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UNITARILY INVARIANT NORMS RELATED TO THE NUMERICAL RADIUS ON $B(H)$

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Abstract. We determine the maximum (minimum) in the class of unitarily invariant norms $\|\cdot\|$ such that $\|T\| \leqslant w(T)$ ($\|T\| \geqslant w(T)$) for every bounded operator T in $B(H)$. Here, H is an infinite dimensional Hilbert space and $w(T)$ denotes the numerical radius of T.

1. Introduction

A *unitarily invariant* norm on $B(H)$, the algebra of all bounded linear operators on a Hilbert space H, is a norm that satisfies $\|UTV\| = \|T\|$, and also a norm is called *weakly unitarily invariant* if $\|UTU^*\| = \|T\|$, where $U,V,T \in B(H)$ and U,V are unitary.

The most familiar example of a weakly unitarily (but not unitarily) invariant norm is the numerical radius $w(\cdot)$ defined as

$$w(T) = \sup\{|\langle Tx,x\rangle| : \ \|x\| \leqslant 1\}.$$

The following inequalities are well known and easily proved:

$$\frac{\|\cdot\|_{\text{op}}}{2} \leqslant w(\cdot) \leqslant \|\cdot\|_{\text{op}},$$

where $\|\cdot\|_{\text{op}}$ is the operator norm on $B(H)$.

Some examples of unitarily invariant norms on $B(H)$ can be found in [3]. When H is of finite dimension n, we shall identify $B(H)$ with M_n, the algebra of all $n\times n$ complex matrices. In this case, the singular value decomposition implies a very nice representation of unitarily invariant norms as symmetric gauge functions [5].

Typical examples of unitarily invariant norms on M_n are Ky Fan p-norms defined by

$$\|A\|_p = \Sigma_{i=1}^n[(s_i(A))^p]^{\frac{1}{p}},$$

where $s_1 \geqslant \ldots \geqslant s_n$ are singular values of A. More details can be found in [2,5].

T. Ando [1] proved that $\max\{\frac{s_1(\cdot)}{2}, \frac{\|\cdot\|_1}{n}\}$ ($s_1(\cdot)$) is the maximum (minimum) in the class of unitarily invariant norms $\|\cdot\|$ such that $\|A\| \leqslant w(A)$ ($\|A\| \geqslant w(A)$) for all $n\times n$ matrices A in M_n. In this paper, we prove that if H is infinite dimensional, then $\max\{\frac{s_1(\cdot)}{2}, \frac{\|\cdot\|_1}{n}\}$ is replaced by $\frac{\|\cdot\|_{\text{op}}}{2}$.

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2. Main result

The following lemma can be obtained from [4, Proposition 3.18] and is used in the next theorem.

LEMMA 2.1. *A norm $\|\cdot\|$ on $B(H)$ is unitarily invariant if and only if for all operators R, T and S the inequality $\|RTS\| \leqslant \|R\|_{op}\|T\|\|S\|_{op}$ holds.*

THEOREM 2.2. *Let H be an infinite dimensional Hilbert space. Then $\frac{1}{2}\|\cdot\|_{op}$ $(\|\cdot\|_{op})$ is the maximum (minimum) in the class of unitarily invariant norms $\|\cdot\|$ satisfying the inequality $\|T\| \leqslant w(T)$ $(w(T) \leqslant \|T\|)$ for all $T \in B(H)$.*

Proof. Let $\{e_\alpha\}_{\alpha\in J}$ be an orthonormal basis for Hilbert space H. Since J is an infinite set, we can choose subsets J_1, J_2 of J such that $J_1 \cap J_2 = \varnothing$ and $card(J_1) = card(J_2) = card(J)$ [6]. Then, there exist bijective maps $f : J_1 \to J_2$ and $g : J_1 \to J$.

Let $\|\cdot\|$ be a unitarily invariant norm satisfying the inequality $\|T\| \leqslant w(T)$ for all $T \in B(H)$. For proving $\|\cdot\| \leqslant \frac{1}{2}\|\cdot\|_{op}$, it is sufficient to prove $\|T\| \leqslant 1$ for every operator T such that $\|T\|_{op} = 2$.

Now, let $\|T\|_{op} = 2$ and $A = \|T\|_{op}I$, where I is the identity operator on H. Also, suppose that $U = \Sigma_{\alpha\in J_1} e_{f(\alpha)} \otimes e_\alpha$, $B = 2\Sigma_{\alpha\in J_1} e_\alpha \otimes e_{f(\alpha)}$, $C = BU = 2\Sigma_{\alpha\in J_1} e_\alpha \otimes e_\alpha$ and $V = \Sigma_{\alpha\in J_1} e_{g(\alpha)} \otimes e_\alpha$. Considering an arbitrary element h on the unit ball of H, with $h = \Sigma_{\alpha\in J} h_\alpha e_\alpha$, we have:

$$\begin{aligned} |\langle B(h), h\rangle| &= |\langle 2\Sigma_{\alpha\in J_1} h_{f(\alpha)} e_\alpha, \Sigma_{\alpha\in J} h_\alpha e_\alpha\rangle| = |\Sigma_{\alpha\in J_1} 2h_{f(\alpha)}\overline{h_\alpha}| \\ &\leqslant \Sigma_{\alpha\in J_1}(|h_{f(\alpha)}|^2 + |h_\alpha|^2) \leqslant \|h\|^2 = 1. \end{aligned}$$

Hence, the inequality $w(B) \leqslant 1$ holds. But V and U are partial isometries and so $\|V\|_{op} = \|U\|_{op} = 1$. Also, since $A = VCV^*$, we have $\|A\| \leqslant \|C\|$. Using Lemma 2.1 we conclude that

$$\|T\| = \|TI\| \leqslant \|T\|_{op}\|I\| = \|A\| \leqslant \|C\| \leqslant \|B\|\|U\|_{op} \leqslant w(B) = 1.$$

Now, let $\|\cdot\|$ be a unitarily invariant norm that satisfies the inequality $w(T) \leqslant \|T\|$ for all $T \in B(H)$. But $\|S\|_{op} = w(S)$, for every hermitian operator S. Therefore

$$\|T\|_{op}^2 = \|TT^*\|_{op} = w(TT^*) \leqslant \|TT^*\| \leqslant \|T\|\|T^*\|_{op} = \|T\|\|T\|_{op}$$

and so $\|T\|_{op} \leqslant \|T\|$ for every $T \in B(H)$. $\square$

REFERENCES

[1] T. ANDO, *Unitarily invariant norms related to the numerical radius*, Linear Algebra Appl. **417** (2006), 3–7.

[2] R. BHATIA, *Matrix Analysis*, Graduate texts in mathematics, Springer-Verlag , 1997.

[3] J. T. CHAN, C. K. LI, C. N. TU, *A class of unitarily invariant norms on* $B(H)$, Proc. Amer. Math. Soc. **129**, 4 (2000), 1065–1076.

[4] J. FANG, D. HADWIN, E. NORDGREN, J. SHEN, *Tracial gauge norms on finite von Neumann algebras satisfying the weak Dixmier property*, J. Funct. Anal. **255**, 1 (2008), 142–183.

[5] R. A. HORN, C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991 .

[6] D. GOLDREI, *Classic Set Theory*, Chapman and Hall, London, 1996.

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