Mathematical
Inequalities
& Applications
Volume 16, Number 3 (2013), 717–728 doi:10.7153/mia-16-54

A CHARACTERIZATION OF THE STABILITY OF A SYSTEM OF THE BANACH SPACE VALUED DIFFERENTIAL EQUATIONS

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(*Communicated by J. Pečarić*)

Abstract. We will consider the Banach space valued differential equation $\mathfrak{y}'(t) = A\mathfrak{y}(t)$, where A is an $n \times n$ complex matrix. We give a necessary and sufficient condition in order that the equation have the Hyers-Ulam stability. As a Corollary, we prove that the Banach space valued linear differential equation with constant coefficients $y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1y'(t) + a_0y(t) = 0$ has the Hyers-Ulam stability if and only if $\mathrm{Re}\,\lambda \neq 0$ for all the solutions λ of the equation $z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 = 0$.

1. Introduction

It seems that the stability problem of functional equations had been first raised by S. M. Ulam (cf. [16, Chapter VI]). "For what metric groups G is it true that an ε-automorphism of G is necessarily near to a strict automorphism? (An ε-automorphism of G means a transformation f of G into itself such that $\rho(f(x \cdot y), f(x) \cdot f(y)) < \varepsilon$ for all $x, y \in G$.)"

D. H. Hyers [6] gave an affirmative answer to the problem as follows. Suppose that $f\colon E_1 \to E_2$ is a mapping between two real Banach spaces E_1 and E_2. If there exists $\varepsilon \geqslant 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon$$

for all $x, y \in E_1$, then the limit

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for each $x \in E_1$, and $T\colon E_1 \to E_2$ is the unique additive mapping such that

$$\|f(x) - T(x)\| \leqslant \varepsilon$$

for all $x \in E_1$. If, in addition, the mapping $\mathbb{R} \ni t \mapsto f(tx)$ is continuous for each fixed $x \in E_1$, then T is linear.

This result is called the *Hyers-Ulam stability* of the *additive* Cauchy equation $g(x+y) = g(x) + g(y)$. Here we note that Hyers [6] calls any solution of this equation a

Mathematics subject classification (2010): Primary 46B99; Secondary 39B72.
Keywords and phrases: Exponential functions, Hyers-Ulam stability, Hyers-Ulam-Rassias stability.
The first and third authors were partly supported by the Grant-in-Aid for Scientific Research (KAKENHI).

"linear" function. Hyers considered only *bounded* Cauchy difference $f(x+y)-f(x)-f(y)$. T. Aoki [2] and Th.M Rassias [14] introduced unbounded Cauchy difference

$$\|f(x+y)-f(x)-f(y)\| \leqslant \varepsilon(\|x\|^p+\|y\|^p)$$

independently, where $0 \leqslant p < 1$. They proved that there exists a unique additive mapping $T\colon E_1 \to E_2$ such that

$$\|f(x)-T(x)\| \leqslant \frac{2\varepsilon}{|2-2^p|}\|x\|^p$$

for all $x \in E_1$. Moreover, Rassias [14] proved that if the mapping $\mathbb{R} \ni t \mapsto f(tx)$ is continuous for each fixed $x \in E_1$, then T is linear.

This result is, what is called, *the Hyers-Ulam-Rassias stability* of the additive Cauchy equation $g(x+y) = g(x)+g(y)$. The stability of various functional equations has been investigated [4, 5, 7, 8, 15].

Alsina and Ger [1] remarked that the Hyers-Ulam stability of the differential equation $y' = y$ holds. In fact, they proved that if $\varepsilon \geqslant 0$ and if f is a differentiable function on an open interval I into $\mathbb{R}$ with $|f'(t)-f(t)| \leqslant \varepsilon$ for all $t \in I$, then there exists a differentiable function $g\colon I \to \mathbb{R}$ such that $g'(t) = g(t)$ and $|f(t)-g(t)| \leqslant 3\varepsilon$ for all $t \in I$. Since then, the stability of several differential equations has been studied (cf. [3, 10, 11, 12, 13]). S.-M. Jung [9] studied the stability of a system of the first order linear differential equations of the form $\mathfrak{y}'(t) = A\mathfrak{y}(t)+\mathfrak{b}(t)$, where A is an $n\times n$ complex matrix and

$$\mathfrak{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}, \qquad \mathfrak{b}(t) = \begin{pmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{pmatrix}$$

for some continuously differentiable functions $y_j\colon \mathbb{R} \to \mathbb{C}$ and continuous functions $b_j\colon \mathbb{R} \to \mathbb{C}$ for $1 \leqslant j \leqslant n$. He gave a *sufficient condition* in order that the equation $\mathfrak{y}'(t) = A\mathfrak{y}(t)+\mathfrak{b}(t)$ have the Hyers-Ulam stability in [9, Theorem 2]. In this paper, we will consider the Banach space valued differential equation $\mathfrak{y}'(t) = A\mathfrak{y}(t)$ and give a necessary and sufficient condition in order that the equation have the Hyers-Ulam stability. As a direct consequence of our main theorem, we can prove that the Banach space valued linear differential equation with constant coefficients $y^{(n)}(t)+a_{n-1}y^{(n-1)}(t)+\cdots+a_1y'(t)+a_0y(t) = 0$ has the Hyers-Ulam stability if and only if $\operatorname{Re}\lambda \neq 0$ for all the solutions λ of the equation $z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0 = 0$, which was proven in [12, Theorem 1.3].

2. Main results

Let X be a complex Banach space with the norm $\|\cdot\|$. Then the direct product X^n is a Banach space with respect to the norm $\|\mathbf{x}\| = \max\{\|x_1\|, \|x_2\|, \cdots, \|x_n\|\}$ for $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in X^n$. If $A = (a_{ij})$ is an $n\times n$ complex matrix, then for $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, $A\mathbf{x}$ will mean $(u_1, u_2, \cdots, u_n) \in X^n$, where $u_i = \sum_{j=1}^n a_{ij}x_j$. Each $n\times n$

complex matrix A is a bounded linear operator from X^n to itself with the operator norm $\|A\|$. We write $C^m(\mathbb{R},X^n)$ for the set of all m-times strongly differentiable functions $\mathfrak{f}\colon \mathbb{R}\to X^n$ such that the m-th derivative $\mathfrak{f}^{(m)}\colon \mathbb{R}\to X^n$ is continuous. For $n\times n$ complex matrix A, define $D_A\colon C^1(\mathbb{R},X^n)\to C(\mathbb{R},X^n)$ by $D_A(\mathfrak{f})(t)=\mathfrak{f}'(t)-A\mathfrak{f}(t)$ $(\mathfrak{f}\in C^1(\mathbb{R},X^n))$. We say that the operator D_A has the Hyers-Ulam stability if there exists a constant $K\geqslant 0$ with the following property: for each $\varepsilon\geqslant 0$ and $\mathfrak{f}\in C^1(\mathbb{R},X^n)$ with $\sup_{t\in\mathbb{R}}\|D_A(\mathfrak{f})(t)\|\leqslant\varepsilon$ there exits $\mathfrak{g}\in C^1(\mathbb{R},X^n)$ such that $D_A(\mathfrak{g})(t)=0$ and $\|\mathfrak{f}(t)-\mathfrak{g}(t)\|\leqslant K\varepsilon$ for all $t\in\mathbb{R}$. We call such a K a HUS constant for D_A. The differential equation $\mathfrak{y}'(t)=A\mathfrak{y}(t)$ is said to have the Hyers-Ulam stability if the operator D_A has the Hyers-Ulam stability. Namely, there exists a constant $K\geqslant 0$ with the following property: for each $\varepsilon\geqslant 0$ and $\mathfrak{f}\in C^1(\mathbb{R},X^n)$ with $\|\mathfrak{f}'(t)-A\mathfrak{f}(t)\|\leqslant\varepsilon$ there exists $\mathfrak{g}\in C^1(\mathbb{R},X^n)$ such that $\mathfrak{g}'(t)=A\mathfrak{g}(t)$ and $\|\mathfrak{f}(t)-\mathfrak{g}(t)\|\leqslant K\varepsilon$ for all $t\in\mathbb{R}$.

LEMMA 2.1. *Let J be an $m\times m$ complex matrix of the following form:*

$$J=\begin{pmatrix}\lambda & 1 & & O\\ & \lambda & \ddots & \\ & & \ddots & 1\\ O & & & \lambda\end{pmatrix}.$$

If $\mathfrak{f}=(f_1,f_2,\cdots,f_m)\in C^1(\mathbb{R},X^m)$ satisfies $\mathfrak{f}'(t)=J\mathfrak{f}(t)$ for all $t\in\mathbb{R}$, then

$$f_j(t)=\sum_{k=0}^{m-j}\frac{f_{j+k}(0)}{k!}t^k e^{\lambda t}\qquad(\forall t\in\mathbb{R})$$

for all $1\leqslant j\leqslant m$.

Proof. Since $f_m'(t)=\lambda f_m(t)$, we obtain that $f_m(t)=f_m(0)e^{\lambda t}$ for all $t\in\mathbb{R}$. Assume that $f_j(t)=\sum_{k=0}^{m-j}f_{j+k}(0)t^k e^{\lambda t}/k!$. We will show that

$$f_{j-1}(t)=\sum_{k=0}^{m-j+1}f_{j-1+k}(0)t^k e^{\lambda t}/k!.$$

Since $\mathfrak{f}'(t)=J\mathfrak{f}(t)$, $f_{j-1}'(t)=\lambda f_{j-1}(t)+f_j(t)$, and therefore

$$\begin{aligned}(f_{j-1}(t)e^{-\lambda t})'&=f_{j-1}'(t)e^{-\lambda t}-\lambda f_{j-1}(t)e^{-\lambda t}\\ &=f_j(t)e^{-\lambda t}=\sum_{k=0}^{m-j}\frac{f_{j+k}(0)}{k!}t^k.\end{aligned}$$

It follows that

$$\begin{aligned}f_{j-1}(t)e^{-\lambda t}&=f_{j-1}(0)+\sum_{k=0}^{m-j}\int_0^t\frac{f_{j+k}(0)}{k!}\tau^k\,d\tau\\ &=f_{j-1}(0)+\sum_{k=0}^{m-j}\frac{f_{j+k}(0)}{(k+1)!}t^{k+1}=\sum_{k=0}^{m-j+1}\frac{f_{j-1+k}(0)}{k!}t^k,\end{aligned}$$

which proves $f_{j-1}(t) = \sum_{k=0}^{m-j+1} f_{j-1+k}(0)t^k e^{\lambda t}/k!$ as claimed. By induction, we obtain that $f_j(t) = \sum_{k=0}^{m-j} f_{j+k}(0)t^k e^{\lambda t}/k!$ for all $1 \leqslant j \leqslant m$. $\square$

LEMMA 2.2. *Let A and B be $n \times n$ complex matrices. If $A = P^{-1}BP$ for some invertible matrix P, then the following are equivalent.*

(i) $\mathfrak{y}'(t) = A\mathfrak{y}(t)$ has the Hyers-Ulam stability.

(ii) $\mathfrak{y}'(t) = B\mathfrak{y}(t)$ has the Hyers-Ulam stability.

Proof. It is enough to show that (i) implies (ii). Suppose that the equation $\mathfrak{y}'(t) = A\mathfrak{y}(t)$ has the Hyers-Ulam stability. Thus, there exists $K \geqslant 0$ such that to each $\varepsilon \geqslant 0$ and $\mathfrak{f} \in C^1(\mathbb{R}, X^n)$ satisfying $\sup_{t\in\mathbb{R}} \|\mathfrak{f}'(t) - A\mathfrak{f}(t)\| \leqslant \varepsilon$ there corresponds $\mathfrak{g} \in C^1(\mathbb{R}, X^n)$ so that $\mathfrak{g}'(t) = A\mathfrak{g}(t)$ and $\|\mathfrak{f}(t) - \mathfrak{g}(t)\| \leqslant K\varepsilon$ for all $t \in \mathbb{R}$. We shall prove that $\mathfrak{y}'(t) = B\mathfrak{y}(t)$ has the Hyers-Ulam stability with a HUS constant $\|P^{-1}\|\,\|P\|K$. Let $\varepsilon \geqslant 0$ and $\mathfrak{f} \in C^1(\mathbb{R}, X^n)$ satisfy $\|\mathfrak{f}'(t) - B\mathfrak{f}(t)\| \leqslant \varepsilon$ for all $t \in \mathbb{R}$. Set $\mathfrak{e}(t) = P^{-1}\mathfrak{f}(t)$ for each $t \in \mathbb{R}$. Then $\mathfrak{e} \in C^1(\mathbb{R}, X^n)$ and $\mathfrak{e}'(t) = P^{-1}\mathfrak{f}'(t)$. Thus

$$\begin{aligned}\|\mathfrak{e}'(t) - A\mathfrak{e}(t)\| &= \|P^{-1}\mathfrak{f}'(t) - (P^{-1}BP)P^{-1}\mathfrak{f}(t)\| \\ &\leqslant \|P^{-1}\|\,\|\mathfrak{f}'(t) - B\mathfrak{f}(t)\| \leqslant \|P^{-1}\|\varepsilon\end{aligned}$$

for all $t \in \mathbb{R}$. By the hypothesis, there exists $\mathfrak{g} \in C^1(\mathbb{R}, X^n)$ such that $\mathfrak{g}'(t) = A\mathfrak{g}(t)$ and $\|\mathfrak{e}(t) - \mathfrak{g}(t)\| \leqslant K\|P^{-1}\|\varepsilon$ for all $t \in \mathbb{R}$. If we define $\mathfrak{h}(t) = P\mathfrak{g}(t)$ for each $t \in \mathbb{R}$, then we obtain that $\mathfrak{h} \in C^1(\mathbb{R}, X^n)$,

$$\mathfrak{h}'(t) = P\mathfrak{g}'(t) = PA\mathfrak{g}(t) = PAP^{-1}\mathfrak{h}(t) = B\mathfrak{h}(t)$$

and that

$$\begin{aligned}\|\mathfrak{f}(t) - \mathfrak{h}(t)\| &= \|P\mathfrak{e}(t) - P\mathfrak{g}(t)\| \leqslant \|P\|\,\|\mathfrak{e}(t) - \mathfrak{g}(t)\| \\ &\leqslant \|P^{-1}\|\,\|P\|K\varepsilon\end{aligned}$$

for all $t \in \mathbb{R}$. Consequently, $\mathfrak{y}'(t) = B\mathfrak{y}(t)$ has the Hyers-Ulam stability with a HUS constant $\|P^{-1}\|\,\|P\|K$, as claimed. $\square$

THEOREM 2.3. *Let A be an $n \times n$ complex matrix. Then the following conditions are equivalent.*

(i) $\mathfrak{y}'(t) = A\mathfrak{y}(t)$ has the Hyers-Ulam stability.

(ii) For each eigenvalue λ of A, $\operatorname{Re}\lambda \neq 0$.

Proof. Let $\{\lambda_1, \lambda_2, \cdots, \lambda_N\}$ be the set of all eigenvalues of A. Suppose that the real part of some eigenvalue of A is 0, say $\operatorname{Re}\lambda_1 = 0$. Choose an $n \times n$ matrix P so that $P^{-1}AP$ is the Jordan canonical form J of A. Without loss of generality, we may assume that J is of the form

$$J = \begin{pmatrix} J_1 & & & O \\ & J_2 & & \\ & & \ddots & \\ O & & & J_N \end{pmatrix},$$

where J_k is the Jordan block with respect to the eigenvalue λ_k for $1 \leqslant k \leqslant N$. Note that each Jordan block J_k is of the form

$$J_k = \begin{pmatrix} J(\lambda_k, m_{k,1}) & & & O \\ & J(\lambda_k, m_{k,2}) & & \\ & & \ddots & \\ O & & & J(\lambda_k, m_{k,r_k}) \end{pmatrix}, \tag{2.1}$$

where $J(\lambda_k, m_{k,l})$ is a Jordan cell, that is $J(\lambda_k, m_{k,l})$ is a $m_{k,l} \times m_{k,l}$ matrix of the form

$$J(\lambda_k, m_{k,l}) = \begin{pmatrix} \lambda_k & 1 & & O \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ O & & & \lambda_k \end{pmatrix} \quad (1 \leqslant k \leqslant N,\ 1 \leqslant l \leqslant r_k).$$

(i) $\Rightarrow$ (ii) Let $\varepsilon > 0$ and choose $x \in X$ so that $\|x\| = 1$. Set $m = m_{1,1}$. Then the Jordan cell $J(\lambda_1, m)$ is an $m \times m$ matrix. We define $\mathfrak{f}_0(t) = (f_1(t), f_2(t), \cdots, f_m(t), 0, \cdots, 0) \in X^n$, where

$$f_j(t) = \frac{\varepsilon t^{m-j+1}}{(m-j+1)!} e^{\lambda_1 t} x \qquad (1 \leqslant j \leqslant m)$$

for all $t \in \mathbb{R}$. Then $\mathfrak{f}_0 \in C^1(\mathbb{R}, X^n)$. On the one hand,

$$f_m'(t) = (\varepsilon t e^{\lambda_1 t} x)' = \varepsilon e^{\lambda_1 t} x + \lambda_1 \varepsilon t e^{\lambda_1 t} x = \varepsilon e^{\lambda_1 t} x + \lambda_1 f_m(t)$$

and, for each $1 \leqslant j \leqslant m-1$,

$$f_j'(t) = \frac{\varepsilon t^{m-j}}{(m-j)!} e^{\lambda_1 t} x + \lambda_1 \frac{\varepsilon t^{m-j+1}}{(m-j+1)!} e^{\lambda_1 t} x = f_{j+1}(t) + \lambda_1 f_j(t).$$

On the other hand, we obtain that

$$\begin{aligned} J\mathfrak{f}_0(t) &= (\lambda_1 f_1(t) + f_2(t), \lambda_1 f_2(t) + f_3(t), \cdots, \lambda_1 f_m(t), 0, \cdots, 0) \\ &= (f_1'(t), f_2'(t), \cdots, f_m'(t) - \varepsilon e^{\lambda_1 t} x, 0, \cdots, 0) \end{aligned}$$

and therefore, $\mathfrak{f}_0'(t) - J\mathfrak{f}_0(t) = (0, \cdots, 0, \varepsilon e^{\lambda_1 t} x, 0, \cdots, 0)$ for all $t \in \mathbb{R}$. Since $\operatorname{Re} \lambda_1 = 0$ and $\|x\| = 1$,

$$\|\mathfrak{f}_0'(t) - J\mathfrak{f}_0(t)\| = \max\{\|\varepsilon e^{\lambda_1 t} x\|, \|0\|, \cdots, \|0\|\} = \varepsilon$$

for all $t \in \mathbb{R}$. We assert that $\sup_{t\in\mathbb{R}} \|\mathfrak{f}_0(t) - \mathfrak{g}(t)\| = \infty$ for all $\mathfrak{g} \in C^1(\mathbb{R}, X^n)$ with $\mathfrak{g}'(t) = J\mathfrak{g}(t)$ for all $t \in \mathbb{R}$. Let $\mathfrak{g}(t) = (g_1(t), g_2(t), \cdots, g_n(t))$. Then we see that $\mathfrak{h} \in C^1(\mathbb{R}, X^m)$, defined by $\mathfrak{h}(t) = (g_1(t), g_2(t), \cdots, g_m(t))$ for $t \in \mathbb{R}$, satisfies $\mathfrak{h}'(t) = J(\lambda_1, m)\mathfrak{h}(t)$ for all $t \in \mathbb{R}$. By Lemma 2.1

$$g_j(t) = \sum_{k=0}^{m-j} \frac{g_{j+k}(0)}{k!} t^k e^{\lambda_1 t} \qquad (\forall t \in \mathbb{R})$$

for all $1 \leqslant j \leqslant m$. It follows that

$$\begin{aligned} \|f_m(t) - g_m(t)\| &= \|\varepsilon t e^{\lambda_1 t} x - g_m(0) e^{\lambda_1 t}\| = \|\varepsilon t x - g_m(0)\| \\ &\geqslant \varepsilon |t| - \|g_m(0)\| \to \infty \quad \text{as } t \to \infty, \end{aligned}$$

here we have used $\operatorname{Re}\lambda_1 = 0$ and $\|x\| = 1$. Consequently

$$\begin{aligned} \sup_{t\in\mathbb{R}} \|\mathfrak{f}_0(t) - \mathfrak{g}(t)\| &= \sup_{t\in\mathbb{R}} \max_{1\leqslant k\leqslant n} \|f_k(t) - g_k(t)\| \\ &\geqslant \sup_{t\in\mathbb{R}} \|f_m(t) - g_m(t)\| = \infty \end{aligned}$$

as claimed. Thus, $\sup_{t\in\mathbb{R}} \|\mathfrak{f}_0'(t) - J\mathfrak{f}_0(t)\| = \varepsilon$ but $\sup_{t\in\mathbb{R}} \|\mathfrak{f}_0(t) - \mathfrak{g}(t)\| = \infty$ for all $\mathfrak{g} \in C^1(\mathbb{R}, X^n)$ with $\mathfrak{g}'(t) = J\mathfrak{g}(t)$. This implies that $\mathfrak{y}'(t) = J\mathfrak{y}(t)$ has no Hyers-Ulam stability. By Lemma 2.2, $\mathfrak{y}'(t) = A\mathfrak{y}(t)$ has no Hyers-Ulam stability.

(ii) $\Rightarrow$ (i) Suppose that $\operatorname{Re}\lambda \neq 0$ for every eigenvalue λ of A. Let J be the Jordan canonical form of A. According to Lemma 2.2, it is enough to prove that $\mathfrak{y}'(t) = J\mathfrak{y}(t)$ has the Hyers-Ulam stability. To do this, let $\varepsilon \geqslant 0$ and $\mathfrak{f} \in C^1(\mathbb{R}, X^n)$ satisfy $\sup_{t\in\mathbb{R}} \|\mathfrak{f}'(t) - J\mathfrak{f}(t)\| \leqslant \varepsilon$. We prove that there exists $\mathfrak{g} \in C^1(\mathbb{R}, X^n)$ such that $\mathfrak{g}'(t) = J\mathfrak{g}(t)$ and $\|\mathfrak{f}(t) - \mathfrak{g}(t)\| \leqslant K\varepsilon$ for all $t \in \mathbb{R}$, where $K \geqslant 0$ is a constant which is independent to ε. Recall that J is of the form

$$J = \begin{pmatrix} J_1 & & & O \\ & J_2 & & \\ & & \ddots & \\ O & & & J_N \end{pmatrix},$$

where J_k is the Jordan block of the form (2.1) for each $1 \leqslant k \leqslant N$. Let $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2, \cdots, \mathfrak{f}_N)$, here $\mathfrak{f}_k \in C^1(\mathbb{R}, X^{n_k})$ for some $n_k \in \mathbb{N}$ with $\sum_{k=1}^N n_k = n$. Then $\sup_{t\in\mathbb{R}} \|\mathfrak{f}_k'(t) - J_k\mathfrak{f}_k(t)\| \leqslant \varepsilon$ for every $1 \leqslant k \leqslant N$. If we find $\mathfrak{g}_k \in C^1(\mathbb{R}, X^{n_k})$ such that $\mathfrak{g}_k'(t) = J_k\mathfrak{g}_k(t)$ and $\|\mathfrak{f}_k(t) - \mathfrak{g}_k(t)\| \leqslant K\varepsilon$ for all $t \in \mathbb{R}$, then $\mathfrak{g} = (\mathfrak{g}_1, \mathfrak{g}_2, \cdots, \mathfrak{g}_N) \in C^1(\mathbb{R}, X^n)$ satisfies $\mathfrak{g}'(t) = J\mathfrak{g}(t)$ and $\|\mathfrak{f}(t) - \mathfrak{g}(t)\| \leqslant K\varepsilon$ for all $t \in \mathbb{R}$. Thus, it is enough to consider the case when J is a Jordan block. By the same reasoning, we may assume that J is a Jordan cell. Consequently, we may assume that J is of the form

$$J = \begin{pmatrix} \lambda & 1 & & O \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ O & & & \lambda \end{pmatrix}.$$

If we write $\mathfrak{f}(t) = (f_1(t), \cdots, f_n(t))$, then $\sup_{t\in\mathbb{R}} \|\mathfrak{f}'(t) - J\mathfrak{f}(t)\| \leqslant \varepsilon$ implies that

$$\sup_{t\in\mathbb{R}} \|f_n'(t) - \lambda f_n(t)\| \leqslant \varepsilon \quad \text{and} \quad \sup_{t\in\mathbb{R}} \|f_j'(t) - \lambda f_j(t) - f_{j+1}(t)\| \leqslant \varepsilon$$

for each $1 \leqslant j \leqslant n-1$. Set

$$h_j(t) = \begin{cases} f_n'(t) - \lambda f_n(t) & j = n \\ f_j'(t) - \lambda f_j(t) - f_{j+1}(t) & 1 \leqslant j \leqslant n-1 \end{cases}$$

for $t \in \mathbb{R}$, and then $\sup_{t\in\mathbb{R}} \|h_j(t)\| \leqslant \varepsilon$ for all $1 \leqslant j \leqslant n$. Then

$$\left\| \int_t^s h_j(\tau) e^{-\lambda\tau} d\tau \right\| \leqslant \left| \int_t^s \|h_j(\tau)\| e^{-\mathrm{Re}\,\lambda\tau} d\tau \right| \leqslant \frac{\varepsilon |e^{-\mathrm{Re}\,\lambda s} - e^{-\mathrm{Re}\,\lambda t}|}{|\mathrm{Re}\,\lambda|} \tag{2.2}$$

for all $s, t \in \mathbb{R}$. We have two possible cases to consider for λ. First suppose that $\mathrm{Re}\,\lambda > 0$. Then, by (2.2), $\int_0^\infty h_j(\tau) e^{-\lambda\tau} d\tau \in X$ exists for all $1 \leqslant j \leqslant n$. Set

$$u_n(t) = \int_t^\infty h_n(\tau) e^{-\lambda\tau} d\tau$$

for each $t \in \mathbb{R}$. Letting $s \to \infty$ in (2.2), we get

$$\|u_n(t)\| \leqslant \frac{\varepsilon e^{-\mathrm{Re}\,\lambda t}}{\mathrm{Re}\,\lambda} \qquad (\forall t \in \mathbb{R}). \tag{2.3}$$

Note, by the definition of h_n, that

$$(f_n(t) e^{-\lambda t})' = f_n'(t) e^{-\lambda t} - \lambda f_n(t) e^{-\lambda t} = h_n(t) e^{-\lambda t}.$$

It follows that

$$f_n(t) e^{-\lambda t} = f_n(0) + \int_0^t h_n(\tau) e^{-\lambda\tau} d\tau$$

for all $t \in \mathbb{R}$. Define $g_n \in C^1(\mathbb{R}, X)$ by

$$g_n(t) = e^{\lambda t} \left(f_n(0) + \int_0^\infty h_n(\tau) e^{-\lambda\tau} d\tau \right)$$

for each $t \in \mathbb{R}$. Then $g_n'(t) = \lambda g_n(t)$ and

$$f_n(t) = g_n(t) - e^{\lambda t} \int_t^\infty h_n(\tau) e^{-\lambda\tau} d\tau = g_n(t) - e^{\lambda t} u_n(t)$$

for all $t \in \mathbb{R}$. According to (2.3),

$$\begin{aligned} \|f_n(t) - g_n(t)\| &= \|e^{\lambda t} u_n(t)\| \leqslant e^{\mathrm{Re}\,\lambda t} \|u_n(t)\| \\ &\leqslant e^{\mathrm{Re}\,\lambda t} \frac{\varepsilon e^{-\mathrm{Re}\,\lambda t}}{\mathrm{Re}\,\lambda} = \frac{\varepsilon}{\mathrm{Re}\,\lambda} \end{aligned}$$

for all $t \in \mathbb{R}$.

Next, we will show that there exists $g_{n-1} \in C^1(\mathbb{R}, X)$ such that

$$g'_{n-1}(t) = \lambda g_{n-1}(t) + g_n(t) \quad \text{and} \quad \|f_{n-1}(t) - g_{n-1}(t)\| \leqslant \varepsilon \sum_{k=1}^{2} \frac{1}{(\operatorname{Re}\lambda)^k}$$

for all $t \in \mathbb{R}$. By the definition of h_{n-1},

$$(f_{n-1}(t)e^{-\lambda t})' = f'_{n-1}(t)e^{-\lambda t} - \lambda f_{n-1}(t)e^{-\lambda t} = (h_{n-1}(t) + f_n(t))e^{-\lambda t},$$

and therefore,

$$f_{n-1}(t)e^{-\lambda t} = f_{n-1}(0) + \int_0^t (h_{n-1}(\tau) + f_n(\tau))e^{-\lambda \tau}\, d\tau.$$

Since $f_n(t) = g_n(t) - e^{\lambda t}u_n(t)$,

$$f_{n-1}(t)e^{-\lambda t} = f_{n-1}(0) + \int_0^t (h_{n-1}(\tau)e^{-\lambda\tau} - u_n(\tau))\, d\tau + \int_0^t g_n(\tau)e^{-\lambda\tau}\, d\tau \qquad (2.4)$$

for all $t \in \mathbb{R}$. By (2.2), $\int_0^\infty h_{n-1}(\tau)e^{-\lambda\tau}\, d\tau$ is well-defined and

$$\left\| \int_t^\infty h_{n-1}(\tau)e^{-\lambda\tau}\, d\tau \right\| \leqslant \frac{\varepsilon e^{-\operatorname{Re}\lambda t}}{\operatorname{Re}\lambda} \qquad (2.5)$$

for all $t \in \mathbb{R}$. According to (2.3),

$$\left\| \int_t^s u_n(\tau)\, d\tau \right\| \leqslant \frac{\varepsilon}{\operatorname{Re}\lambda} \left| \int_t^s e^{-\operatorname{Re}\lambda\tau}\, d\tau \right| = \frac{\varepsilon |e^{-\operatorname{Re}\lambda s} - e^{-\operatorname{Re}\lambda t}|}{(\operatorname{Re}\lambda)^2}$$

for all $s, t \in \mathbb{R}$, which shows the existence of $\int_0^\infty u_n(\tau)\, d\tau \in X$. Furthermore,

$$\left\| \int_t^\infty u_n(\tau)\, d\tau \right\| \leqslant \frac{\varepsilon e^{-\operatorname{Re}\lambda t}}{(\operatorname{Re}\lambda)^2} \qquad (\forall t \in \mathbb{R}). \qquad (2.6)$$

It follows from (2.5) and (2.6) that

$$\left\| \int_t^\infty (h_{n-1}(\tau)e^{-\lambda\tau} - u_n(\tau))\, d\tau \right\| \leqslant \frac{\varepsilon e^{-\operatorname{Re}\lambda t}}{\operatorname{Re}\lambda} + \frac{\varepsilon e^{-\operatorname{Re}\lambda t}}{(\operatorname{Re}\lambda)^2} \qquad (2.7)$$

for all $t \in \mathbb{R}$. Define $g_{n-1} \in C^1(\mathbb{R}, X)$ by

$$g_{n-1}(t) = e^{\lambda t} \left(f_{n-1}(0) + \int_0^\infty (h_{n-1}(\tau)e^{-\lambda\tau} - u_n(\tau))\, d\tau + \int_0^t g_n(\tau)e^{-\lambda\tau}\, d\tau \right)$$

for all $t \in \mathbb{R}$. Then we see that $g'_{n-1}(t) = \lambda g_{n-1}(t) + g_n(t)$ and, by (2.4),

$$f_{n-1}(t) = g_{n-1}(t) - e^{\lambda t} \int_t^\infty (h_{n-1}(\tau)e^{-\lambda\tau} - u_n(\tau))\, d\tau$$

for all $t \in \mathbb{R}$. From (2.7)

$$\begin{aligned}\|f_{n-1}(t)-g_{n-1}(t)\| &= \left\| e^{\lambda t}\int_t^{\infty}(h_{n-1}(\tau)e^{-\lambda\tau}-u_n(\tau))\,d\tau\right\| \\ &\leqslant e^{\operatorname{Re}\lambda t}\left\|\int_t^{\infty}(h_{n-1}(\tau)e^{-\lambda\tau}-u_n(\tau))\,d\tau\right\| \\ &\leqslant e^{\operatorname{Re}\lambda t}\left(\frac{\varepsilon e^{-\operatorname{Re}\lambda t}}{\operatorname{Re}\lambda}+\frac{\varepsilon e^{-\operatorname{Re}\lambda t}}{(\operatorname{Re}\lambda)^2}\right)=\varepsilon\sum_{k=1}^{2}\frac{1}{(\operatorname{Re}\lambda)^k}\end{aligned}$$

for all $t \in \mathbb{R}$.

By using induction, we will prove that for each $1 \leqslant j \leqslant n-2$ there exits $g_j \in C^1(\mathbb{R},X)$ such that

$$g_j'(t)=\lambda g_j(t)+g_{j+1}(t) \quad \text{and} \quad \|f_j(t)-g_j(t)\|\leqslant\varepsilon\sum_{k=1}^{n-j+1}\frac{1}{(\operatorname{Re}\lambda)^k} \tag{2.8}$$

for all $t\in\mathbb{R}$. Let $2\leqslant j\leqslant n-1$ and suppose that there exists $g_j\in C^1(\mathbb{R},X)$ satisfying (2.8). Set $v_j(t)=f_j(t)-g_j(t)$ for each $t\in\mathbb{R}$ and $\delta_j=\varepsilon\sum_{k=1}^{n-j+1}(\operatorname{Re}\lambda)^{-k}$. Then $\sup_{t\in\mathbb{R}}\|v_j(t)\|\leqslant\delta_j$. By the definition of h_{j-1}, $(f_{j-1}(t)e^{-\lambda t})'=(h_{j-1}(t)+f_j(t))e^{-\lambda t}$, and thus

$$f_{j-1}(t)e^{-\lambda t}=f_{j-1}(0)+\int_0^t(h_{j-1}(\tau)+f_j(\tau))e^{-\lambda\tau}\,d\tau$$

for all $t\in\mathbb{R}$. Since $f_j(t)=v_j(t)+g_j(t)$,

$$f_{j-1}(t)e^{-\lambda t}=f_{j-1}(0)+\int_0^t(h_{j-1}(\tau)+v_j(\tau))e^{-\lambda\tau}\,d\tau+\int_0^t g_j(\tau)e^{-\lambda\tau}\,d\tau \tag{2.9}$$

for all $t\in\mathbb{R}$. By (2.2), $\int_0^\infty h_{j-1}(\tau)e^{-\lambda\tau}\,d\tau$ exists and

$$\left\|\int_t^\infty h_{j-1}(\tau)e^{-\lambda\tau}\,d\tau\right\|\leqslant\frac{\varepsilon e^{-\operatorname{Re}\lambda t}}{\operatorname{Re}\lambda} \tag{2.10}$$

for all $t\in\mathbb{R}$. Since $\sup_{t\in\mathbb{R}}\|v_j(t)\|\leqslant\delta_j$,

$$\left\|\int_t^s v_j(\tau)e^{-\lambda\tau}\,d\tau\right\|\leqslant\delta_j\left|\int_t^s e^{-\operatorname{Re}\lambda\tau}\,d\tau\right|=\frac{\delta_j|e^{-\operatorname{Re}\lambda s}-e^{-\operatorname{Re}\lambda t}|}{\operatorname{Re}\lambda}$$

for all $s,t\in\mathbb{R}$. Thus $\int_0^\infty v_j(\tau)e^{-\lambda\tau}\,d\tau\in X$ exists and

$$\left\|\int_t^\infty v_j(\tau)e^{-\lambda\tau}\,d\tau\right\|\leqslant\frac{\delta_j e^{-\operatorname{Re}\lambda t}}{\operatorname{Re}\lambda}\qquad(\forall t\in\mathbb{R}). \tag{2.11}$$

According to (2.10) and (2.11),

$$\begin{aligned}&\left\|\int_t^\infty(h_{j-1}(\tau)+v_j(\tau))e^{-\lambda\tau}\,d\tau\right\|\leqslant\frac{\varepsilon e^{-\operatorname{Re}\lambda t}}{\operatorname{Re}\lambda}+\frac{\delta_j e^{-\operatorname{Re}\lambda t}}{\operatorname{Re}\lambda}\\ &=(\varepsilon+\delta_j)\frac{e^{-\operatorname{Re}\lambda t}}{\operatorname{Re}\lambda}=\varepsilon\left(1+\sum_{k=1}^{n-j+1}(\operatorname{Re}\lambda)^{-k}\right)\frac{e^{-\operatorname{Re}\lambda t}}{\operatorname{Re}\lambda}=\varepsilon\sum_{k=1}^{n-j+2}\frac{e^{-\operatorname{Re}\lambda t}}{(\operatorname{Re}\lambda)^k}\end{aligned} \tag{2.12}$$

for all $t \in \mathbb{R}$. Define $g_{j-1} \in C^1(\mathbb{R},X)$ by

$$g_{j-1}(t) = e^{\lambda t}\left(f_{j-1}(0) + \int_0^{\infty}(h_{j-1}(\tau)+v_j(\tau))e^{-\lambda\tau}\,d\tau + \int_0^t g_j(\tau)e^{-\lambda\tau}\,d\tau\right)$$

for each $t \in \mathbb{R}$. Then we see that $g'_{j-1}(t) = \lambda g_{j-1}(t) + g_j(t)$ and, by (2.9),

$$f_{j-1}(t) = g_{j-1}(t) - e^{\lambda t}\int_t^{\infty}(h_{j-1}(\tau)+v_j(\tau))e^{-\lambda t}\,d\tau$$

for all $t \in \mathbb{R}$. From (2.12)

$$\begin{aligned}\|f_{j-1}(t) - g_{j-1}(t)\| &= \left\|e^{\lambda t}\int_t^{\infty}(h_{j-1}(\tau)+v_j(\tau))e^{-\lambda t}\,d\tau\right\| \\ &= e^{\mathrm{Re}\,\lambda t}\left\|\int_t^{\infty}(h_{j-1}(\tau)+v_j(\tau))e^{-\lambda t}\,d\tau\right\| \\ &\leqslant e^{\mathrm{Re}\,\lambda t}\varepsilon\sum_{k=1}^{n-j+2}\frac{e^{-\mathrm{Re}\,\lambda t}}{(\mathrm{Re}\,\lambda)^k} = \varepsilon\sum_{k=1}^{n-j+2}\frac{1}{(\mathrm{Re}\,\lambda)^k}\end{aligned}$$

for all $t \in \mathbb{R}$. By induction, we obtain that, for each $1 \leqslant j \leqslant n-1$, there exists $g_j \in C^1(X,\mathbb{R})$ such that $g'_j(t) = \lambda g_j(t) + g_{j+1}(t)$ and $\|f_j(t) - g_j(t)\| \leqslant \delta_j$ for all $t \in \mathbb{R}$. Therefore, $\mathfrak{g} = (g_1, g_2, \cdots, g_n) \in C^1(\mathbb{R},X^n)$ satisfies that $\mathfrak{g}'(t) = J\mathfrak{g}(t)$ and $\|\mathfrak{f}(t) - \mathfrak{g}(t)\| \leqslant \varepsilon\sum_{k=1}^n(\mathrm{Re}\,\lambda)^{-k}$ for all $t \in \mathbb{R}$, as claimed.

In the second case, $\mathrm{Re}\,\lambda < 0$. Define $\mathfrak{u} \in C^1(\mathbb{R},X^n)$ by $\mathfrak{u}(t) = \mathfrak{f}(-t)$ for $t \in \mathbb{R}$. Then $\|\mathfrak{u}'(t) + J\mathfrak{u}(t)\| = \|-\mathfrak{f}'(-t) + J\mathfrak{f}(-t)\| \leqslant \varepsilon$ for all $t \in \mathbb{R}$. Moreover, the eigenvalue of $-J$ is $-\lambda$, and thus $\mathrm{Re}\,(-\lambda) > 0$. Thus, $\mathfrak{y}'(t) = -J\mathfrak{y}(t)$ has the Hyers-Ulam stability, and therefore there exists $\mathfrak{v} \in C^1(\mathbb{R},X^n)$ such that $\mathfrak{v}'(t) = -J\mathfrak{v}(t)$ and $\|\mathfrak{u}(t) - \mathfrak{v}(t)\| \leqslant M\varepsilon$ for all $t \in \mathbb{R}$, where $M = \sum_{k=1}^n(-\mathrm{Re}\,\lambda)^{-k}$. If we set $\mathfrak{g}(t) = \mathfrak{v}(-t)$ for $t \in \mathbb{R}$, then we obtain that $\mathfrak{g} \in C^1(\mathbb{R},X^n)$, $\mathfrak{g}'(t) = -\mathfrak{v}'(-t) = J\mathfrak{v}(-t) = J\mathfrak{g}(t)$ and $\|\mathfrak{f}(t) - \mathfrak{g}(t)\| = \|\mathfrak{u}(-t) - \mathfrak{v}(-t)\| \leqslant M\varepsilon$ for all $t \in \mathbb{R}$.

From the above, we proved that if $\mathrm{Re}\,\lambda \neq 0$, then there exists $\mathfrak{g} \in C^1(\mathbb{R},X^n)$ such that $\mathfrak{g}'(t) = J\mathfrak{g}(t)$ and $\|\mathfrak{f}(t) - \mathfrak{g}(t)\| \leqslant \varepsilon\sum_{k=1}^n|\mathrm{Re}\,\lambda|^{-k}$ for all $t \in \mathbb{R}$. Thus, $\mathfrak{y}'(t) = J\mathfrak{y}(t)$ has the Hyers-Ulam stability. By Lemma 2.2, we conclude that $\mathfrak{y}'(t) = A\mathfrak{y}(t)$ has the Hyers-Ulam stability. □

Let $a_0, a_1, \cdots, a_{n-1} \in \mathbb{C}$ and $C^n(\mathbb{R},X)$ the set of all n-times strongly differentiable mappings whose n-th derivative is continuous. Define the operator $\mathscr{P}\colon C^n(\mathbb{R},X) \to C(\mathbb{R},X)$ by

$$\mathscr{P}(f)(t) = f^{(n)}(t) + a_{n-1}f^{(n-1)}(t) + \cdots + a_1f'(t) + a_0f(t)$$

for $f \in C^n(\mathbb{R},X)$ and $t \in \mathbb{R}$. We say that the operator $\mathscr{P}$ has the Hyers-Ulam stability if there exists a constant $K \geqslant 0$ with the following property: for each $\varepsilon \geqslant 0$ and $f \in C^n(\mathbb{R},X)$ with $\sup_{t\in\mathbb{R}}\|\mathscr{P}(f)(t)\| \leqslant \varepsilon$ there exists $g \in C^n(\mathbb{R},X)$ such that $\mathscr{P}(g)(t) = 0$ and $\|f(t) - g(t)\| \leqslant K\varepsilon$ for all $t \in \mathbb{R}$. The n-th order linear differential equation $y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1y'(t) + a_0y(t) = 0$ is said to have the Hyers-Ulam stability if the operator $\mathscr{P}$ has the Hyers-Ulam stability. As a Corollary to Theorem 2.3, we have the following, which was proven in [12, Theorem 1.3].

COROLLARY 2.4. *Let $n \in \mathbb{N}$ and $a_0, a_1, \cdots, a_{n-1} \in \mathbb{C}$, where $\mathbb{N}$ is the set of all natural numbers. Then the following conditions are equivalent.*

(i) The differential equation $y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = 0$ has the Hyers-Ulam stability.

(ii) $\operatorname{Re}\lambda \neq 0$ *for each $\lambda \in \mathbb{C}$ with $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$.*

Proof. Let $f \in C^n(\mathbb{R}, X)$. Set $\mathfrak{f}(t) = (f^{(n-1)}(t), \cdots, f'(t), f(t)) \in X^n$ for each $t \in \mathbb{R}$ and

$$A = \begin{pmatrix} -a_{n-1} & \cdots & -a_1 & -a_0 \\ 1 & & O & 0 \\ & \ddots & & \vdots \\ O & & 1 & 0 \end{pmatrix}.$$

Then $\|\mathscr{P}(f)(t)\| = \|\mathfrak{f}'(t) - A\mathfrak{f}(t)\|$ for all $t \in \mathbb{R}$. Moreover, $\mathscr{P}(g)(t) = 0$ for all $t \in \mathbb{R}$ if and only if $\mathfrak{g}'(t) = A\mathfrak{g}(t)$ for all $t \in \mathbb{R}$, where $g \in C^n(\mathbb{R}, X)$ and $\mathfrak{g}(t) = (g^{(n-1)}(t), \cdots, g'(t), g(t)) \in X^n$. Consequently, the operator $\mathscr{P}$ has the Hyers-Ulam stability if and only if $\mathfrak{y}'(t) = A\mathfrak{y}(t)$ has the Hyers-Ulam stability. According to Theorem 2.3, $\mathfrak{y}'(t) = A\mathfrak{y}(t)$ has the Hyers-Ulam stability if and only if $\operatorname{Re}\lambda \neq 0$ for each eigenvalue λ of A. Finally, we note that $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$. In fact, the characteristic polynomial $|zE - A|$ of A is $z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ since

$$\begin{aligned} |zE - A| &= \begin{pmatrix} z + a_{n-1} & a_{n-2} & \cdots & a_0 \\ -1 & z & & O \\ & \ddots & \ddots & \\ O & & -1 & z \end{pmatrix} \\ &= (z + a_{n-1}) \begin{vmatrix} z & & & O \\ -1 & z & & \\ & \ddots & \ddots & \\ O & & -1 & z \end{vmatrix} + \begin{vmatrix} a_{n-2} & \cdots & a_1 & a_0 \\ -1 & z & & O \\ & \ddots & \ddots & \\ O & & -1 & z \end{vmatrix} \\ &= (z + a_{n-1})z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0, \end{aligned}$$

where we have used twice the induction. The proof is complete. □

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(Received September 14, 2011)

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