to get the results of Theorem 3 we need to compare inequality (4.1) for $K(x) = x^\gamma \varphi(x)$ and $K(x) = x^{\gamma-1}\psi(x)$, where $\psi(x) = x\varphi(x)$ (see Lemma 4 below). First we state a useful remark that guides us how to prove Lemma 4 (see [2, Lemma 3.1]).

REMARK 2. Let $\varphi(x)$, $0 \leqslant x < \infty$, be a differentiable positive convex, increasing function and $\varphi(0) = \varphi'(0) = 0$ and let $\psi(x) = x\varphi(x)$. Because $\frac{\psi'(x)}{x}$ is increasing and $\psi(0) = \psi'(0) = 0$, then $\psi(x)$ is superquadratic. In particular, if $\varphi(x)$, $0 \leqslant x < \infty$, is a differentiable positive superquadratic function, then (according to Lemma 1) φ is convex increasing and $\varphi(0) = \varphi'(0) = 0$. Therefore also $\psi(x)$ is positive, increasing, convex and superquadratic.

LEMMA 4. *Let $K(x) = x^\gamma \varphi(x) = x^{\gamma-1}\psi(x)$, $\gamma \geqslant 1$, where φ is a differentiable positive superquadratic function and $\psi(x) = x\varphi(x)$. Then the bound obtained for $K(x) = x^\gamma \varphi(x)$ is stronger than the bound obtained for $K(x) = x^{\gamma-1}\psi(x)$, that is:*

$$K(y) - K(x) \geqslant \varphi(x)(y^\gamma - x^\gamma) + \varphi'(x) y^\gamma (y-x) + y^\gamma \varphi(|y-x|) \tag{4.6}$$

implies that

$$K(y) - K(x) \geqslant \psi(x)\left(y^{\gamma-1} - x^{\gamma-1}\right) + \psi'(x) y^{\gamma-1}(y-x) + y^{\gamma-1}\psi(|y-x|). \tag{4.7}$$

Moreover, if $K(x) = x^n \varphi(x)$, $\psi_k(x) = x^k \varphi(x)$, n is an integer, $k = 1, 2, \ldots, n$, and $\varphi(x)$ is nonnegative superquadratic, then the inequalities

$$\begin{aligned} &\int_\Omega K(f(s))\,d\mu(s) \; - K\left(\int_\Omega f(s)\,d\mu(s)\right) \\ &\quad \geqslant \int_\Omega \left[\varphi(x)(f^n(s) - x^n) + C_\varphi(x) f^n(s)(f(s) - x)\right. \\ &\qquad\qquad\qquad\qquad \left. + f^n(s)\varphi(|f(s) - x|)\right] d\mu(s) \end{aligned}$$

$$\geqslant \int_\Omega \Big[\psi_k(x)\left(f^{n-k}(s)-x^{n-k}\right)+C_{\psi_k}(x)f^{n-k}(s)(f(s)-x) \\ +f^{n-k}(s)\psi_k(|f(s)-x|)\Big]d\mu(s)$$

$$\geqslant \int_\Omega \psi_n(|f(s)-x|)\,d\mu(s)\geqslant 0 \tag{4.8}$$

hold for all probability measure spaces (Ω,μ) *of* μ*-integrable nonnegative functions* f *, where* $x=\int_\Omega f(s)\,d\mu(s)$.

Furthermore if $\varphi(x)$ *is positive, increasing, convex, subquadratic and* $\varphi(0)=\varphi'(0)=0$, *then* $x\varphi(x)$ *is superquadratic and*

$$\int_\Omega [\varphi(x)(f^n(s)-x^n)+C_\varphi(x)f^n(s)(f(s)-x) \\ +f^n(s)\varphi(|f(s)-x|)]\,d\mu(s)$$

$$\geqslant \int_\Omega K(f(s))\,d\mu(s)\ -K\left(\int_\Omega f(s)\,d\mu(s)\right)$$

$$\geqslant \int_\Omega \Big[\psi_k(x)\left(f^{n-k}(s)-x^{n-k}\right)+C_{\psi_k}(x)f^{n-k}(s)(f(s)-x) \\ +f^{n-k}(s)\psi_k(|f(s)-x|)\Big]d\mu(s)$$

$$\geqslant \int_\Omega \psi_n(|f(s)-x|)\,d\mu(s)\geqslant 0;\qquad k=1,\dots,n. \tag{4.9}$$

In particular, if $\varphi(x)=x^p,\ x\geqslant 0,\ p\geqslant 1,$ *then (4.8) is satisfied when* $p\geqslant 2$ *and (4.9) is satisfied when* $1\leqslant p\leqslant 2$. *When* $p=2$ *equality holds in the first inequality of (4.8) and in the first inequality of (4.9).*

Proof. We will show that when φ is differentiable, nonnegative superquadratic or φ is differentiable, positive increasing, convex and $\varphi(0)=\varphi'(0)=0$, where $\psi(x)=x\varphi(x)$, then

$$\varphi(x)(y^\gamma-x^\gamma)+\varphi'(x)y^\gamma(y-x)+y^\gamma\varphi(|y-x|) \\ \geqslant \psi(x)\left(y^{\gamma-1}-x^{\gamma-1}\right)+\psi'(x)y^{\gamma-1}(y-x)+y^{\gamma-1}\psi(|y-x|). \tag{4.10}$$

After some manipulations we see that in order to prove (4.10) it is sufficient to show that

$$\varphi'(x)(x-y)^2+y\varphi(|y-x|)-|y-x|\,\varphi(|y-x|)\geqslant 0. \tag{4.11}$$

Case a: $y\geqslant x\geqslant 0$. In this case we have to prove that

$$\varphi'(x)(x-y)^2+x\varphi(|y-x|)\geqslant 0, \tag{4.12}$$

which is satisfied because $x,\ \varphi,\ \varphi'$ are nonnegative.

Case b: $x\geqslant y\geqslant 0$. Then (4.11) becomes

$$(x-y)\left[(x-y)\varphi'(x)-\varphi(x-y)\right]+y\varphi(x-y)\geqslant 0. \tag{4.13}$$

Since φ' is increasing, from $x \geqslant y \geqslant 0$ we get that

$$(x-y)\varphi'(x) - \varphi(x-y) \geqslant (x-y)\varphi'(x-y) - \varphi(x-y) \geqslant 0. \tag{4.14}$$

The last inequality in (4.14) holds since $\varphi(0) = \varphi'(0) = 0$ and as φ is convex. Hence (4.13) holds.

From Remark 2 we know that if φ is nonnegative and superquadratic so is $x^k\varphi(x) = \psi_k(x)$, k is a nonnegative integer. Since (4.3) is obtained from (4.1), then, by using (4.10), we get (4.8). If φ is positive, subquadratic, convex, increasing, differentiable and $\varphi(0) = \varphi'(0) = 0$, then, as we showed, (4.10) still holds and as in this case that (4.4) holds we get that for a nonnegative integer n, the left hand-side of (4.9) is nonnegative. However, according to Remark 2, $\psi_k(x) = x^k\varphi(x)$, $k = 1,2...$, is superquadratic. Therefore also the right hand-side of (4.9) holds. The assertions on $\varphi(x) = x^p$, $p \geqslant 1$, in the statement of this lemma hold because these functions are differentiable, nonnegative increasing and superquadratic when $p \geqslant 2$ and subquadratic when $1 \leqslant p \leqslant 2$. This completes the proof of the lemma. □

REMARK 3. As explained in Remark 2, given a positive, increasing and convex $F(x)$ where $F(0) = F'(0) = 0$ we get that $\varphi(x) = xF(x)$ is superquadratic and therefore that $K(x) = x^\gamma\varphi(x)$ satisfies inequality (4.6), $\gamma \in \mathbb{R}_+$. Hence, from (4.5) and (4.6) we derive the following: Let $F(x)$, $0 \leqslant x < b$, be a differentiable positive increasing convex function and $F(0) = F^{'}(0) = 0$. Then $K(x) = x^{\gamma+1}F(x)$, $\gamma \in \mathbb{R}_+$, because of the convexity of $F(x)$ satisfies the inequality

$$y^{\gamma+1}F(y) - x^{\gamma+1}F(x) \geqslant F(x)\left(y^{\gamma+1} - x^{\gamma+1}\right) + F'(x)y^{\gamma+1}(y-x),$$

and because of the superquadracity of $\varphi(x) = xF(x)$ the inequality

$$\begin{aligned} &y^{\gamma+1}F(y) - x^{\gamma+1}F(x) \\ &\quad \geqslant xF(x)(y^\gamma - x^\gamma) + (xF(x))'y^\gamma(y-x) + y^\gamma|y-x|F(|y-x|). \end{aligned}$$

REMARK 4. From Example 3 in [1], we can see that if φ is superquadratic but φ or φ' is not increasing and $\psi(x) = x\varphi(x)$, then ψ is superterzatic and can be subquadratic.

Indeed $\varphi(x) = x^2\ln x$, $0 < x < \infty$, $\varphi(0) = 0$ is superquadratic and $\psi(x) = x^3\ln x$, $0 < x < \infty$, $\psi(0) = \psi'(0) = 0$, is subquadratic on $\left[0, e^{-3/2}\right]$ and therefore is superterzatic and subquadratic on $\left[0, e^{-3/2}\right]$. It is easy to verify that this follows in general if φ is negative, decreasing, concave superquadratic function, satisfying $\varphi(0) = \varphi'(0) = 0$. In this case we get the following inequalities:

$$\begin{aligned} &(y-x)\varphi(x) + (y-x)y\varphi'(x) + y\varphi(|y-x|) \\ &\qquad \leqslant \psi(y) - \psi(x) \leqslant (y-x)\psi'(x) + \psi(|y-x|), \end{aligned}$$

where $\psi(x) = x\varphi(x)$. In other words

$$(y-x)\varphi(x) + (y-x)y\varphi'(x) + y\varphi(|y-x|)$$

$$\leqslant y\varphi(y) - x\varphi(x)$$
$$\leqslant (y-x)x\varphi'(x) + (y-x)\varphi(x) + (|y-x|)\varphi(|y-x|).$$

In such cases we get that the superterzacity of ψ gives a lower bound and the subqudracity of ψ gives an upper bound to $\psi(y) - \psi(x)$.

5. Some new scales of refined Hardy type inequalities

In this section we use the ideas and techniques of [11], and implement them on the functions $K(x) = x^{\gamma}\varphi(x)$, $\gamma \in \mathbb{R}_+$, where $\varphi(x)$ is superquadratic. First we prove a proposition in a similar way to Proposition 2.1a in [11].

PROPOSITION 1. *Let* $0 < b \leqslant \infty$, $u : (0,\infty) \to \mathbb{R}$, *be a nonnegative weight function such that* $\frac{u(x)}{x^2}$ *is locally integrable on* $(0,\infty)$ *and let the weight function* v *be defined by*

$$v(t) = t\int_t^b \frac{u(x)}{x^2}dx, \quad t \in (0,b). \tag{5.1}$$

If the function φ *is integrable and superquadratic on* $[0,b)$ *and* $K(x) = x^{\gamma}\varphi(x)$, $\gamma \in \mathbb{R}_+$, *then*

$$\begin{aligned}
&\int_0^b K(f(x))\frac{v(x)}{x}dx - \int_0^b K\left(\frac{1}{x}\int_0^x (f(t)dt)\right)\frac{u(x)}{x}dx] \\
&\geqslant \int_0^b\int_t^b \left(f^{\gamma}(t) - \left(\frac{1}{x}\int_0^x f(\tau)d\tau\right)^{\gamma}\right)\varphi\left(\frac{1}{x}\int_0^x f(\tau)d\tau\right)\frac{u(x)}{x^2}dxdt \\
&\quad + \int_0^b\int_t^b f^{\gamma}(t)\left(f(t) - \frac{1}{x}\int_0^x f(\tau)d\tau\right)C_{\varphi}\left(\frac{1}{x}\int_0^x f(\tau)d\tau\right)\frac{u(x)}{x^2}dxdt \\
&\quad + \int_0^b\int_t^b f^{\gamma}(t)\varphi\left(\left|f(t) - \frac{1}{x}\int_0^x f(\tau)d\tau\right|\right)\frac{u(x)}{x^2}dxdt,
\end{aligned} \tag{5.2}$$

holds for all nonnegative locally integrable functions f. *If* φ *is subquadratic, then the reverse of inequality (5.2) holds.*

COROLLARY 2. *For* $\gamma = 0$ *(5.2) coincides with the statement in Proposition 2.1a in* [11], *that is*

$$\begin{aligned}
&\int_0^b \varphi(f(x))\frac{v(x)}{x}dx - \int_0^b \varphi\left(\frac{1}{x}\int_0^x f(t)dt\right)\frac{u(x)}{x}dx \\
&\qquad \geqslant \int_0^b\int_t^b \varphi\left(\left|f(t) - \frac{1}{x}\int_0^x f(\tau)d\tau\right|\right)\frac{u(x)}{x^2}dxdt.
\end{aligned} \tag{5.3}$$

Proof. Let us choose the probability measure $d\mu(t) = \frac{1}{x}dt$, $0 \leqslant t \leqslant x$ in (4.3). Then

$$\frac{1}{x}\int_0^x K(f(t))dt - K\left(\frac{1}{x}\int_0^x f(t)dt\right)$$

$$\begin{aligned}&\geqslant \varphi\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{1}{x}\int_0^x\left(f^{\gamma}(t)-\left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^{\gamma}\right)dt\\&+C_{\varphi}\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{1}{x}\int_0^x f^{\gamma}(t)\left(f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)dt\\&+\frac{1}{x}\int_0^x f^{\gamma}(t)\,\varphi\left(\left|f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right|\right)dt.\end{aligned}\tag{5.4}$$

Multiplying (5.4) by $\frac{u(x)}{x}$ and integrating on $0\leqslant x\leqslant b$, we get that

$$\begin{aligned}&\int_0^b\int_0^x K(f(t))\,dt\frac{u(x)}{x^2}dx-\int_0^b K\left(\frac{1}{x}\int_0^x f(t)\,dt\right)\frac{u(x)}{x}dx\\&\quad\geqslant\int_0^b\int_0^x\left(f^{\gamma}(t)-\left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^{\gamma}\right)\varphi\left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)\frac{u(x)}{x^2}dtdx\\&\quad+\int_0^b\int_0^x f^{\gamma}(t)\left(f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)C_{\varphi}\left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)\frac{u(x)}{x^2}dtdx\\&\quad+\int_0^b\int_0^x f^{\gamma}(t)\,\varphi\left(\left|f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right|\right)\frac{u(x)}{x^2}dtdx.\end{aligned}\tag{5.5}$$

Now using (5.1) and Fubini's theorem we find that

$$\begin{aligned}&\int_0^b\int_0^x K(f(t))\frac{u(x)}{x^2}dtdx\\&\quad=\int_0^b\frac{1}{t}\int_t^b K(f(t))\frac{tu(x)}{x^2}dxdt\\&\quad=\int_0^b K(f(t))\frac{v(t)}{t}dt=\int_0^b K(f(x))\frac{v(x)}{x}dt.\end{aligned}\tag{5.6}$$

(5.5) and (5.6) lead to (5.2). When φ is subquadratic the proof is similar and therefore is omitted. The proof is complete. □

EXAMPLE 2. From Proposition 1 for $\varphi(x)=x^p,\ p\geqslant 2$ (therefore $C_{\varphi}(x)=\varphi'(x)=px^{p-1}$), choosing $u(x)=1$ and $\gamma\in\mathbb{R}$, we find that

$$\begin{aligned}&\int_0^b\left(1-\frac{x}{b}\right)f^{p+\gamma}(x)\frac{dx}{x}-\int_0^b\left(\frac{1}{x}\int_0^x f(t)\,dt\right)^{p+\gamma}\frac{dx}{x}\\&\quad\geqslant\int_0^b\int_t^b\left(f^{\gamma}(t)-\left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^{\gamma}\right)\left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^{p}\frac{dx}{x^2}dt\\&\quad+\int_0^b\int_t^b f^{\gamma}(t)\left(f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)p\left(\frac{1}{x}\int_0^x f(\tau)\,d\tau\right)^{p-1}\frac{dx}{x^2}dt\\&\quad+\int_0^b\int_t^b f^{\gamma}(t)\left(\left|f(t)-\frac{1}{x}\int_0^x f(\tau)\,d\tau\right|\right)^{p}\frac{dx}{x^2}dt.\end{aligned}\tag{5.7}$$

The reverse inequality holds when $1<p\leqslant 2$.

By using (5.7) we are now ready to derive our new scales of refined Hardy type inequalities.

THEOREM 3. *Let $p \geqslant 2,\ k>1,\ 0<b\leqslant\infty$, and $\gamma\in\mathbb{R}_+$, and let the function f be nonnegative and locally integrable on $(0,b)$. Then*

$$
\begin{aligned}
&\left(\frac{p+\gamma}{k-1}\right)^{p+\gamma}\int_0^b\left(1-\left(\frac{x}{b}\right)^{\frac{k-1}{p+\gamma}}\right)x^{p+\gamma-k}f^{p+\gamma}(x)\,dx-\int_0^b x^{-k}\left(\int_0^x f(t)\,dt\right)^{p+\gamma}dx\\
&\geqslant\left(\frac{k-1}{p+\gamma}\right)\int_0^b\int_t^b\left(\left(f(t)\frac{p+\gamma}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p+\gamma}}\right)^{\gamma}-\left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^{\gamma}\right)\\
&\quad\times\left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^{p}x^{\left(1-\frac{k-1}{p+\gamma}\right)(p+\gamma-1)}t^{\frac{k-1}{p+\gamma}-1}\frac{dx}{x^2}dt\\
&\quad+\left(\frac{k-1}{p+\gamma}\right)^{1-\gamma}\int_0^b\int_t^b\left(f(t)\,t^{1-\frac{k-1}{p+\gamma}}\right)^{\gamma}\left(f(t)\frac{p+\gamma}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p+\gamma}}-\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)\\
&\quad\times p\left(\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right)^{p-1}x^{\left(1-\frac{k-1}{p+\gamma}\right)(p+1)}t^{\frac{k-1}{p+\gamma}-1}\frac{dx}{x^2}dt\\
&\quad+\left(\frac{k-1}{p+\gamma}\right)^{1-\gamma}\int_0^b\int_t^b\left(f(t)\,t^{1-\frac{k-1}{p+\gamma}}\right)^{\gamma}\left(\left|f(t)\frac{p+\gamma}{k-1}\left(\frac{t}{x}\right)^{1-\frac{k-1}{p+\gamma}}-\frac{1}{x}\int_0^x f(\sigma)\,d\sigma\right|\right)^{p}\\
&\quad\times x^{\left(1-\frac{k-1}{p+\gamma}\right)(p+1)}t^{\frac{k-1}{p+\gamma}-1}\frac{dx}{x^2}dt. \qquad (5.8)
\end{aligned}
$$

Moreover, if γ is a nonnegative integer, then the right hand side of (5.8) is nonnegative. If $1<p\leqslant 2$, then inequality (5.8) is reversed. Equality holds when $p=2$. When $\gamma=0$, inequality (5.8) coincide with (1.5).

Proof. The proof follows the steps of the proof of Theorem 3.1 in [11].

We denote the right hand side of (5.7) by R and replace the parameter b by $b^{\frac{k-1}{p+\gamma}}$ and $f(x)$ by $f\left(x^{\frac{p+\gamma}{k-1}}\right)x^{\frac{p+\gamma}{k-1}-1}$. Then

$$
\begin{aligned}
R=&\int_0^{b^{\frac{k-1}{p+\gamma}}}\int_t^{b^{\frac{k-1}{p+\gamma}}}\left(f^{\gamma}\left(t^{\frac{p+\gamma}{k-1}}\right)t^{\left(\frac{p+\gamma}{k-1}-1\right)\gamma}-\left(\frac{1}{x}\int_0^x f\left(\tau^{\frac{p+\gamma}{k-1}}\right)\tau^{\frac{p+\gamma}{k-1}-1}d\tau\right)^{\gamma}\right)\\
&\times\left(\frac{1}{x}\int_0^x f\left(\tau^{\frac{p+\gamma}{k-1}}\right)\tau^{\frac{p+\gamma}{k-1}-1}d\tau\right)^{p}\frac{dx}{x^2}dt\\
&+\int_0^{b^{\frac{k-1}{p+\gamma}}}\int_t^{b^{\frac{k-1}{p+\gamma}}}\left(f\left(t^{\frac{p+\gamma}{k-1}}\right)t^{\frac{p+\gamma}{k-1}-1}\right)^{\gamma}\left(f\left(t^{\frac{p+\gamma}{k-1}}\right)t^{\frac{p+\gamma}{k-1}-1}-\frac{1}{x}\int_0^x f\left(\tau^{\frac{p+\gamma}{k-1}}\right)\tau^{\frac{p+\gamma}{k-1}-1}d\tau\right)\\
&\times p\left(\frac{1}{x}\int_0^x f\left(\tau^{\frac{p+\gamma}{k-1}}\right)\tau^{\frac{p+\gamma}{k-1}-1}d\tau\right)^{p-1}\frac{dx}{x^2}dt
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{b^{\frac{k-1}{p+\gamma}}}\int_t^{b^{\frac{k-1}{p+\gamma}}} f^{\gamma}\left(t^{\frac{p+\gamma}{k-1}}\right)t^{\left(\frac{p+\gamma}{k-1}-1\right)\gamma}\\
&\times\left(\left|f\left(t^{\frac{p+\gamma}{k-1}}\right)t^{\frac{p+\gamma}{k-1}-1}-\frac{1}{x}\int_0^x f\left(\tau^{\frac{p+\gamma}{k-1}}\right)\tau^{\frac{p+\gamma}{k-1}-1}d\tau\right|\right)^p\frac{dx}{x^2}dt. \qquad (5.9)
\end{aligned}
$$

We use now the substitutions

$$
y=x^{\frac{p+\gamma}{k-1}} \qquad \text{and} \qquad s=t^{\frac{p+\gamma}{k-1}} \Leftrightarrow x=y^{\frac{k-1}{p+\gamma}}\ t=s^{\frac{k-1}{p+\gamma}}
$$

from which it follows that

$$
t=b^{\frac{k-1}{p+\gamma}} \Rightarrow s=b, \quad x=b^{\frac{k-1}{p+\gamma}} \Rightarrow y=b,
$$

$$
dt=\tfrac{k-1}{p+\gamma}s^{\frac{k-1}{p+\gamma}-1}ds, \quad \tfrac{k-1}{p+1}ds=t^{\frac{p+\gamma}{k-1}-1}dt, \quad dx=y^{\frac{k-1}{p+\gamma}-1}\tfrac{k-1}{p+\gamma}dy,
$$

$$
dy=\tfrac{p+\gamma}{k-1}x^{\frac{p+\gamma}{k-1}-1}dx, \quad \textit{and} \quad t^{\frac{p+\gamma}{k-1}-1}=s^{1-\frac{k-1}{p+\gamma}}.
$$

By using these substitutions we get from (5.9) that

$$
\begin{aligned}
R=&\left(\frac{k-1}{p+\gamma}\right)^{p+\gamma+2}\int_0^b\int_s^b\left(\left(\frac{p+\gamma}{k-1}f(s)\left(\frac{s}{y}\right)^{1-\frac{k-1}{p+\gamma}}\right)^{\gamma}-\left(\frac{1}{y}\int_0^y f(\sigma)d\sigma\right)^{\gamma}\right)\\
&\times\left(\frac{1}{y}\int_0^y f(\sigma)d\sigma\right)^p y^{\left(1-\frac{k-1}{p+\gamma}\right)(p+\gamma+1)}s^{\frac{k-1}{p+\gamma}-1}\frac{dy}{y^2}ds\\
&+p\left(\frac{k-1}{p+\gamma}\right)^{p+2}\int_0^b\int_s^b\left(\frac{p+\gamma}{k-1}f(s)\left(\frac{s}{y}\right)^{\left(1-\frac{k-1}{p+\gamma}\right)}-\frac{1}{y}\int_0^y f(\sigma)d\sigma\right)\\
&\times\left(f(s)s^{\left(1-\frac{k-1}{p+\gamma}\right)}\right)^{\gamma}\left(\frac{1}{y}\int_0^y f(\sigma)d\sigma\right)^{p-1}y^{\left(1-\frac{k-1}{p+\gamma}\right)(p+1)}s^{\frac{k-1}{p+\gamma}-1}\frac{dy}{y^2}ds\\
&+\left(\frac{k-1}{p+\gamma}\right)^{p+2}\int_0^b\int_s^b\left(\left|\frac{p+\gamma}{k-1}f(s)\left(\frac{s}{y}\right)^{1-\frac{k-1}{p+\gamma}}-\frac{1}{y}\int_0^y f(\sigma)d\sigma\right|\right)^p\\
&\times\left(f(s)s^{1-\frac{k-1}{p+\gamma}}\right)^{\gamma}y^{\left(1-\frac{k-1}{p+\gamma}\right)(p+1)}s^{\frac{k-1}{p+\gamma}-1}\frac{dy}{y^2}ds. \qquad (5.10)
\end{aligned}
$$

Now we make the same changes on the left hand side of (5.7), denoted by L, that is, we replace b by $b^{\frac{k-1}{p+\gamma}}$ and $f(x)$ by $f\left(x^{\frac{p+\gamma}{k-1}}\right)x^{\frac{p+\gamma}{k-1}-1}$ and by the substitution $y=x^{\frac{p+\gamma}{k-1}}$ we get that

$$
\begin{aligned}
L=&\int_0^b\frac{k-1}{p+\gamma}\left(1-\left(\frac{y}{b}\right)^{\frac{k-1}{p+\gamma}}\right)y^{p+\gamma-k}(f(y))^{p+\gamma}dy\\
&-\left(\frac{k-1}{p+\gamma}\right)^{p+\gamma+1}\int_0^b y^{-k}\left(\int_0^y f(s)ds\right)^{p+\gamma}dy. \qquad (5.11)
\end{aligned}
$$

Therefore from (5.7), (5.9)-(5.11), after dividing L and R by $\left(\frac{k-1}{p+\gamma}\right)^{p+\gamma+1}$, we get the first inequality in (5.8).

We obtain inequality (5.8) by starting with inequality (4.3) and chosing $\varphi(x) = x^p$, $p \geqslant 2$, $x \geqslant 0$. In Lemma 4 we proved that the right handside of (4.3) is nonnegative when γ is a nonnegative integer. Therefore, also the right hand-side of (5.8) is nonnegative when γ is a nonnegative integer.

The reverse of inequality (5.8) holds for $1 < p \leqslant 2$ because in this case the function $\varphi(x) = x^p$, $x > 0$, is subquadratic. The equality in the case $p = 2$ follows from the fact that there is equality in (2.1) for $\varphi(x) = x^2$ and therefore also in (4.1) and (4.3) for $p = 2$, $\gamma \in \mathbb{R}_+$. This extends the equality case proved in [11, Theorem 3.1].

When $\gamma = 0$ the first two double-integrals in (5.8) are indeed zero: The first double-integral is evidently zero since the integrand equals zero in this case. We get that the second double-integral equals zero for $\gamma = 0$ by changing the order of integration in this case, and then integrating between $t = 0$ and $t = x$ we get by a direct computation that the integral equals zero, and therefore also the double integral equals zero. The last integral in (5.8) coincides with the corresponding term in (1.5): This is obtained by choosing $\gamma = 0$, $k - 1 = p - \alpha - 1$, and by the fact that

$$\frac{x^{\left(1-\frac{k-1}{p}\right)(p+1)}}{x^2} = x^{p-k-\frac{k-1}{p}}.$$

This completes the proof of the theorem. □

REFERENCES

[1] S. ABRAMOVICH, S. IVELIĆ AND J. PEČARIĆ, *On convex, superquadratic and superterzatic functions Jensen and Jensen-Steffensen inequalities*, to appear in Inequalities and Applications **10**, International Series of Numerical Mathematics, 161, Birkhauser, Basel, 2012.

[2] S. ABRAMOVICH, G. JAMESON AND G. SINNAMON, *Refining Jensen's inequality*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **47** (95), (2004), 3–14.

[3] S. ABRAMOVICH, AND L. E. PERSSON, *Some new refined Hardy type inequalities with breaking points $p = 2$ or $p = 3$*, Research report 2012, submitted (9 pages).

[4] G. H. HARDY, *Note on a theorem of Hilbert*, Math. Z. **6** (1920), 314–317.

[5] G. H. HARDY, *Notes on some points in the integral calculus*, LX, An inequality between integrals, Messanger Math. **54** (1925), 150–156.

[6] G. H. HARDY, *Notes on some points in the integral calculus*, LXIV, Further inequalities between integrals, Messanger of Math. **57** (1928), 12–16.

[7] C. O. IMORU, *On some integral inequalities related to Hardy's*, Canad. Math. Bull **20**, 3 (1977), 307–312.

[8] V. KOKITASIHVILI, A. MESHKI AND L. E. PERSSON, *Weighted norm inequalities for integral transform with product kernels*, Methematics Research Development Series, Nova Science Publishers Inc., New York, 2010.

[9] A. KUFNER, L. MALIGRANDA AND L. E. PERSSON, *The Hardy inequality. About its history and related results*, Vydavatelsky Servis, Pilzen, 2007.

[10] A. KUFNER AND L. E. PERSSON, *Weighted inequalities of Hardy type*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[11] J. A. OGUNTUASE AND L. E. PERSSON, *Refinement of Hardy's inequalities via superquadratic and subquadratic functions*, J. Math. Anal. Appl. **339** (2008), 1305–1312.

[12] J. A. OGUNTUASE AND L. E. PERSSON, *Refinement of Hardy's inequality for "all" p. Banach and function spaces II*, Yokohama Publ., Yokohama, 2008, 129–144.
[13] J. A. OGUNTUASE AND L. E. PERSSON, *Hardy type inequalities via convexity – the journey so far*, Aust. J. Math. Anal. Appl. **7**, 2 (2010), Art. 18, 19pp.
[14] L. E. PERSSON AND N. SAMKO, *What should have happened if Hardy had discovered this?*, J. Inequal. Appl. **29** (2012), 20 pp.

(Received March 28, 2012)

S. Abramovich
Department of Mathematics
University of Haifa
Haifa, Israel
e-mail: abramos@math.haifa.ac.il

L.-E. Persson
Department of Mathematics
Luleå University of Technology, SE
971871, Luleå, Sweden
and
Narvik University College
P.O.Box 385, N-8505
Narvik, Norway
e-mail: larserik@ltu.se