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ON INEQUALITY $R_p < R$ OF THE PEDAL TRIANGLE

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Abstract. In this paper we give a simple proof of the pedal triangle inequality $R_p < R$, where R is the circumradius of a triangle and R_p is the circumradius of the pedal triangle of an interior point with respect to this triangle. We also establish a stronger result and a refinement of inequality $R_p < R$. Some related interesting conjectures checked by the computer are put forward.

1. Introduction and main results

Let P be an interior point of the triangle ABC, let D, E, F be the feet of perpendiculars from P to the sides BC, CA, AB. Denote by S, R, r, s the area, circumradius inradius, and semi-perimeter of the triangle ABC respectively, and denote by S_p, R_p, r_p the area, circumradius and inradius of the pedal triangle DEF, respectively. As usual, we put

$$BC = a, \quad CA = b, \quad AB = c,$$

$$PD = r_1, \quad PE = r_2, \quad PF = r_3,$$

$$PA = R_1, \quad PB = R_2, \quad PC = R_3.$$

It is well known that the following inequality holds between S_p and S:

$$S_p \leqslant \frac{1}{4}S, \tag{1.1}$$

with equality if and only if P coincides with the circumcenter O of $\triangle ABC$. This inequality is a direct consequence of the following Gergonne formula (see [1]):

$$S_p = \frac{1}{4}\left(1 - \frac{PO^2}{R^2}\right)S, \tag{1.2}$$

which holds actually for arbitrary interior point P of the circumcircle of $\triangle ABC$.

Of course, we can consider other inequalities between the pedal triangle DEF and the original triangle ABC. For instance, compare R_p with R we can find the following inequality:

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THEOREM 1.1. *For any interior point P of $\triangle ABC$, we have*

$$R_p < R, \tag{1.3}$$

in which the constant 1 in front of R is the best possible.

In fact, inequality (1.3) is given in [2] by the author without proof twenty years ago. But it seems that nobody has been studying this inequality since then. In this paper we shall give a simple proof of inequality (1.3).

Recently, the author found $R_p < R$ can be strengthened to (1.4) below.

THEOREM 1.2. *For any interior point P of $\triangle ABC$, we have*

$$R_p + \frac{3\sqrt{3}r_1r_2r_3}{4S_p} \leqslant R, \tag{1.4}$$

with equality if and only if $\triangle ABC$ is equilateral and P is its center.

In my recent paper [3], I have proved a linear inequality:

$$R_1 + R_2 + R_3 - r_1 - r_2 - r_3 \geqslant 6r_p. \tag{1.5}$$

This and (1.3) inspire us to obtain the following refinement of inequality $R_p < R$:

THEOREM 1.3. *For any interior point P of $\triangle ABC$, we have*

$$R_p < \frac{1}{2}(R_1 + R_2 + R_3 - r_1 - r_2 - r_3) < R, \tag{1.6}$$

in which the constant $\frac{1}{2}$ is the best possible.

2. Proofs of the theorems

2.1. Proof of Theorem 1.1

Proof. In order to prove inequality (1.3), we first show that the weighted inequality which involves three sides and the area of $\triangle ABC$:

$$(xa + yb + zc)(ayz + bzx + cxy) > 2(y+z)(z+x)(x+y)S \tag{2.1}$$

holds for all positive real numbers x, y, z.

Since

$$\begin{aligned}
&(xa + yb + zc)(ayz + bzx + cxy) - 2(y+z)(z+x)(x+y)S \\
&= xyz(a^2 + b^2 + c^2) + a(bz + cy)x^2 + b(cx + az)y^2 + c(ay + bx)z^2 \\
&\quad -2\left[2xyz + x(y^2 + z^2) + y(z^2 + x^2) + z(x^2 + y^2)\right] S \\
&= xyz(a^2 + b^2 + c^2 - 4S) + (bc - 2S)x(y^2 + z^2) \\
&\quad +(ca - 2S)y(z^2 + x^2) + (ab - 2S)z(x^2 + y^2) > 0,
\end{aligned}$$

hence inequality (2.1) holds true.

Putting $x = r_1$, $y = r_2$, $z = r_3$ in (2.1), then using the identity:

$$ar_1 + br_2 + cr_3 = 2S, \tag{2.2}$$

we get

$$ar_2r_3 + br_3r_1 + cr_1r_2 > (r_2 + r_3)(r_3 + r_1)(r_1 + r_2).$$

Noticing the following identity:

$$ar_2r_3 + br_3r_1 + cr_1r_2 = 4RS_p, \tag{2.3}$$

which is easily proved, we have

$$(r_2 + r_3)(r_3 + r_1)(r_1 + r_2) < 4RS_p. \tag{2.4}$$

Since also

$$(r_2 + r_3)(r_3 + r_1)(r_1 + r_2) > EF \cdot FD \cdot DE = 4R_pS_p,$$

here we used the known formula $abc = 4SR$ for the pedal $\triangle DEF$. Thus, the claimed inequality $R_p < R$ follows from (2.4) at once.

We now show the constant in (1.3) is the best possible. Suppose that the following inequality:

$$(xa + yb + zc)(ayz + bzx + cxy) > k(y + z)(z + x)(x + y)S$$

holds for positive numbers x, y, z and k. For $y = z = 1$ and let $x \to 0$, then we have $(b + c)a > 2kS$, thus

$$b + c > kh_a, \tag{2.5}$$

where h_a is the altitude of BC. If we take $b = c$ and let $a \to 0$, then $b \to h_a$, $c \to h_a$ and $k < 2$ follows further from (2.5). This means that the constant 2 on the right hand side of (2.1) is optimal. Thus, the constant 1 in front of R in $R_p < R$ is the best possible. The proof of Theorem 1.1 is completed. □

2.2. Proof of Theorem 1.2

To prove Theorem 1.2, we need several lemmas.

LEMMA 2.1. *Let x, y, z be three real numbers such that $y + z > 0$, $z + x > 0$, $x + y > 0$ and $yz + zx + xy > 0$. Then the following inequality:*

$$xa^2 + yb^2 + zc^2 \geqslant 4\sqrt{yz + zx + xy}S \tag{2.6}$$

holds for any $\triangle ABC$. Equality in (2.6) holds if and only if $x : y : z = (b^2 + c^2 - a^2) : (c^2 + a^2 - b^2) : (a^2 + b^2 - c^2)$.

Proof. Indeed, the weighted inequality (2.6) is equivalent to the following famous Neuberg-Pedoe inequality for two triangles (see, e.g., [4]):

$$a^2(b'^2+c'^2-a'^2)+b^2(c'^2+a'^2-b'^2)+c^2(a'^2+b'^2-c'^2)\geqslant 16SS', \tag{2.7}$$

where a', b', c' are the sides of $\triangle A'B'C'$ and S' is its area. Equality in (2.7) holds if and only if the two triangles are similar.

When real numbers x, y, z satisfy $y+z>0$, $z+x>0$, $x+y>0$ and $yz+zx+xy>0$, it is easy to prove that $\sqrt{y+z}$, $\sqrt{z+x}$, $\sqrt{x+y}$ form a triangle $A_0B_0C_0$ with area $\frac{1}{2}\sqrt{yz+zx+xy}$ (we omit the details). If we apply Neuberg-Pedoe inequality to $\triangle A_0B_0C_0$ and $\triangle ABC$, then inequality (2.6) follows immediately (In contrast, (2.7) can be deduced easily from (2.6)). Moreover the equality in (2.6) holds only when $a:b:c=\sqrt{y+z}:\sqrt{z+x}:\sqrt{x+y}$. This implies $x:y:z=(b^2+c^2-a^2):(c^2+a^2-b^2):(a^2+b^2-c^2)$. The proof of Lemma 2.1 is completed. □

Next, we give a lemma which is very interesting itself.

LEMMA 2.2. *For any interior point P of $\triangle ABC$, we have*

$$\frac{a}{r_1}+\frac{b}{r_2}+\frac{c}{r_3}-\frac{a}{R_1}-\frac{b}{R_2}-\frac{c}{R_3}\geqslant 3\sqrt{3}, \tag{2.8}$$

with equality if and only if $\triangle ABC$ is equilateral and P is its center.

Proof. Since the area of the quadrilateral is less than or equal to the half of product of two diagonals, so we have

$$S_{\triangle PCA}+S_{\triangle PAB}\leqslant \frac{1}{2}aR_1,$$

with equality only if $PA\perp BC$. Hence

$$br_2+cr_3\leqslant aR_1, \tag{2.9}$$

and two similar relations are valid. Therefore, to prove inequality (2.8) we need to prove that

$$\frac{a}{r_1}+\frac{b}{r_2}+\frac{c}{r_3}-\frac{a^2}{br_2+cr_3}-\frac{b^2}{cr_3+ar_1}-\frac{c^2}{ar_1+br_2}\geqslant 3\sqrt{3}. \tag{2.10}$$

Next, we shall show first that the equivalent weighted inequality:

$$(x+y+z)\left(\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}-\frac{a^2}{y+z}-\frac{b^2}{z+x}-\frac{c^2}{x+y}\right)\geqslant 6\sqrt{3}S, \tag{2.11}$$

i.e.

$$\frac{y+z-x}{x(y+z)}a^2+\frac{z+x-y}{y(z+x)}b^2+\frac{x+y-z}{z(x+y)}c^2\geqslant \frac{6\sqrt{3}}{x+y+z}S \tag{2.12}$$

holds for any positive real numbers x, y, z. Putting

$$x_1 = \frac{y+z-x}{x(y+z)}, \quad y_1 = \frac{z+x-y}{y(z+x)}, \quad z_1 = \frac{x+y-z}{z(x+y)},$$

we have

$$y_1 + z_1 = \frac{x(y^2+z^2+xy+xz)}{yz(z+x)(x+y)} > 0,$$

etc., and

$$y_1z_1 + z_1x_1 + x_1y_1 = \frac{2(x+y+z)}{(y+z)(z+x)(x+y)} > 0.$$

Thus, according to Lemma 2.1, to prove (2.12) it remains to prove that

$$4\sqrt{\frac{2(x+y+z)}{(y+z)(z+x)(x+y)}} \geqslant \frac{6\sqrt{3}}{x+y+z},$$

which is equivalent to

$$8(x+y+z)^3 \geqslant 27(y+z)(z+x)(x+y).$$

This follows from the arithmetic-geometric mean inequality obviously. Hence, inequalities (2.12) and (2.11) are proved.

In (2.11), putting $x = ar_1$, $y = br_2$, $z = cr_3$, then using the preceding identity (2.2) we get (2.10) immediately. Thus, inequality (2.8) follows from (2.10) by (2.9). Clearly, the equality in (2.12) occurs if and only if $x = y = z$, $a = b = c$. Further, we know that the equality condition of (2.8) is just as mentioned in Lemma 2.2. This completes the proof of Lemma 2.2. □

LEMMA 2.3. *For any interior point P, the following inequality holds:*

$$aR_2R_3 + bR_3R_1 + cR_1R_2 \geqslant abc, \tag{2.13}$$

with equality if and only if P coincide with one vertex of $\triangle ABC$ or $\triangle ABC$ is acute-angled and P is its orthocenter.

Inequality (2.13) is due to T. Hayashi and is actually valid for arbitrary point P (see [4], [5]).

LEMMA 2.4. *If the following inequality:*

$$f(a,b,c,R_1,R_2,R_3,r_1,r_2,r_3) \geqslant 0 \tag{2.14}$$

holds for arbitrary point P of the plane of $\triangle ABC$, then the inequality holds after making transformation K:

$$\begin{aligned}&(a,b,c,R_1,R_2,R_3,r_1,r_2,r_3)\\ &\to \left(\frac{aR_1}{2r_2r_3R}, \frac{bR_2}{2r_3r_1R}, \frac{cR_3}{2r_1r_2R}, \frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}, \frac{1}{R_1}. \frac{1}{R_2}, \frac{1}{R_3}\right).\end{aligned}$$

The K transformation is called reciprocation transformation (see e.g., [2], [4], [6], [7], [8]).

LEMMA 2.5. *For any interior point P of $\triangle ABC$, we have*

$$ar_1R_1^2 + br_2R_2^2 + cr_3R_3^2 = 8R^2S_p. \tag{2.15}$$

The identity (2.15) is very important and is equivalent to

$$ar_1R_1^2 + br_2R_2^2 + cr_3R_3^2 = 2R(ar_2r_3 + br_3r_1 + cr_1r_2), \tag{2.16}$$

which is given in [8] by M. S. Klamkin. In [2], the author pointed out that the more general identity:

$$\vec{S}_{\triangle PBC}PA^2 + \vec{S}_{\triangle PCA}PB^2 + \vec{S}_{\triangle PAB}PC^2 = 4R^2\vec{S}_{\triangle DEF} \tag{2.17}$$

(where $\vec{S}_{\triangle PBC}$ denotes the directed area of $\triangle PBC$ etc., see e.g. [3]) holds for any point P in the plane. A new proof of (2.17) is given recently by the author in [9].

Next, we prove Theorem 1.2.

Proof. By Lemma 2.3, we have

$$\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} \geqslant \frac{abc}{R_1R_2R_3}. \tag{2.18}$$

Coupling (2.18) with inequality (2.8) of Lemma 2.2 yields

$$\frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3} - \frac{abc}{R_1R_2R_3} \geqslant 3\sqrt{3}, \tag{2.19}$$

in which the equality condition is the same as in (2.8). Applying Lemma 2.4 to inequality (2.19), we obtain

$$\frac{aR_1^2}{2r_2r_3R} + \frac{bR_2^2}{2r_3r_1R} + \frac{cR_3^2}{2r_1r_2R} - \frac{abcR_1R_2R_3}{8r_1r_2r_3R^3} \geqslant 3\sqrt{3}.$$

As $a = 2R\sin A$ etc., so that

$$\frac{1}{2R}\left(ar_1R_1^2 + br_2R_2^2 + cr_3R_3^2\right) - R_1R_2R_3\sin A\sin B\sin C \geqslant 3\sqrt{3}r_1r_2r_3.$$

By Lemma 2.5 and the following identity:

$$R_1R_2R_3\sin A\sin B\sin C = 4S_pR_p, \tag{2.20}$$

which is gotten by using $abc = 4SR$ to the pedal triangle DEF, we get

$$4(R - R_p)S_p \geqslant 3\sqrt{3}r_1r_2r_3,$$

which is equivalent with inequality (1.4) of Theorem 1.2. Clearly, the equality in (1.4) holds only when $\triangle ABC$ is equilateral and P is its center. This completes the proof of Theorem 1.2. □

2.3. Proof of Theorem 1.3

LEMMA 2.6. *The ternary quadratic inequality:*

$$p_1x^2 + p_2y^2 + p_3z^2 \geqslant q_1yz + q_2zx + q_3xy \tag{2.21}$$

holds for all real numbers x, y, z *if and only if* $p_1 \geqslant 0,\ p_2 \geqslant 0,\ p_3 \geqslant 0,\ 4p_2p_3 - q_1^2 \geqslant 0,$ $4p_3p_1 - q_2^2 \geqslant 0,\ 4p_1p_2 - q_3^2 \geqslant 0,$ *and*

$$4p_1p_2p_3 - (q_1q_2q_3 + p_1q_1^2 + p_2q_2^2 + p_3q_3^2) \geqslant 0. \tag{2.22}$$

The above conclusion is well known (see, e.g., [10]). Because it gives the necessary and sufficient conditions of the general ternary quadratic inequality, the importance is self-evident. In recent years, we have used it in a number of articles (see, e.g., [11]–[13]).

LEMMA 2.7. *For all real numbers* u, v, w *and positive real numbers* x, y, z *we have*

$$M_1 \equiv d_1u^2 + d_2v^2 + d_3w^2 - (e_1vw + e_2wu + e_3uv) > 0, \tag{2.23}$$

where

$$\begin{aligned} d_1 &= (x+y+z)(y^2+z^2),\\ d_2 &= (x+y+z)(z^2+x^2),\\ d_3 &= (x+y+z)(x^2+y^2),\\ e_1 &= 2x[x(x+y+z) - 2yz],\\ e_2 &= 2y[y(x+y+z) - 2zx],\\ e_3 &= 2z[z(x+y+z) - 2xy]. \end{aligned}$$

Proof. First we can verify that

$$\begin{aligned} 4d_2d_3 - e_1^2 = {}& 4(y^2+4yz+z^2)x^4 + 4(y+z)^2y^2z^2 + 8(y+z)^3x^3\\ &+4(y^2+3yz+z^2)(y^2-yz+z^2)x^2 + 8(y+z)xy^2z^2 \end{aligned}$$

Since $y^2 - yz + z^2 > 0$, then $4d_2d_3 - e_1^2 > 0$. Analogously, we have $4d_3d_1 - e_2^2 > 0$, $4d_1d_2 - e_3^2 > 0$. Thus, according to Lemma 2.6, to prove (2.23) we need to prove that

$$4d_1d_2d_3 - (d_1e_1^2 + d_2e_2^2 + d_3e_3^2 + e_1e_2e_3) > 0. \tag{2.24}$$

It is easily verified that

$$4d_1d_2d_3 - (d_1e_1^2 + d_2e_2^2 + d_3e_3^2 + e_1e_2e_3) = 16xyz(Q_1 + Q_2), \tag{2.25}$$

where

$$\begin{aligned} Q_1 &= (3y^2 - 4yz + 3z^2)x^4 + 2(y+z)(3y^2 - 2yz + 3z^2)x^3,\\ Q_2 &= (3y^4 + 2y^3z + 13y^2z^2 + 2yz^3 + 3z^4)x^2\\ &\quad -2yz(y+z)(2y^2 - 3yz + 2z^2)x + 3(y+z)^2y^2z^2. \end{aligned}$$

Note that $3y^2-4yz+3z^2>0$ and $3y^2-2yz+3z^2>0$, hence $Q_1>0$. Again, it is easy to compute the quadratic discriminant of Q_2 on x:

$$\Delta_1=-4y^2z^2(5y^4+18y^3z+22z^2y^2+18z^3y+5z^4)(y+z)^2<0.$$

Hence $Q_2>0$. Therefore, we deduce (2.24) is true from (2.25). This completes the proof of Lemma 2.7. □

LEMMA 2.8. *For any positive real numbers $x,\ y,\ z,\ u,\ v,\ w$, we have*

$$M_2\equiv f_1u^2+f_2v^2+f_3w^2-(g_1vw+g_2wu+g_3uv)>0, \tag{2.26}$$

where

$$\begin{aligned}
f_1&=(y+z)(y^2+xy+zx+z^2),\\
f_2&=(z+x)(z^2+yz+yx+x^2),\\
f_3&=(x+y)(x^2+zx+zy+y^2),\\
g_1&=2x^3+2(y+z)x^2-(y^2+z^2)x-2yz(y+z),\\
g_2&=2y^3+2(z+x)y^2-(z^2+x^2)y-2zx(z+x),\\
g_3&=2z^3+2(x+y)z^2-(x^2+y^2)z-2xy(x+y).
\end{aligned}$$

Proof. In fact, inequality (2.26) is not valid for all real numbers $u,\ v,\ w$. So we can not apply Lemma 2.6 directly to prove it. After analysing, we find the following identity:

$$M_2=E_1+E_2+E_3+M_1, \tag{2.27}$$

where M_1 is the same as in Lemma 2.7, and

$$\begin{aligned}
E_1&=2yz(y+z)vw+2zx(z+x)wu+2xy(x+y)uv,\\
E_2&=vwx(y-z)^2+wuy(z-x)^2+uvz(x-y)^2,\\
E_3&=xyz[(v-w)^2+(w-u)^2+(u-v)^2].
\end{aligned}$$

Clearly, $E_1>0,\ E_2\geqslant 0$ and $E_3\geqslant 0$ hold for positive real numbers $x,\ y,\ z,\ u,\ v,\ w$. Therefore, by identity (2.27) and the inequality $M_1\geqslant 0$ of Lemma 2.7, we see that $M_2>0$ holds for the positive real numbers. Thus we complete the proof of Lemma 2.8. □

LEMMA 2.9. *For any triangle ABC and positive real numbers $x,\ y,\ z$, we have*

$$\frac{s-a}{x}+\frac{s-b}{y}+\frac{s-c}{z}\geqslant\frac{s(xa+yb+zc)}{yza+zxb+xyc}, \tag{2.28}$$

with equality if and only if $x=y=z$.

In [2], the author proved inequality (2.28) by using the polar moment of the inertia inequality of M. S. Klamkin (for the latter see [14], [15], [16]). We give a direct proof of Lemma 2.9 here.

Proof. It is not difficult to verify the identity:

$$\begin{aligned}&(ax+by+cz)[(b+c-a)x+(c+a-b)y+c(a+b-c)z]\\&\quad-(a+b+c)(ayz+bzx+cxy)\\&=\frac{1}{2}(c+a-b)(a+b-c)(y-z)^2+\frac{1}{2}(a+b-c)(b+c-a)(z-x)^2\\&\quad+\frac{1}{2}(b+c-a)(c+a-b)(x-y)^2.\end{aligned}\tag{2.29}$$

Hence the following inequality holds for all real numbers x, y, z:

$$\begin{aligned}&(ax+by+cz)[(b+c-a)x+(c+a-b)y+c(a+b-c)z]\\&\geqslant(a+b+c)(ayz+bzx+cxy),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&(ax+by+cz)[(s-a)x+(s-b)y+c(s-c)z]\\&\geqslant s(ayz+bzx+cxy).\end{aligned}\tag{2.30}$$

If $x>0$, $y>0$, $z>0$, by replacing $x\to\frac{1}{x}$, $y\to\frac{1}{y}$, $z\to\frac{1}{z}$ in (2.30) and then multiplying both sides by xyz, we get (2.28) immediately. From (2.29), it is seen that the equality in (2.28) holds if and only if $x=y=z$. Lemma 2.9 is proved. □

REMARK 2.1. In a recent paper [17], the author established the following geometric inequality:

$$R_1^k+R_2^k+R_3^k\geqslant(2R_p)^k+2(4r_p)^k,\tag{2.31}$$

where $k\geqslant 1$. We also used Lemma 2.9 to prove this inequality there.

LEMMA 2.10. *Under the K transformation in Lemma 2.4, we have the following transformational relations:*

$$S\to\frac{S}{2r_1r_2r_3R},\quad R\to\frac{R_1R_2R_3}{4r_1r_2r_3R},\quad S_p\to\frac{S}{2R_1R_2R_3R_p}.$$

In [2], the author has given the transformational relations for the elements S, R, S_p, R_p under the five transformations (including the above lemma 2.10). These can be proved by using previous identities (2.2), (2.3), (2.15) and (2.20) etc.

We now prove Theorem 1.3.

Proof. First, we prove the right hand inequality of the double inequality (1.6):

$$\frac{1}{2}(R_1+R_2+R_3-r_1-r_2-r_3)<R.\tag{2.32}$$

From inequality $PE+PF>EF$ and the fact $EF=R_1\sin A$, we get

$$R_1<\frac{r_2+r_3}{\sin A}, \tag{2.33}$$

and have two analogues. Thus, it is enough to show that

$$\frac{r_2+r_3}{\sin A}+\frac{r_3+r_1}{\sin B}+\frac{r_1+r_2}{\sin C}-r_1-r_2-r_3<2R.$$

Namely,

$$r_1\left(\frac{1}{\sin B}+\frac{1}{\sin C}-1\right)+r_2\left(\frac{1}{\sin C}+\frac{1}{\sin A}-1\right) +r_3\left(\frac{1}{\sin A}+\frac{1}{\sin B}-1\right)<2R. \tag{2.34}$$

Noticing identity $ar_1+br_2+cr_3=2S$, it remains to prove that

$$\frac{1}{R}\left(\frac{1}{\sin B}+\frac{1}{\sin C}-1\right)\leqslant\frac{a}{S} \tag{2.35}$$

and two analogous inequalities. Because of symmetry, we only need to prove (2.35). Since $a=2R\sin A$ and $S=2R^2\sin A\sin B\sin C$, thus (2.35) is equivalent to the following trigonometric inequality:

$$\frac{1}{\sin B}+\frac{1}{\sin C}-1\leqslant\frac{1}{\sin B\sin C},$$

i.e.,

$$\sin B\sin C-\sin B-\sin C+1\geqslant 0,$$

which is equivalent to the evident inequality:

$$(1-\sin B)(1-\sin C)\geqslant 0.$$

This completes the proof of (2.32).

Secondly, we prove the left hand inequality of (1.6):

$$R_p<\frac{1}{2}(R_1+R_2+R_3-r_1-r_2-r_3). \tag{2.36}$$

We shall first prove the following weighted inequality:

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-\frac{a}{y+z}-\frac{b}{z+x}-\frac{c}{x+y} >\frac{1}{s}\left[\frac{a(s-a)}{x}+\frac{b(s-b)}{y}+\frac{c(s-c)}{z}\right]. \tag{2.37}$$

where $x > 0$, $y > 0$, $z > 0$ and $s = (a+b+c)/2$. It is not difficult to obtain the following identity:

$$\begin{aligned}
&\frac{a}{x}+\frac{b}{y}+\frac{c}{z}-\frac{a}{y+z}-\frac{b}{z+x}-\frac{c}{x+y} \\
&\quad -\frac{1}{s}\left[\frac{a(s-a)}{x}+\frac{b(s-b)}{y}+\frac{c(s-c)}{z}\right] \\
&= \frac{m_1a^2+m_2b^2+m_3c^2-(n_1bc+n_2ca+n_3ab)}{xyz(y+z)(z+x)(x+y)(a+b+c)},
\end{aligned} \tag{2.38}$$

where

$$\begin{aligned}
m_1 &= yz(z+x)(x+y)(2y+2z-x), \\
m_2 &= zx(x+y)(y+z)(2z+2x-y), \\
m_3 &= xy(y+z)(z+x)(2x+2y-z), \\
n_1 &= xyz(y+z)(y+z+2x), \\
n_2 &= xyz(z+x)(z+x+2y), \\
n_3 &= xyz(x+y)(x+y+2z).
\end{aligned}$$

Therefore, to prove (2.38) we need to prove that

$$M_3 \equiv m_1a^2+m_2b^2+m_3c^2-(n_1bc+n_2ca+n_3ab) > 0. \tag{2.39}$$

If we put $s-a=u$, $s-b=v$, $s-c=w$, then $a=v+w$, $b=w+u$, $c=u+v$, and

$$\begin{aligned}
M_3 = {} & m_1(v+w)^2+m_2(w+u)^2+m_3(u+v)^2 \\
& -[n_1(w+u)(u+v)+n_2(u+v)(v+w)+n_3(v+w)(w+u)].
\end{aligned} \tag{2.40}$$

Substituting m_1, m_2, m_3, n_1, n_2, n_3 into (2.40), we obtain further

$$M_3 = 2\left[k_1u^2+k_2v^2+k_3w^2-(t_1vw+t_2wu+t_3uv)\right], \tag{2.41}$$

where

$$\begin{aligned}
k_1 &= x^2(y+z)(y^2+xy+zx+z^2), \\
k_2 &= y^2(z+x)(z^2+yz+yx+x^2), \\
k_3 &= z^2(x+y)(x^2+zx+zy+y^2), \\
t_1 &= yz[2x^3+2(y+z)x^2-(y^2+z^2)x-2yz(y+z)], \\
t_2 &= zx[2y^3+2(z+x)y^2-(z^2+x^2)y-2zx(z+x)], \\
t_3 &= xy[2z^3+2(x+y)z^2-(x^2+y^2)z-2xy(x+y)].
\end{aligned}$$

Therefore, we have to prove that

$$k_1u^2+k_2v^2+k_3w^2-(t_1vw+t_2wu+t_3uv) > 0. \tag{2.42}$$

Replacing $u \to u/x$, $v \to /y$, $w \to /z$ in (2.42), the inequality turns into the inequality (2.26) of Lemma 2.8. Hence, inequality $M_3 > 0$ holds true and the inequality (2.37) is proved.

We now make substitutions $x \to xa$, $y \to yb$, $z \to zc$ in (2.37), then

$$\begin{aligned} &\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{a}{yb+zc}-\frac{b}{zc+xa}-\frac{c}{xa+yb} \\ &> \frac{1}{s}\left(\frac{s-a}{x}+\frac{s-b}{y}+\frac{s-c}{z}\right). \end{aligned} \tag{2.43}$$

This inequality and the inequality (2.28) of Lemma 2.9 imply that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{a}{yb+zc}-\frac{b}{zc+xa}-\frac{c}{xa+yb} > \frac{xa+yb+zc}{yza+zxb+xyc}. \tag{2.44}$$

For $x = r_1$, $y = r_2$, $z = r_3$ in (2.44), using identities (2.2), (2.3) and the previous inequality (2.9) $br_2 + cr_3 \leqslant aR_1$ etc., one has

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{R_1}-\frac{1}{R_2}-\frac{1}{R_3} > \frac{S}{2RS_p}. \tag{2.45}$$

Applying K transformation of Lemma 2.4 to inequality (2.45) and then using Lemma 2.10, we immediately obtain

$$R_1+R_2+R_3-r_1-r_2-r_3 > 2R_p.$$

Hence the desired inequality (2.36) is proved.

Finally, we show that the constant $\frac{1}{2}$ in (1.6) is optimal.

Suppose that the following inequalities:

$$k_1(R_1+R_2+R_3-r_1-r_2-r_3) < R \tag{2.46}$$

and

$$R_p < k_2(R_1+R_2+R_3-r_1-r_2-r_3) \tag{2.47}$$

hold for any interior point P. Considering an isosceles triangle ABC whose sides are 1, 1, $2x$ $(0 < x < 1)$ and AH_a is the altitude of BC. If we let $A \to 0, B \to \frac{\pi}{2}, C \to \frac{\pi}{2}$ and let $P \to A$, then $x \to 0$ and $R \to AH_a = \frac{1}{2}$, $R_p \to AH_a = \frac{1}{2}$, $R_1+R_2+R_3 \to 2$, $r_1+r_2+r_3 \to 1$. In this case, (2.46) and (2.47) become $k_1 < \frac{1}{2}$, $\frac{1}{2} < k_2$ respectively. The double inequality $k_1 < \frac{1}{2} < k_2$ means that the constant $\frac{1}{2}$ in (2.32) and (2.36) are both the best possible. The proof of Theorem 1.3 is completed. □

3. Some related conjectures

In this section, we propose some interesting related conjectures.

Considering the stronger inequalities of $R_p < R$, the author first conjectures that the inequality $R_p + 2r_p \leqslant R$ holds. However, through verifying by the computer, we find it is not true. But it is likely that the following strict inequality holds:

CONJECTURE 3.1. For any interior point P of $\triangle ABC$, we have

$$R_p + \sqrt{2} r_p < R. \tag{3.1}$$

On the other hand, we conjecture

$$R_p^2 + 12 r_p^2 \leqslant R^2, \tag{3.2}$$

which is clearly better than $R_p < R$. More generally, we propose the following exponential generalization:

CONJECTURE 3.2. Let $k \geqslant 2$ be a real number, then we have

$$R_p^k + 2^k(2^k - 1) r_p^k \leqslant R^k. \tag{3.3}$$

For the inequality (1.4) of Theorem 1.2, we propose the following stronger conjecture:

CONJECTURE 3.3. For any interior point P of $\triangle ABC$, we have

$$R_p + \frac{s r_1 r_2 r_3}{4 r S_p} \leqslant R. \tag{3.4}$$

The known inequality $s \geqslant 3\sqrt{3} r$ shows (3.4) is stronger than (1.4). If (3.4) is true, then by inequality (1.1) and $S = rs$ we can get

$$R_p + \frac{r_1 r_2 r_3}{r^2} \leqslant R. \tag{3.5}$$

This weaker inequality has not yet been proven and inspires the author to pose the following similar inequality:

$$R_p + \frac{R_1 R_2 R_3}{8 R_p^2} \leqslant R. \tag{3.6}$$

(with equality only when P is the circumcenter of $\triangle ABC$). By the previous identity (2.21) and the known formula:

$$S = 2R^2 \sin A \sin B \sin C, \tag{3.7}$$

we see that (3.6) is equivalent to

$$R_p^2 + \frac{S_p}{S} R^2 \leqslant R R_p. \tag{3.8}$$

Again, by Gergonne formula (1.2), the above inequality is equivalent to

$$4 R R_p \geqslant 4 R_p^2 + R^2 - PO^2.$$

Then the following interesting conjecture is arisen:

CONJECTURE 3.4. For any interior point P of $\triangle ABC$, we have

$$PO \geqslant | R - 2R_p |, \tag{3.9}$$

where O is the circumcenter of $\triangle ABC$.

Comparing (3.5) with (3.6), we give the following conjecture:

CONJECTURE 3.5. For any interior point P of $\triangle ABC$, we have

$$\frac{R_1R_2R_3}{r_1r_2r_3} \geqslant 8\frac{R_p^2}{r^2}. \tag{3.10}$$

By (2.20) and (3.7), we conclude that the inequality $R_p < R$ is equivalent to

$$\frac{R_1R_2R_3}{8R^3} < \frac{S_p}{S}, \tag{3.11}$$

which prompts the author to present the stronger inequality:

CONJECTURE 3.6. For any interior point P of $\triangle ABC$, we have

$$\frac{8r_1r_2r_3 + R_1R_2R_3}{8R^3} \leqslant \frac{S_p}{S}. \tag{3.12}$$

If (3.12) is valid, then by the area inequality (1.1) we get

$$8r_1r_2r_3 + R_1R_2R_3 \leqslant 2R^3. \tag{3.13}$$

The author further thinks that the sharp inequlaity (3.14) below also hods.

CONJECTURE 3.7. For any interior point P of $\triangle ABC$, we have

$$(8r_1r_2r_3)^4 + (R_1R_2R_3)^4 \leqslant 2R^{12}. \tag{3.14}$$

For strict inequality (2.45), we propose the following:

CONJECTURE 3.8. For any interior point P of $\triangle ABC$, we have

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{R_1} - \frac{1}{R_2} - \frac{1}{R_3} \geqslant \frac{1}{R} + \frac{S}{2RS_p}. \tag{3.15}$$

If the above inequality holds true, then using (1.1) we get

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{R_1} - \frac{1}{R_2} - \frac{1}{R_3} \geqslant \frac{3}{R}, \tag{3.16}$$

which is also not proven till now. On the other hand, it makes the author to put forward

CONJECTURE 3.9. For any interior point P of $\triangle ABC$, we have

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{R_1}-\frac{1}{R_2}-\frac{1}{R_3}\geqslant\frac{3}{2r}. \tag{3.17}$$

Euler's inequality $R \geqslant 2r$ shows that (3.17) is better than (3.16).

Finally, for Theorem 1.3, we propose the following conjecture:

CONJECTURE 3.10. Let $k>1$ be a positive real number, then for any interior point P of $\triangle ABC$ we have

$$R_p^k<\frac{1}{2^k}\left(R_1^k+R_2^k+R_3^k-r_1^k-r_2^k-r_3^k\right)<R^k. \tag{3.18}$$

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