Mathematical
Inequalities
& Applications
Volume 16, Number 3 (2013), 729–741 doi:10.7153/mia-16-55

COMMUTATORS FOR MULTIPLIERS ON BESOV DUNKL SPACES

SALLAM HASSANI AND MOHAMED SIFI

(*Communicated by J. Pečarić*)

Abstract. In this paper, we first study the boundedness properties of the Dunkl multiplier of the interval $[a,b]$ associated with the reflection group $\mathbb{Z}_2$. Next, we prove that the commutator $[T,T_\mu]$ is bounded on the Besov Dunkl spaces $BD_p^{\sigma,q}$, if T is a bounded linear operator on $BD_p^{\sigma_j,q_j}$ $(j=0,1$ and $0<\sigma_1<\sigma<\sigma_0)$ and T_μ is a dyadic admissible multiplier. These results are obtained for the multi-dimensional Dunkl transform associated to the reflection group $\mathbb{Z}_2^d$.

1. Introduction

Dunkl operators provide an essential tool to extend Fourier analysis on Euclidean spaces. Since their invention in 1989, these operators have largely contributed in the setting of root systems and associated reflection groups, to the development of harmonic analysis. One of the most important issues related to the harmonic analysis is the problem of multipliers and commutators. In [15], Rochberg and Weiss developed the study of the commutators of bounded linear operators and certain operators, generally unbounded and nonlinear, associated with the complex interpolation method, with very interesting application to estimates for commutators of singular integrals with pointwise multipliers. A similar construction was done in [12] by Jawerth, Rochberg and Weiss for the real method, and further results and applications to classical analysis have been obtained in [5], [7] and [8], among others.

Let m be a bounded measurable function, the Dunkl multiplier P_m associated with m is defined, for a suitable function f, by

$$P_m f(x) = \mathscr{F}_\kappa(m\mathscr{F}_\kappa f)(-x),$$

where $\mathscr{F}_\kappa$ denotes the Dunkl transform (for more details see the next section). This operator reduces to the well-known Fourier multiplier in the case where the multiplicity function κ is equal to 0.

Several results in classical Fourier multipliers have been extended to the setting of Dunkl transform. In the particular case, when $m=\chi_{[-1,1]}$, where χ_A denotes the characteristic function of the set A, Betancor, Ciaurri and Varona give a condition ensuring the boundedness of $P_{[-1,1]}$ (cf. [4]). The same conclusion is obtained by Nowak

Mathematics subject classification (2010): Primary 33C52; secondary 42B10, 43A32, 33C80, 22E30.
Keywords and phrases: Dunkl transform, Dunkl multipliers, commutators, Besov Dunkl spaces.
The authors are supported by the DGRST research project 04/UR/15-02 and CMCU program 10G 1503.

and Stempak in [13] via the Dunkl transplantation operator. More generally, they have successfully linked Fourier multipliers to the Dunkl one. The first part of this paper is devoted to the study of L^p boundedness of the Dunkl multiplier associated with all interval $[a,b]$.

In the second part of the paper, we prove a commutator theorem. To be more precise, we study the boundedness of the commutator $[T,T_\mu] = TT_\mu - T_\mu T$, where T is any bounded linear operator on Besov Dunkl spaces $BD_p^{\sigma_j,q_j}$ $(j=0,1$ and $0<\sigma_1<\sigma_0)$ and T_μ is an admissible dyadic multiplier.

The description of Besov Dunkl classes as approximation spaces, the calculation of almost optimal approximation elements in combination with real interpolation and the cancelation properties of the commutators will be the main tools used in the proof of the commutator theorem.

2. Preliminaries

This section is devoted to the preliminaries and background. We recall some notations and results in Dunkl theory (cf. [9], [10], [11], [17], [18], [20]).

Let $e_1,...,e_d$ be the standard basis of $\mathbb{R}^d$. We denote by σ_j (for each j from 1 to d) the reflection with respect to the hyperplane perpendicular to e_j, that is to say for every $x=(x_1,...,x_d)\in\mathbb{R}^d$

$$\sigma_j(x) = x - 2\frac{\langle x,e_j\rangle}{\|e_j\|^2}e_j,$$

of course $\langle .,.\rangle$ is the usual inner product on $\mathbb{R}^d\times\mathbb{R}^d$ and $\|.\|$ is the associated norm. Let G be the finite reflection group generated by $\{\sigma_j : j=1,...,d\}$, so G is isomorphic to $\mathbb{Z}_2^d$.

Let $\kappa_1,...,\kappa_d$ be nonnegative real numbers. Associated with these objects are the Dunkl operators D_j, $j=1,...,d$ which have been introduced in [10] by Dunkl. They are given for $x\in\mathbb{R}^d$, by

$$D_jf(x_j) = \partial_j f(x_j) + \kappa_j\frac{f(x_j)-f(-x_j)}{x_j},$$

where ∂_j denotes the usual partial derivative.

The Dunkl operators are antisymmetric with respect to the measure $h_\kappa^2(x)dx$ with density

$$h_\kappa^2(x) = \prod_{j=1}^{d}|x_j|^{2\kappa_j}, \quad \kappa=(\kappa_1,\kappa_2,...,\kappa_d). \tag{1}$$

Let us note that h_κ is homogeneous of degree $\gamma_\kappa = \sum_{j=1}^{d}\kappa_j$.

For $y\in\mathbb{C}$ and $j\in\{1,...,d\}$, the simultaneous eigenfunction problem

$$D_jf(x) = yf(x), \quad x\in\mathbb{R},$$

has a unique solution $f(x) = E_{\kappa_j}(x,y)$ such that $E_{\kappa_j}(\lambda,0) = 1$, which is given in terms of Bessel functions (cf. [16]).

Specifically

$$E_{\kappa_j}(x,y) = j_{\kappa_j - \frac{1}{2}}(ixy) + \frac{xy}{2\kappa_j + 1} j_{\kappa_j + \frac{1}{2}}(ixy),$$

where

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1) \sum_{n \geqslant 0} (-1)^n \frac{(z/2)^{2n}}{n!\Gamma(n+\alpha+1)},$$

are the normalized Bessel functions (cf. [19]).

For $x \in \mathbb{R}^d$ and $y \in \mathbb{C}^d$, let

$$E_\kappa(x,y) = \prod_{j=1}^{d} E_{\kappa_j}(x_j, y_j), \tag{2}$$

then the map $(x,y) \mapsto E_\kappa(x,y)$ can be extended to a holomorphic function on $\mathbb{C}^d \times \mathbb{C}^d$ and the following properties hold:

(i) $E_\kappa(x,y) = E_\kappa(y,x), \quad x,y \in \mathbb{C}^d$.

(ii) $|E_\kappa(ix,y)| \leqslant 1, \quad x,y \in \mathbb{R}^d$.

(iii) $E_\kappa(\lambda x,y) = E_\kappa(x,\lambda y), \quad x,y \in \mathbb{C}^d, \lambda \in \mathbb{C}$.

The Dunkl kernel E_κ is of particular interest as it gives rise to an integral transform which is taken with respect to the measure $h_\kappa^2(x)dx$. More precisely, for $f \in L^1(\mathbb{R}^d, h_\kappa^2)$, the Dunkl transform of f, denoted by $\mathscr{F}_\kappa f$, is defined by

$$\mathscr{F}_\kappa f(x) = c_\kappa \int_{\mathbb{R}^d} f(y) E_\kappa(x,-iy) h_\kappa^2(y) dy, \quad x \in \mathbb{R}^d,$$

where c_κ is the Mehta-type constant

$$c_\kappa^{-1} = \int_{\mathbb{R}^d} e^{-\frac{\|x\|^2}{2}} h_\kappa^2(x) dx = \prod_{j=1}^{d} c_{\kappa_j}^{-1}.$$

Let us point out that the Dunkl transform coincides with the Euclidean Fourier transform when $\kappa_1 = \ldots = \kappa_d = 0$ and that it is more or less a Hankel transform when $d = 1$.

We list some known properties of the Dunkl transform:

(i) The Dunkl transform is a topological automorphism of the Schwartz space $\mathscr{S}(\mathbb{R}^d)$.

(ii) (Plancherel Theorem) The Dunkl transform extends to an isometric automorphism of $L^2(\mathbb{R}^d, h_\kappa^2)$.

(iii) (Inversion formula) For every $f \in \mathscr{S}(\mathbb{R}^d)$, and more generally for every $f \in L^1(\mathbb{R}^d, h_\kappa^2)$ such that $\mathscr{F}_\kappa f \in L^1(\mathbb{R}^d, h_\kappa^2)$, we have

$$f(x) = \mathscr{F}_\kappa^2 f(-x), \quad x \in \mathbb{R}^d.$$

3. Dunkl multipliers

In order to study the boundedness of the Dunkl multiplier $P_{[a,b]}$ we need the following lemma, where $L^p(\mathbb{R}^d, h_\kappa^2)$, $p \in [1,+\infty]$, denotes the space of measurable functions on $\mathbb{R}^d$ such that

$$\|f\|_{p,\kappa} = \left(\int_{\mathbb{R}^d} |f(y)|^p h_\kappa^2(y)dy\right)^{\frac{1}{p}} < +\infty, \quad \text{if} \quad 1 \leqslant p < +\infty,$$

$$\|f\|_{\infty,\kappa} = ess \sup_{y\in\mathbb{R}^d} |f(y)| < +\infty, \quad \text{otherwise},$$

and we use the shorter notation $\|.\|_{p,\kappa}$ instead of $\|.\|_{L^p(\mathbb{R}^d,h_\kappa^2)}$.

LEMMA 1. *Let $\kappa > 0$, $a < b$ and $f \in L^2(\mathbb{R}, h_\kappa^2) \cap L^p(\mathbb{R}, h_\kappa^2)$. Then*

$$P_{[a,b]}f(x) = c_\kappa^2 \int_{\mathbb{R}} \mathscr{K}_\kappa(x,y)f(y)h_\kappa^2(y)dy,$$

where

$$\begin{aligned}\mathscr{K}_\kappa(x,y) &= |b|^{2\kappa+1} sgn(b) \int_0^1 E_\kappa(ibr,x)E_\kappa(-ibr,y)h_\kappa^2(r)dr \\ &- |a|^{2\kappa+1} sgn(a) \int_0^1 E_\kappa(iar,x)E_\kappa(-iar,y)h_\kappa^2(r)dr. \end{aligned} \tag{3}$$

Proof. By the definition of the Dunkl multiplier, we have

$$\begin{aligned}P_{[a,b]}f(x) &= \mathscr{F}_\kappa\Big(\chi_{[a,b]}(r)\mathscr{F}_\kappa f(r)\Big)(-x) \\ &= c_\kappa \int_a^b E_\kappa(ir,x)\mathscr{F}_\kappa f(r)h_\kappa^2(r)dr \\ &= c_\kappa^2 \int_a^b E_\kappa(ir,x)\Big(\int_{\mathbb{R}} E_\kappa(-ir,y)f(y)h_\kappa^2(y)dy\Big)h_\kappa^2(r)dr \\ &= c_\kappa^2 \int_{\mathbb{R}} \Big(\int_a^b E_\kappa(ir,x)E_\kappa(-ir,y)h_\kappa^2(r)dr\Big)f(y)h_\kappa^2(y)dy,\end{aligned}$$

where in the last step we have used Fubini's theorem. Then, the multiplier $P_{[a,b]}$ can be written as :

$$P_{[a,b]}f(x) = c_\kappa^2 \int_{\mathbb{R}} \mathscr{K}_\kappa(x,y)f(y)h_\kappa^2(y)dy$$

with kernel

$$\mathscr{K}_\kappa(x,y) = \int_a^b E_\kappa(ir,x)E_\kappa(-ir,y)h_\kappa^2(r)dr. \tag{4}$$

Let us decompose (4) as a difference of two terms $\mathscr{K}_\kappa^1(x,y) - \mathscr{K}_\kappa^2(x,y)$ where

$$\mathscr{K}_\kappa^1(x,y) = \int_0^b E_\kappa(ir,x)E_\kappa(-ir,y)h_\kappa^2(r)dr \tag{5}$$

and

$$\mathscr{K}_\kappa^2(x,y) = \int_0^a E_\kappa(ir,x)E_\kappa(-ir,y)h_\kappa^2(r)dr. \qquad (6)$$

After performing a change of variables in (5) and (6), we obtain (3). Thus the lemma is proved. □

Our first result will be the following one.

THEOREM 1. *Let $\kappa > 0$ and $a < b$. If $\frac{1+2\kappa}{1+\kappa} < p < \frac{1+2\kappa}{\kappa}$, then*

$$\|P_{[a,b]}f\|_{p,\kappa} \leqslant C\|f\|_{p,\kappa}, \quad f \in L^p(\mathbb{R}, h_\kappa^2)$$

where C is a positive constant.

Proof. Take $f \in L^2(\mathbb{R}, h_\kappa^2) \cap L^p(\mathbb{R}, h_\kappa^2)$, by using Lemma 1 and Fubini's theorem, we have

$$\begin{aligned} P_{[a,b]}f(x) &= c_\kappa^2\Big(|b|^{2\kappa+1}sgn(b)\int_0^1 E_\kappa(ibr,x)\int_{\mathbb{R}} E_\kappa(-ibr,y)f(y)h_\kappa^2(y)dyh_\kappa^2(r)dr \\ &\quad - |a|^{2\kappa+1}sgn(a)\int_0^1 E_\kappa(iar,x)\int_{\mathbb{R}} E_\kappa(-iar,y)f(y)h_\kappa^2(y)dyh_\kappa^2(r)dr\Big) \\ &= c_\kappa|b|^{2\kappa+1}sgn(b)\int_0^1 E_\kappa(ibr,x)\mathscr{F}_\kappa f(br)h_\kappa^2(r)dr \\ &\quad - c_\kappa|a|^{2\kappa+1}sgn(a)\int_0^1 E_\kappa(iar,x)\mathscr{F}_\kappa f(ar)h_\kappa^2(r)dr. \end{aligned}$$

After performing a change of variables, we obtain

$$\mathscr{F}_\kappa f(\xi r) = |\xi|^{-(2\kappa+1)}\mathscr{F}_\kappa f_\xi(r), \quad \xi = a, b,$$

where $f_\xi(r) = f(\frac{r}{\xi})$. Then we can reformulate $P_{[a,b]}$ as follows

$$\begin{aligned} P_{[a,b]}f(x) &= c_\kappa sgn(b)\int_0^1 E_\kappa(ibr,x)\mathscr{F}_\kappa f_b(r)h_\kappa^2(r)dr \\ &\quad - c_\kappa sgn(a)\int_0^1 E_\kappa(iar,x)\mathscr{F}_\kappa f_a(r)h_\kappa^2(r)dr \\ &= sgn(b)\mathscr{F}_\kappa\Big(\chi_{[0,1]}(r)\mathscr{F}_\kappa f_b(r)\Big)(-bx) \\ &\quad - sgn(a)\mathscr{F}_\kappa\Big(\chi_{[0,1]}(r)\mathscr{F}_\kappa f_a(r)\Big)(-ax) \\ &= sgn(b)(P_{[0,1]}f_b)(bx) - sgn(a)(P_{[0,1]}f_a)(ax). \end{aligned} \qquad (7)$$

Using that, for $1 < p < \infty$ and $\alpha \in \mathbb{R}$, the condition $-1 < \alpha < p-1$ is sufficient to ensure the boundedness of the Fourier multiplier associated with $\chi_{[0,1]}$ from $L^p(\mathbb{R}, |x|^\alpha dx)$ into itself. Choosing $\alpha = 2\kappa(1-\frac{p}{2})$ and applying Corollary 3.3 and Proposition 3.4 in [13], then, for $\frac{1+2\kappa}{1+\kappa} < p < \frac{1+2\kappa}{\kappa}$ the Dunkl multiplier $P_{[0,1]}$ is bounded from $L^p(\mathbb{R}, h_\kappa^2)$ into itself.

As $L^2(\mathbb{R},h_\kappa^2)\cap L^p(\mathbb{R},h_\kappa^2)$ is dense in $L^p(\mathbb{R},h_\kappa^2)$, then $P_{[a,b]}$ can be extended to a bounded operator from $L^p(\mathbb{R},h_\kappa^2)$ into itself and the equality (7) is still valid for all $f\in L^p(\mathbb{R},h_\kappa^2)$.

Taking L^p norm, we obtain

$$\|P_{[a,b]}f\|_{p,\kappa}\leqslant |b|^{\frac{-2\kappa-1}{p}}\|P_{[0,1]}f_b\|_{p,\kappa}+|a|^{\frac{-2\kappa-1}{p}}\|P_{[0,1]}f_a\|_{p,\kappa}.$$

Hence, we have

$$\|P_{[a,b]}f\|_{p,\kappa}\leqslant C\big(|b|^{\frac{-2\kappa-1}{p}}\|f_b\|_{p,\kappa}+|a|^{\frac{-2\kappa-1}{p}}\|f_a\|_{p,\kappa}\big).$$

Since,

$$\|f_\xi\|_{p,\kappa}=|\xi|^{\frac{2\kappa+1}{p}}\|f\|_{p,\kappa},\quad \xi=a,b.$$

We deduce that

$$\|P_{[a,b]}f\|_{p,\kappa}\leqslant C\|f\|_{p,\kappa}.$$

Thus, we get the proof of the theorem. □

NOTATIONS. For $\kappa=(\kappa_1,\kappa_2,...,\kappa_d)\in\mathbb{R}_+^d$, we denote by

$$p_1(\kappa)=\frac{1+2\max(\kappa_1,...,\kappa_d)}{1+\max(\kappa_1,...,\kappa_d)};\quad p_2(\kappa)=\frac{1+2\min(\kappa_1,...,\kappa_d)}{\min(\kappa_1,...,\kappa_d)}.$$

THEOREM 2. *Let $Q_{a,b}=[a_1,b_1]\times[a_2,b_2]\times...\times[a_d,b_d]$ be such that $a_j<b_j$, $j=1,...,d$. Then, if $p_1(\kappa)<p<p_2(\kappa)$, we have*

$$\|P_{Q_{a,b}}f\|_{p,\kappa}\leqslant C\|f\|_{p,\kappa},\quad f\in L^p(\mathbb{R}^d,h_\kappa^2),$$

where C is a positive constant.

REMARK 1. We note that Betancor, Ciaurri and Varona [4], obtained the same result in the case when $d=1$ and $[a,b]=[-1,1]$. Next, Nowak and Stempak [13], have successfully linked Fourier multipliers to the Dunkl ones, via the Dunkl transplantation operator. However, this approach is heavily connected to the weighted estimates and can be applied in the case when $d=1$ and $[a,b]=[-r,r]$.

Proof. Using Fubini's theorem, the Dunkl multiplier $P_{Q_{a,b}}f$ can be written as:

$$P_{Q_{a,b}}f(x)=c_\kappa^2\int_{\mathbb{R}^d}\mathscr{K}_\kappa(x,y)f(y)h_\kappa^2(y)dy,$$

where

$$\mathscr{K}_\kappa(x,y)=\int_{Q_{a,b}}E_\kappa(-iy,r)E_\kappa(ix,r)h_\kappa^2(r)dr.$$

Using (1) and (2), it is easy to show that

$$P_{Q_{a,b}} = P^{(1)}_{[a_1,b_1]} \circ P^{(2)}_{[a_2,b_2]} \circ \ldots \circ P^{(d)}_{[a_d,b_d]},$$

where $P^{(j)}_{[a_j,b_j]}, 1 \leqslant j \leqslant d$, denotes the Dunkl multiplier applied to the $j-th$ coordinate.

According to Theorem 1, if $\frac{1+2\kappa_j}{1+\kappa_j} < p < \frac{1+2\kappa_j}{\kappa_j}$, then the mapping $f \to P^{(j)}_{[a_j,b_j]} f$ can be extended to a bounded operator from $L^p(\mathbb{R}, h^2_\kappa)$ into itself. This concludes the proof of the theorem. □

4. The commutator theorem

In this section, we study the commutators of bounded linear operator in Besov Dunkl spaces and an admissible dyadic multiplier, associated with the real interpolation method. Before recalling some results about real interpolation theory (cf. [2], [3]) and Besov Dunkl spaces (cf. [1]), we specify few notations. If A and B are two Banach spaces, we write $T : A \to B$ to mean that T is a bounded linear operator between A and B, while by $P \simeq Q$ we mean that $P \leqslant cQ$ and $Q \leqslant cP$ for some constant $c > 0$ independent of the variables involved.

Let $\bar{A} = (A_0, A_1)$ be a couple of Banach spaces and for any $x \in \Sigma(\bar{A}) = A_0 + A_1$ and $t > 0$, let us denote

$$K(t,x) = K(t,x;\bar{A}) = \inf\{\|x_0\|_{A_0} + t\|x_1\|_{A_1}; \quad x = x_0 + x_1\},$$

the Peetre's K-functional.

If $0 < \theta < 1$ and $1 \leqslant q \leqslant \infty$, we denote $\bar{A}_{\theta,q}$ the corresponding interpolation space defined by the real K-method, endowed with the norm

$$\|x\|_{\bar{A}_{\theta,q}} = \|t^{-\theta} K(t,x)\|_{L^q(\frac{dt}{t})}.$$

If $\bar{A}$, $\bar{B}$ are two Banach couples, we denote by $\mathscr{L}(\bar{A};\bar{B})$ the set all linear operator $T : \Sigma(\bar{A}) \to \Sigma(\bar{B})$ such that

(i) $T(A_j) \subset B_j \ (j = 0, 1)$.

(ii) $\|T\| = \max(\|T\|_{A_0,B_0}; \|T\|_{A_1,B_1}) < \infty$.

If $T \in \mathscr{L}(\bar{A};\bar{B})$, then $T : \bar{A}_{\theta,q} \to \bar{B}_{\theta,q}$.

If

$$Sf(t) = \int_0^t f(s)\frac{ds}{s} + t\int_t^\infty f(s)\frac{ds}{s^2}$$

is the Calderòn operator, we set

$$\sigma(\bar{A}) = \{x \in \Sigma(\bar{A});\ \|x\|_{\sigma(\bar{A})} = S(K(.,x))(1) < \infty\}.$$

Observe that $\sigma(\bar{A})$ is a linear subspace of $\Sigma(\bar{A})$ which contain all real interpolation space $\bar{A}_{\theta,q}$ and moreover $\|Tx\|_{\sigma(\bar{B})} \leqslant \|T\|\ \|x\|_{\sigma(\bar{A})}$, $(T \in \mathscr{L}(\bar{A};\bar{B}))$.

Let $\mathscr{W}$ be a Hausdorff topological linear space and A a Banach subspace of $\mathscr{W}$, with continuous embedding $A \hookrightarrow \mathscr{W}$. Let us also consider a fixed approximation family $\{V(r)\}_{r>0}$ (cf. [5]), which is a family of nonempty subsets of $\mathscr{W}$, with the following properties $V(s) \subset V(r)$ when $s<r$, $-V(r)=V(r)$ and $V(s)+V(r) \subset V(s+r)$.

It is clear that $0 \in \bigcap_{r>0} V(r)$ and that $V=\bigcup_{r>0} V(r)$ is an abelian group that will be endowed with the semi-norm

$$\|x\|_V = \inf\{r>0; x \in V(r)\}.$$

Then, as in ([14]), we can define the approximation spaces $E^{t,q}$ of all $f \in V+A$ by the condition

$$\|f\|_{E^{t,q}} = \|r^{1/t}E(r,f)\|_{L^q(\frac{dr}{r})} < \infty,$$

with

$$E(r,f) = \inf_{g \in V(r)} \|f-g\|_A \simeq \|f-f_r\|_A$$

if $f_r \in V(r)$ and $\|f-f_r\|_A \leqslant cE(r,f)$ for some constant $c>1$ independent of $r>0$ and f.

Let σ, p, q be such that $0<\sigma<\infty$ and $1 \leqslant p,q < \infty$. The Besov Dunkl space $BD_p^{\sigma,q}$ $\left(\text{or } BD_p^{\sigma,q}(\mathbb{R}^d, h_\kappa^2)\right)$, is the approximation space $E^{1/\sigma,q}$ when $A=L^p(\mathbb{R}^d, h_\kappa^2)$ and $V(r)=\{g \in \mathscr{S}'(\mathbb{R}^d),\ supp \mathscr{F}_\kappa g \subset [-r,r]^d\}$, where $\mathscr{F}_\kappa g$ is the Dunkl transform of the distribution g. Hence,

$$BD_p^{\sigma,q} = \left\{ f \in L^p(\mathbb{R}^d, h_\kappa^2); \|f\|_{BD_p^{\sigma,q}} = \left(\int_0^\infty [r^\sigma E(r,f)]^q \frac{dr}{r} \right)^{\frac{1}{q}} < \infty \right\}.$$

In order to give the proof of a commutator theorem, we need two lemmas. The first one is just a description of real interpolation for couples of Besov Dunkl spaces which has been recently proved by Abdelkefi, Anker, Sassi and Sifi (cf. [1]).

LEMMA 2. *Let* $0<\sigma_0, \tilde{\sigma}_0 < \infty$, $1 \leqslant p,q,q_0,q_1 < \infty$, $0<\theta<1$ *and* $\sigma=(1-\theta)\sigma_0+\theta\tilde{\sigma}_0$. *Then*

$$(BD_p^{\sigma_0,q_0}, BD_p^{\tilde{\sigma}_0,q_1})_{\theta,q} = BD_p^{\sigma,q}.$$

The second lemma give us a quasi-optimal decomposition for the K functional for a couple of Besov Dunkl spaces. More precisely, we have

LEMMA 3. *Let* $0<\sigma_0, \tilde{\sigma}_0 < \infty$, $1 \leqslant q_0, \tilde{q}_0 < \infty$, $p_1(\kappa) < p < p_2(\kappa)$ *and assume that* $\rho = \sigma_0 - \tilde{\sigma}_0 > 0$. *Then*

$$K(t^\rho, f; BD_p^{\sigma_0,q_0}, BD_p^{\tilde{\sigma}_0,\tilde{q}_0}) \simeq \|P_t f\|_{BD_p^{\sigma_0,q_0}} + t^\rho \|f-P_t f\|_{BD_p^{\tilde{\sigma}_0,\tilde{q}_0}},$$

where $P_t f$ *is the Dunkl multiplier of* $[-t,t]^d$.

Proof. We proceed in the same manner as in [6]. Let $g_t \in \mathscr{S}'(\mathbb{R}^d)$, satisfying

$$supp \mathscr{F}_\kappa g_t \subset [-t,t]^d; \quad \text{and} \quad \|f-g_t\|_{p,\kappa} \leqslant 2E(t,f).$$

For $p_1(\kappa) < p < p_2(\kappa)$, we have that

$$\begin{aligned}\|f-P_t f\|_{p,\kappa} &\leqslant \|f-g_t\|_{p,\kappa} + \|g_t - P_t f\|_{p,\kappa}\\ &= \|f-g_t\|_{p,\kappa} + \|P_t(g_t-f)\|_{p,\kappa}\\ &\leqslant CE(t,f).\end{aligned}$$

Hence, we can write thanks to Theorem 4 of [5]

$$K(t^\rho, f; BD_p^{\sigma_0,q_0}, BD_p^{\tilde{\sigma}_0,\tilde{q}_0}) \simeq \|P_t f\|_{BD_p^{\sigma_0,q_0}} + t^\rho \|f-P_t f\|_{BD_p^{\tilde{\sigma}_0,\tilde{q}_0}},$$

and the lemma is therefore proved. □

The following proposition, whose proof follows by combining Theorem 3 and Corollary 1 of [5], will be useful in order to prove our main result.

PROPOSITION 1. *Let* $\{t_j\}_{j\in\mathbb{Z}} \subset]0,+\infty[$ *be an increasing sequence such that* $t_j \leqslant 1$ *if* $j<0$ *and* $t_j \geqslant 1$ *if* $j \geqslant 0$, $t_j \uparrow +\infty$ *as* $j \uparrow +\infty$ *and* $t_j \downarrow 0$ *as* $j \downarrow -\infty$, *and let* $\{\mu_k\}_{k\in\mathbb{Z}}$ *be any sequence of complex numbers such that*

$$\sup_{n\in\mathbb{Z}} \sum_{t_k\in[2^{n-1},2^n[} |\mu_{k+1}-\mu_k| < +\infty.$$

For a given couple of Banach spaces $\bar{X}$ *and for every* t_j *let* $x = x_0(t_j)+x_1(t_j)$ *be a decomposition such that*

$$\|x_0(t_j)\|_{X_0} + t_j\|x_1(t_j)\|_{X_1} \leqslant CK(t_j,x), \ \ (x\in\sigma(\bar{X}))$$

where $C>1$ *is a constant.*

Let us define

$$T_\mu x = \sum_{k=1}^{\infty} \mu_k (x_0(t_k) - x_0(t_{k-1})).$$

Then, if $T_\mu : \sigma(\bar{X})) \to \Sigma(\bar{X})$, $T_\mu : \sigma(\bar{Y})) \to \Sigma(\bar{Y})$ *there exists a constant* $C>0$ *such that*

$$K(t,[T,T_\mu](x)) \leqslant C\|T\| S\big(K(.,x)\big)$$

for any $x\in\sigma(\bar{X})$ *and* $T\in\mathscr{L}(\bar{X};\bar{Y}))$.

Let now introduce the dyadic admissible multiplier.

DEFINITION 1. Let $\{\mu_j\}_{j\geqslant 0}$ be any sequence of complex numbers. A dyadic multiplier will be a function

$$\mu = \{\mu_j\}_{j\geqslant 0} = \sum_{j=0}^{\infty} \mu_j \chi_{C_j}$$

which is constant on every $C_j = Q_j \setminus Q_{j-1}$ where $Q_j = [-2^j, 2^j]$.

A dyadic multiplier is said to be admissible if

$$\sup_{j\geqslant 0} |\mu_j - \mu_{j-1}| < \infty \quad (\mu_{-1} = 0).$$

We now state the commutator theorem, whose proof is based on the abstract method of [12].

THEOREM 3. *Let* $1 \leqslant q, q_0, q_1, \tilde{q}_0, \tilde{q}_1 < \infty$, $\sigma_0 > \tilde{\sigma}_0 > 0$, $\sigma_1 > \tilde{\sigma}_1 > 0$ *such that* $\sigma_0 - \tilde{\sigma}_0 = \sigma_1 - \tilde{\sigma}_1$, $\sigma = (1-\theta)\sigma_0 + \theta\tilde{\sigma}_0$, $\tilde{\sigma} = (1-\theta)\sigma_1 + \theta\tilde{\sigma}_1$ $(0 < \theta < 1)$, *and assume that* $p_1(\kappa) < p < p_2(\kappa)$.

If μ *is an admissible dyadic multiplier, then*

$$\|[T, T_\mu]\|_{\mathscr{L}(BD_p^{\sigma,q}; BD_p^{\tilde{\sigma},q})} \leqslant C\|T\|,$$

where T *is any bounded linear operator between* $(BD_p^{\sigma_0,q_0}; BD_p^{\tilde{\sigma}_0,\tilde{q}_0})$ *and* $(BD_p^{\sigma_1,q_1}; BD_p^{\tilde{\sigma}_1,\tilde{q}_1})$, *and*

$$\|T\| = \max\left(\|T\|_{\mathscr{L}(BD_p^{\sigma_0,q_0}; BD_p^{\sigma_1,q_1})}, \|T\|_{\mathscr{L}(BD_p^{\tilde{\sigma}_0,\tilde{q}_0}; BD_p^{\tilde{\sigma}_1,\tilde{q}_1})}\right).$$

Proof. Let μ be a dyadic admissible multiplier, i.e. $\mu = \sum_{k=0}^{\infty} \mu_k \chi_{C_k}$, with $\mu_0 = 0$ and $\sup_k |\mu_{k+1} - \mu_k| < \infty$. We obtain

$$T_\mu f = \sum_{k=1}^{\infty} \mu_k (P_{2^k} f - P_{2^{k-1}} f).$$

Indeed,

$$\mu_k \mathscr{F}_\kappa (P_{2^k} f - P_{2^{k-1}} f) = \mu_k (\chi_{Q_k} - \chi_{Q_{k-1}}) \mathscr{F}_\kappa f = \mu_k \chi_{C_k} \mathscr{F}_\kappa f,$$

Now, by denoting

$$\lambda_0 = \mu_1 - \mu_0 = \mu_1,\ \lambda_1 = \mu_2 - \mu_1, \ldots,\ \lambda_k = \mu_{k+1} - \mu_k, \ldots$$

we obtain

$$\|\lambda\|_\infty = \sup_k |\lambda_k| = \sup_k |\mu_{k+1} - \mu_k| < \infty,$$

since

$$\lambda_0 = \mu_1,\ \lambda_0 + \lambda_1 = \mu_2, \ldots, \sum_{j=0}^{k} \lambda_j = \mu_{k+1}, \ldots$$

We are led to

$$T_\mu f = \sum_{k=1}^{\infty} \Big(\sum_{j=0}^{k-1} \lambda_j\Big)(P_{2^k} f - P_{2^{k-1}} f) = \sum_{j=0}^{\infty} \lambda_j \sum_{k>j} (P_{2^k} f - P_{2^{k-1}} f),$$

that is to say

$$T_\mu f = \sum_{j=0}^{\infty} \lambda_j (f - P_{2^j} f).$$

Moreover

$$T_\mu : \sigma(\bar{A}) \to \Sigma(\bar{A}), \quad \text{if } \bar{A} = (BD_p^{\sigma_0,q_0}; BD_p^{\tilde{\sigma}_0,\tilde{q}_0}),$$

Indeed, by Lemma 3, if $\rho = \sigma_0 - \tilde{\sigma}_0$, then we have

$$\begin{aligned} \|T_\mu f\|_{\Sigma(\bar{A})} &\leqslant \|\lambda\|_\infty \sum_{j\geqslant 0} \|f - P_{2^j} f\|_{BD_p^{\tilde{\sigma}_0,\tilde{q}_0}} \\ &\leqslant C\|\lambda\|_\infty \sum_{j\geqslant 0} \frac{K(2^{\rho j}, f; \bar{A})}{2^{\rho j}} \\ &\leqslant C\|\lambda\|_\infty \frac{2^\rho}{\rho \ln 2} \int_1^\infty \frac{K(s, f; \bar{A})}{s} \frac{ds}{s} \\ &\leqslant C\|\lambda\|_\infty \frac{2^\rho}{\rho \ln 2} \|f\|_{\sigma(\bar{A})}. \end{aligned}$$

Similarly

$$T_\mu : \sigma(\bar{B}) \to \Sigma(\bar{B}), \quad \text{if } \bar{B} = (BD_p^{\sigma_1,q_1}; BD_p^{\tilde{\sigma}_1,\tilde{q}_1}).$$

Now, given $T \in \mathscr{L}(\bar{A}; \bar{B})$ and $f \in \sigma(\bar{A})$, let $\rho = \sigma_0 - \tilde{\sigma}_0$, and we consider the sequence $t_k = 2^{\rho k}$ $(k \geqslant 0)$ by Lemma 3, $(P_{2^k} f, f - P_{2^k} f)$ is an almost optimal decomposition of f for the couple $(BD_p^{\sigma_0,q_0}; BD_p^{\tilde{\sigma}_0,\tilde{q}_0})$ and we obtain

$$\|P_{2^k} f\|_{BD_p^{\sigma_0,q_0}} + t_k \|f - P_{2^k} f\|_{BD_p^{\tilde{\sigma}_0,\tilde{q}_0}} \leqslant cK(t_k, f; \bar{A}).$$

The Dunkl multiplier can be reformulated as follows

$$T_\mu f = \sum_{k=1}^{\infty} \mu_k (x_0(t_k) - x_0(t_{k-1})),$$

where

$$x_0(t_k) = P_{2^k} f.$$

On the other hand since $2^\rho > 1$, Remark 3 in [5], yields to

$$\begin{aligned} \sup_{n\in\mathbb{Z}} \sum_{t_k \in [2^{n-1}, 2^n[} |\mu_{k+1} - \mu_k| &\simeq \sup_{n\in\mathbb{Z}} \sum_{t_k \in [2^{\rho(n-1)}, 2^{\rho n}[} |\mu_{k+1} - \mu_k| \\ &= \sup_k |\mu_{k+1} - \mu_k| = c < +\infty. \end{aligned}$$

Thanks to Proposition 1, we are led to

$$K(t, [T, T_\mu] f; \bar{B}) \leqslant C\|T\| S(K(., f; \bar{A})).$$

If in Hardy's inequalities for averages [2], we have

$$\|t^{-\theta}\int_0^t g(s)ds\|_{L^q(\frac{dt}{t})} \leqslant \frac{1}{\theta}\|t^{1-\theta}g(t)\|_{L^q(\frac{dt}{t})}, \quad (\theta > 0)$$

and

$$\|t^{1-\theta}\int_t^\infty g(s)\frac{ds}{s}\|_{L^q(\frac{dt}{t})} \leqslant \frac{1}{1-\theta}\|t^{1-\theta}g(t)\|_{L^q(\frac{dt}{t})}, \quad (\theta < 1),$$

we take $g(s) = \frac{K(.,f;\bar{A})}{s}$, then

$$\begin{aligned}
\|[T,T_\mu]f\|_{\bar{B}_{\theta,q}} &= \|t^{-\theta}K(t,[T,T_\mu]f;\bar{B})\|_{L^q(\frac{dt}{t})} \\
&\leqslant C\|t^{-\theta}S(K(.,f;\bar{A}))\|_{L^q(\frac{dt}{t})} \\
&\leqslant C\|t^{-\theta}\int_0^t K(s,f;\bar{A}))\frac{ds}{s}\|_{L^q(\frac{dt}{t})} \\
&+ C\|t^{-\theta}t\int_t^\infty \frac{K(s,f;\bar{A}))}{s}\frac{ds}{s}\|_{L^q(\frac{dt}{t})} \\
&\leqslant M\|t^{1-\theta}\frac{K(s,f;\bar{A}))}{t}\|_{L^q(\frac{dt}{t})} \\
&= M\|f\|_{\bar{A}_{\theta,q}},
\end{aligned}$$

where $M = \frac{C}{\theta(1-\theta)}$ and $C = C(\rho, \|\lambda\|_\infty, \|T\|)$.

Lemma 2 gives $\bar{A}_{\theta,q} = BD_p^{\sigma,q}$ and $\bar{B}_{\theta,q} = BD_p^{\widetilde{\sigma},q}$. Then the theorem is proved. □

REFERENCES

[1] C. ABDELKEFI, J. PH. ANKER, F. SASSI AND M. SIFI, *Besov type spaces on $\mathbb{R}^d$ and integrability for the Dunkl transform*, Symmetry, Integrability and Geometry: Methods and Applications **5** (2009), Paper 019, 15 pages.

[2] C. BENNETT AND R. SHARPLEY, *Interpolation of Operators*, Academic Press, New York, 1988.

[3] J. BERGH AND J. LÖFSTRÖM, *Interpolation Spaces, An Introduction*, Springer, Berlin, Heidelberg, New York, 1976.

[4] J. J. BETANCOR, Ò. CIAURRI AND J. L. VARONA, *The multiplier of the interval $[-1,1]$ for the Dunkl transform on the real line*, J. Funct. Anal. **242** (2007), 327–336.

[5] J. CERDÀ, N. YA. KRUGLJAK AND J. MARTÍN, *Commutators for approximation spaces and Marcinkiewicz type multipliers*, J. Approx. Theory **100** (1999), 251–265.

[6] J. CERDÀ AND J. MARTÍN, *Commutators for Fourier multipliers on Besov spaces*, J. Approx. Theory **129** (2004), 119–128.

[7] M. CWIKEL, B. JAWERTH AND M. MILMAN, *The domain spaces of quasilogarithmic operators*, Trans. Amer. Mat. Soc. **317** (1989), 599–609.

[8] M. CWIKEL, B. JAWERTH, M. MILMAN AND R. ROCHBERG, *Differential estimates and commutators in interpolation theory*, Analysis at Urbana, Vol. II (Urbana, IL, 1986–1987), 170–220, London Math. Soc. Lecture Note Ser., 138, Cambridge Univ. Press, Cambridge, 1989.

[9] M. F. E. DE JEU, *The Dunkl transform*, Invent. Math. **113**, 1 (1993), 147–162.

[10] C. F. DUNKL, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. **311**, 1 (1989), 167–183.

[11] C. F. DUNKL, *Hankel transforms associated to finite reflection groups*, in: Proc. of special session on hypergeometric functions on domains of positivity, Jack polynomials and applications, Proceeding, Tampa 1991, Contemporary Mathematics **138** (1992), 123–138.

[12] B. JAWERTH, R. ROCHBERG AND G. WEISS, *Commutators and other second order estimates in real interpolation theory*, Ark. Mat. **24** (1986), 191–219.

[13] A. NOWAK AND K. STEMPAK, *Relating transplantation and multipliers for Dunkl and Hankel transforms*, Math. Nachr. **11** (2008), 1604–1611.

[14] J. PEETRE AND G. SPARR, *Interpolation of Normed Abelian Groups*, Ann. Math. Pura Appl. **92** (1972), 217–262.

[15] R. ROCHBERG AND G. WEISS, *Derivatives of analytic families of Banach spaces*, Ann. of Math. **118** (1983), 315–347.

[16] M. RÖSLER, *Bessel-type signed hypergroups on* $\mathbb{R}$*,* Probability measures on groups and related structures, XI (Oberwolfach, 1994), 292–304, World Sci., Publ., River Edge, NJ, 1995.

[17] M. RÖSLER, *Dunkl operators: theory and applications*, In Orthogonal polynomials and special functions (Leuven 2002), **1817** of Lectures Notes in Math. (2003), Springer Berlin, 93–135.

[18] K. TRIMÈCHE, *Paley-Wiener theorems for the Dunkl transform and Dunkl translation operators*, Integral Transforms Spec. Funct. **13**, 1 (2002), 17–38.

[19] G. N. WATSON, *A Treatise on the theory of Bessel Functions*, Cambridge University Press, Cambridge, England (1944).

[20] Y. XU, *Orthogonal polynomials for a family of product weight functions on the spheres* , Canad. J. Math. **49** (1997), 175–192.

(Received October 14, 2011)

Sallam Hassani
Department of Mathematics
Faculty of Sciences of Tunis
University Tunis El Manar
2090 Tunis, Tunisia
e-mail: sallem.h@gmail.com

Mohamed Sifi
Department of Mathematics
Faculty of Sciences of Tunis
University Tunis El Manar
2090 Tunis, Tunisia,
e-mail: mohamed.sifi@fst.rnu.tn