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SCHUR POWER CONVEXITY OF THE DARÓCZY MEANS

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Abstract. In this paper, the Schur convexity is generalized to Schur f-convexity, which contains the Schur geometrical convexity, harmonic convexity and so on. When $f:\mathbb{R}_+\to\mathbb{R}$ is defined by $f(x)=(x^m-1)/m$ if $m\neq 0$ and $f(x)=\ln x$ if $m=0$, the necessary and sufficient conditions for f-convexity (is called Schur m-power convexity) of Daróczy means are given, which improve, generalize and unify Shi et al.'s results.

1. Introduction

Let $a,b\in\mathbb{R}_+:=(0,\infty)$. The classical Heronian mean of a and b is defined as

$$H(a,b)=\frac{a+\sqrt{ab}+b}{3}. \tag{1.1}$$

In 2001, this mean was generalized by Janous [16] as follows:

$$H_w(a,b)=\begin{cases}\frac{a+w\sqrt{ab}+b}{w+2}, & 0\leqslant w<\infty,\\ \sqrt{ab}, & w=\infty.\end{cases} \tag{1.2}$$

Recently, Jia and Cao [17] considered the p-order Heronian mean defined by

$$H_p(a,b)=\left(\frac{a^p+(ab)^{p/2}+b^p}{3}\right)^{1/p}. \tag{1.3}$$

Several variants as well as interesting applications of Heronian mean, generalized Heronian mean and p-order Heronian mean can be found in the recent papers [20, 17, 42, 43, 36, 11, 37, 38, 39].

Shi et al. [28] generalized (1.3) into the weighted form, that is,

$$H_{p,w}(a,b)=\begin{cases}\left(\frac{a^p+w(ab)^{p/2}+b^p}{w+2}\right)^{1/p}, & p\neq 0,\\ \sqrt{ab}, & p=0,\end{cases} \tag{1.4}$$

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where $w \geqslant 0$. In an oral communication [3], Daróczy also defined this class of means as the p-modification (see [31]) of $H_w(a,b)$ for $w \geqslant -1$. Naturally, this class of means $H_{p,w}(a,b)$ was called Daróczy means by Burai.

It is easy to prove that $H_{p,w}(a,b)$ for $w \geqslant -1$ is a mean of positive numbers a and b indeed. If $H_w(a^p,b^p) > 0$ for all $a,b > 0$, however, then the value range of the parameter w can be extended to $(-2,\infty)$. Hence in what follows, we call $H_{p,w}(a,b)$ defined by (1.4) Daróczy means, where $p \in \mathbb{R}$, $w > -2$. But it should be noted that $H_{p,w}(a,b)$ is not a mean of positive numbers a and b when $-2 < w < -1$.

In 2008, Burai [3] gave necessary as well as sufficient condition for the comparability of Daróczy means for $w \geqslant -1$. Shi et al. [28] discussed the Schur convexity of Daróczy means for $w \geqslant 0$ and proved the following results.

THEOREM 1. ([28, Theorem 3.1]) *For fixed* $p \in \mathbb{R}$ *and* $w \geqslant 0$,

1) $H_{p,w}(a,b)$ *is increasing for* $(a,b) \in \mathbb{R}^2_+$;

2) if $(p,w) \in \{p \leqslant 1, w \geqslant 0\} \cup \{1 < p \leqslant 3/2, w \geqslant 1\} \cup \{3/2 < p \leqslant 2, w \geqslant 2\}$, *then,* $H_{p,w}(a,b)$ *is Schur concave for* $(a,b) \in \mathbb{R}^2_+$;

3) if $p \geqslant 2, 0 \leqslant w \leqslant 2$, *then,* $H_{p,w}(a,b)$ *is Schur convex for* $(a,b) \in \mathbb{R}^2_+$.

THEOREM 2. ([28, Theorem 3.2]) *For fixed* $p \in \mathbb{R}$ *and* $w \geqslant 0$,

1) if $p < 0$, *then* $H_{p,w}(a,b)$ *is Schur-geometrically concave for* $(a,b) \in \mathbb{R}^2_+$;

2) if $p > 0$, *then* $H_{p,w}(a,b)$ *is Schur-geometrically convex for* $(a,b) \in \mathbb{R}^2_+$.

As far as the Schur convexity of Daróczy means is concerned, the above results seem to be the best till now.

The purpose of this paper is to study Schur m-power convexity of Daróczy means $H_{p,w}(a,b)$ and give necessary and sufficient conditions for it, which improve, generalize and unify Shi et al.'s results.

Our main results are as follows.

THEOREM 3. *For fixed* $p \in \mathbb{R}$, $m > 0$ *and* $w > -2$, *Daróczy mean* $H_{p,w}(a,b)$ *is Schur* m*-power convex with respect to* $(a,b) \in \mathbb{R}^2_+$ *if and only if* $(p,w) \in \Omega_1$, *where*

$$\Omega_1 = \left\{-2 < w \leqslant 0,\ p \geqslant \frac{w+2}{2}m\right\} \cup \left\{w > 0,\ p \geqslant \max\left(\frac{w+2}{2}m, 2m\right)\right\}. \tag{1.5}$$

THEOREM 4. *For fixed* $p \in \mathbb{R}$, $m > 0$ *and* $w > -2$, *Daróczy mean* $H_{p,w}(a,b)$ *is Schur* m*-power concave with respect to* $(a,b) \in \mathbb{R}^2_+$ *if and only if* $(p,w) \in \Omega_2$, *where*

$$\Omega_2 = \{-2 < w < 0,\ p < 0\} \cup \left\{w \geqslant 0,\ p \leqslant \min\left(\frac{w+2}{2}m, 2m\right)\right\}. \tag{1.6}$$

THEOREM 5. *For fixed* $p \in \mathbb{R}$, $m < 0$ *and* $w > -2$, *Daróczy mean* $H_{p,w}(a,b)$ *is Schur* m*-power convex with respect to* $(a,b) \in \mathbb{R}^2_+$ *if and only if* $(p,w) \in E_1$, *where*

$$E_1 = \{-2 < w < 0,\ p > 0\} \cup \left\{w \geqslant 0,\ p \geqslant \max\left(\frac{w+2}{2}m, 2m\right)\right\}. \tag{1.7}$$

THEOREM 6. *For fixed* $p \in \mathbb{R}$, $m < 0$ *and* $w > -2$, *Daróczy mean* $H_{p,w}(a,b)$ *is Schur concave with respect to* $(a,b) \in \mathbb{R}_+^2$ *if and only if* $(p,w) \in E_2$, *where*

$$E_2 = \left\{-2 < w \leqslant 0,\ p \leqslant \frac{w+2}{2}m\right\} \cup \left\{w > 0,\ p \leqslant \min\left(\frac{w+2}{2}m, 2m\right)\right\}. \tag{1.8}$$

THEOREM 7. *For fixed* $p \in \mathbb{R}$, $m = 0$ *and* $w > -2$, *Daróczy mean* $H_{p,w}(a,b)$ *is Schur* m*-power convex (Schur* m*-power concave) with respect to* $(a,b) \in \mathbb{R}_+^2$ *if and only if* $p \geqslant (\leqslant) 0$.

The organization of the paper is as follows. In section 2, based on the notion and lemmas of Schur convexity, we introduce the definition of Schur f-convex and Schur f-concave function, and prove criterion theorem for Schur f-convexity. As a special case, the definition and decision theorem of Schur power convexity are deduced. In section 3, some lemmas are given. Our main results are proved in section 4.

2. Schur f-convexity and Schur Power convexity

Schur convexity was introduced by Schur in 1923 [21], and it has many important applications in analytic inequalities [2, 13, 40], linear regression [30], graphs and matrices [8], combinatorial optimization [15], information-theoretic topics [10], Gamma functions [22], stochastic orderings [26], reliability [14], and other related fields.

Concerning the Schur convexities of well-known means such as Stolarsky means and Gini means, we can refer to [4, 5, 6, 12, 19, 23, 24, 25, 27, 29].

For convenience of readers, we begin by recalling some definitions and lemmas of Schur convexity.

DEFINITION 1. ([21, 32]) Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ $(n \geqslant 2)$.

(i) $\mathbf{x}$ is said to be majorized by $\mathbf{y}$ (in symbol $\mathbf{x} \prec \mathbf{y}$) if

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]} \text{ for } 1 \leqslant k \leqslant n-1, \quad \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}, \tag{2.1}$$

where $x_{[1]} \geqslant x_{[2]} \cdots \geqslant x_{[n]}$ and $y_{[1]} \geqslant y_{[2]} \cdots \geqslant y_{[n]}$ are rearrangements of $\mathbf{x}$ and $\mathbf{y}$ in a decreasing order.

(ii) $\mathbf{x} \geqslant \mathbf{y}$ means $x_i \geqslant y_i$ for all $i = 1, 2, \cdots, n$. Let $\mathbf{U} \subseteq \mathbb{R}^n$ $(n \geqslant 2)$. The function $\phi : \mathbf{U} \mapsto \mathbb{R}$ is said to be increasing if $\mathbf{x} \geqslant \mathbf{y}$ implies $\phi(\mathbf{x}) \geqslant \phi(\mathbf{y})$. ϕ is said to be decreasing if and only if $-\phi$ is increasing.

(iii) $\mathbf{U} \subseteq \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \cdots, \alpha x_n + \beta y_n) \in \mathbf{U}$ for all $\mathbf{x}$, $\mathbf{y}$ and all $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$.

(iv) Let $\mathbf{U} \subseteq \mathbb{R}^n$ $(n \geqslant 2)$ be a set with nonempty interior. Then $\phi : \mathbf{U} \mapsto \mathbb{R}$ is said to be Schur convex if $\mathbf{x} \prec \mathbf{y}$ on $\mathbf{U}$ implies $\phi(\mathbf{x}) \leqslant \phi(\mathbf{y})$. ϕ is said to be Schur concave if $-\phi$ is Schur convex.

DEFINITION 2. ([21]) (i) $\mathbf{U} \subseteq \mathbb{R}^n$ $(n \geqslant 2)$ is called symmetric set, if $\mathbf{x} \in \mathbf{U}$ implies $\mathbf{xP} \in \mathbf{U}$ for every $n \times n$ permutation matrix $\mathbf{P}$.

(ii) The function $\phi : \mathbf{U} \mapsto \mathbb{R}$ is called symmetric one if for every permutation matrix $\mathbf{P}$, $\phi(\mathbf{xP}) = \phi(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{U}$.

The following lemma gives a useful characterization of Schur-convexity [32, Theorem 6.4 and Note] (also can refer to [21]).

LEMMA 1. *Let $\mathbf{U} = I^n (I \subseteq \mathbb{R})$ be a symmetric set with nonempty interior $\mathbf{U}^0$ and let $\phi : \mathbf{U} \mapsto \mathbb{R}$ be continuous on $\mathbf{U}$ and differentiable in $\mathbf{U}^0$. Then ϕ is Schur convex (Schur concave) on $\mathbf{U}$ if and only if ϕ is symmetric on $\mathbf{U}$ and*

$$(x_i - x_j)\left(\frac{\partial \phi(\mathbf{x})}{\partial x_i} - \frac{\partial \phi(\mathbf{x})}{\partial x_j}\right) \geqslant (\leqslant) 0 \tag{2.2}$$

holds for $(x_1, x_2, ..., x_n) \in \mathbf{U}^0$, $i \neq j$, $i, j = 1, 2, ..., n$.

Now we define the Schur f-convexity as follows.

DEFINITION 3. Let $\mathbf{U} = I^n (I \subseteq \mathbb{R})$ and let f be a strictly monotone function defined on I. Denote by

$$f(\mathbf{x}) = (f(x_1), f(x_2), ..., f(x_n)) \text{ and } f(\mathbf{y}) = (f(y_1), f(y_2), ..., f(y_n)).$$

(i) $\mathbf{U}$ is called a f-convex set if $(f^{-1}(\alpha f(x_1) + \beta f(y_1)), \cdots, f^{-1}(\alpha f(x_n) + \beta f(y_n)))$ $\in \mathbf{U}$ for all $\mathbf{x}, \mathbf{y} \in \mathbf{U}$ and all $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$.

(ii) Let $\mathbf{U}$ be a set with nonempty interior. Then function $\phi : \mathbf{U} \mapsto \mathbb{R}$ is said to be Schur f-convex on $\mathbf{U}$ if $f(\mathbf{x}) \prec f(\mathbf{y})$ on $\mathbf{U}$ implies $\phi(\mathbf{x}) \leqslant \phi(\mathbf{y})$.

ϕ is said to be Schur f-concave if $-\phi$ is Schur f-convex.

REMARK 1. Let $\mathbf{U} = I^n (I \subseteq \mathbb{R})$ and let f be a strictly monotone function defined on I and $f(\mathbf{U}) = \{f(\mathbf{x}) : \mathbf{x} \in \mathbf{U}\}$. Then function $\phi : \mathbf{U} \mapsto \mathbb{R}$ is Schur f-convex (Schur f-concave) if and only if $\phi \circ f^{-1}$ is Schur convex (Schur concave) on $f(\mathbf{U})$.

Indeed, if function $\phi : \mathbf{U} \mapsto \mathbb{R}$ is Schur f-convex, then $\forall \mathbf{x}', \mathbf{y}' \in f(\mathbf{U})$, there are $\mathbf{x}, \mathbf{y} \in \mathbf{U}$ such that $\mathbf{x}' = f(\mathbf{x}), \mathbf{y}' = f(\mathbf{y})$. If $f(\mathbf{x}) \prec f(\mathbf{y})$, that is, $\mathbf{x}' \prec \mathbf{y}'$, then $\phi(\mathbf{x}) \leqslant \phi(\mathbf{y})$, that is, $\phi((f^{-1}(\mathbf{x}')) \leqslant \phi((f^{-1}(\mathbf{y}'))$. This shows that $\phi \circ f^{-1}$ is Schur convex on $f(\mathbf{U})$. Conversely, if $\phi \circ f^{-1}$ is Schur convex on $f(\mathbf{U})$, then $\forall \mathbf{x}, \mathbf{y} \in \mathbf{U}$ such that $f(\mathbf{x}) \prec f(\mathbf{y})$, we have $\phi((f^{-1}(f(\mathbf{x}))) \leqslant \phi((f^{-1}(f(\mathbf{y})))$, that is, $\phi(\mathbf{x}) \leqslant \phi(\mathbf{y})$. This indicates ϕ is Schur f-convex on $\mathbf{U}$.

In the same way, we can show that ϕ is Schur f-concave on $\mathbf{U}$ if and only if $\phi \circ f^{-1}$ is Schur concave on $f(\mathbf{U})$.

REMARK 2. Let $\mathbf{U} \subseteq \mathbb{R}^n$ $(n \geqslant 2)$ be a symmetric set and the function $\phi : \mathbf{U} \mapsto \mathbb{R}$ be Schur f-convex (Schur f-concave). Then ϕ is symmetric on $\mathbf{U}$.

In fact, for any $\mathbf{x} \in \mathbf{U}$ and every permutation matrix $\mathbf{P}$, we have $\mathbf{xP} \in \mathbf{U}$. Note $\mathbf{xP}$ is another permutation of $\mathbf{x}$, hence $f(\mathbf{x}) \prec f(\mathbf{xP}) \prec f(\mathbf{x})$. Since ϕ is Schur f-convex (Schur f-concave), we have $\phi(\mathbf{x}) \leqslant (\geqslant) \phi(\mathbf{xP}) \leqslant (\geqslant) \phi(\mathbf{x})$, that is, $\phi(\mathbf{xP}) = \phi(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{U}$. This shows that ϕ is symmetric on $\mathbf{U}$.

By Lemma 1 and Remark 1, 2, we have the following

THEOREM 8. *Assume that* $\mathbf{U} = I^n (I \subseteq \mathbb{R})$ *is a symmetric set with nonempty interior* $\mathbf{U}^0$, $\phi : \mathbf{U} \mapsto \mathbb{R}$ *is continuous on* $\mathbf{U}$ *and differentiable in* $\mathbf{U}^0$, *and* f *is a strictly monotone and derivable function defined on* I. *Then* ϕ *is Schur* f*-convex (Schur* f*-concave) on* $\mathbf{U}$ *if and only if* ϕ *is symmetric on* $\mathbf{U}$ *and*

$$(f(x_i) - f(x_j))\left(\frac{1}{f'(x_i)}\frac{\partial \phi(\mathbf{x})}{\partial x_i} - \frac{1}{f'(x_j)}\frac{\partial \phi(\mathbf{x})}{\partial x_j}\right) \geqslant (\leqslant) 0 \tag{2.3}$$

holds for $(x_1, x_2, \dots, x_n) \in \mathbf{U}^0$, $i \neq j$, $i, j = 1, 2, \dots, n$.

Proof. We easily check that $\phi \circ f^{-1}$ is symmetric on $f(\mathbf{U})$ if and only if ϕ is symmetric on $\mathbf{U}$.

By Remark 1 and Lemma 1, $\phi \circ f^{-1}$ is Schur convex (Schur concave) if and only if $\phi \circ f^{-1}$ is symmetric on $f(\mathbf{U})$ and

$$(y_i - y_j)\left(\frac{\partial\left(\phi \circ f^{-1}\right)}{\partial y_i} - \frac{\partial(\phi \circ f^{-1})}{\partial y_j}\right) \geqslant (\leqslant) 0$$

holds for $(y_1, y_2, \dots, y_n) \in f\left(\mathbf{U}^0\right)$, $i \neq j$, $i, j = 1, 2, \dots, n$. Making the substitution $\mathbf{x} = f^{-1}(\mathbf{y})$ yields desired result. □

Putting $f(x) = 1, \ln x, x^{-1}$ in Definition 3 yield the Schur convexity, Schur geometrical convexity and Schur harmonic convexity. It is clear that the Schur f-convexity is a generalization of the Schur convexity mentioned above. In general, we have

DEFINITION 4. Let $f : \mathbb{R}_+ \mapsto \mathbb{R}$ be defined by $f(x) = (x^m - 1)/m$ if $m \neq 0$ and $f(x) = \ln x$ if $m = 0$. Then function $\phi : \mathbf{U}(\subseteq \mathbb{R}^n_+) \mapsto \mathbb{R}$ is said to be Schur m-power convex on $\mathbf{U}$ if $f(\mathbf{x}) \prec f(\mathbf{y})$ on $\mathbf{U}$ implies $\phi(\mathbf{x}) \leqslant \phi(\mathbf{y})$.

ϕ is said to be Schur m-power concave if $-\phi$ is Schur m-power convex.

For Schur power convexity, by Theorem 8 we have

COROLLARY 1. *Let* $\mathbf{U} \subseteq \mathbb{R}^n_+$ *be a symmetric set with nonempty interior* $\mathbf{U}^0$ *and* $\phi : \mathbf{U} \mapsto \mathbb{R}$ *be continuous on* $\mathbf{U}$ *and differentiable in* $\mathbf{U}^0$. *Then* ϕ *is Schur* m*-power convex (Schur* m*-power concave) on* $\mathbf{U}$ *if and only if* ϕ *is symmetric on* $\mathbf{U}$ *and*

$$\frac{x_i^m - x_j^m}{m}\left(x_i^{1-m}\frac{\partial \phi(\mathbf{x})}{\partial x_i} - x_j^{1-m}\frac{\partial \phi(\mathbf{x})}{\partial x_j}\right) \geqslant (\leqslant) 0 \ \textit{if} \ m \neq 0, \tag{2.4}$$

$$(\ln x_i - \ln x_j)\left(x_i\frac{\partial \phi(\mathbf{x})}{\partial x_i} - x_j\frac{\partial \phi(\mathbf{x})}{\partial x_j}\right) \geqslant (\leqslant) 0 \ \textit{if} \ m = 0 \tag{2.5}$$

holds for $(x_1, x_2, \dots, x_n) \in \mathbf{U}^0$, $i \neq j$, $i, j = 1, 2, \dots, n$.

3. Lemmas

LEMMA 2. *For fixed* $p \in \mathbb{R}$, $m > 0$ *and* $w > -2$, *Daróczy mean* $H_{p,w}(a,b)$ *is Schur* m*-power convex (Schur* m*-power concave) with respect to* $(a,b) \in \mathbb{R}_+^2$ *if and only if* $g(t) \geqslant (\leqslant) 0$ *for all* $t > 0$, *where*

$$g(t) := g_{p,w}(t) = 2\sinh((p-m)t) - w\sinh(mt). \tag{3.1}$$

Proof. Let $m \neq 0$ and $H_{p,w} := H_{p,w}(a,b)$ defined by (1.1).
For $p \neq 0$ we get

$$\frac{\partial \ln H_{p,w}}{\partial a} = \frac{1}{H_{p,w}} \frac{\partial H_{p,w}}{\partial a} = \frac{a^{p-1} + \frac{w}{2} a^{\frac{p}{2}-1} b^{\frac{p}{2}}}{a^p + w(ab)^{p/2} + b^p},$$
$$\frac{\partial \ln H_{p,w}}{\partial b} = \frac{1}{H_{p,w}} \frac{\partial H_{p,w}}{\partial b} = \frac{\frac{w}{2} a^{\frac{p}{2}} b^{\frac{p}{2}-1} + b^{p-1}}{a^p + w(ab)^{p/2} + b^p},$$

thus the expression $\Delta_{p,w}$ corresponding to (2.4) in case of $\phi = H_{p,w}$ can be written as

$$\begin{aligned}\Delta_{p,w} &:= \frac{a^m - b^m}{m}\left(a^{1-m}\frac{\partial H_{p,w}}{\partial a} - b^{1-m}\frac{\partial H_{p,w}}{\partial b}\right)\\ &= \frac{a^m - b^m}{m} H_{p,w} \frac{(a^{p-m} - b^{p-m}) + \frac{w}{2}\left(a^{-m+p/2}b^{p/2} - a^{p/2}b^{-m+p/2}\right)}{a^p + w(ab)^{p/2} + b^p}.\end{aligned}$$

Substituting $\ln\sqrt{a/b} = t$ and using $\sinh x = \frac{1}{2}(e^x - e^{-x})$, $\cosh x = \frac{1}{2}(e^x + e^{-x})$ lead to

$$\begin{aligned}\Delta_{p,w} &= H_{p,w}\frac{a^m - b^m}{m}(ab)^{(p-m)/2}\frac{2\sinh(p-m)t - w\sinh mt}{w + 2\cosh pt}\\ &:= \frac{H_{p,w}\frac{a^m - b^m}{m}(ab)^{(p-m)/2}}{w + 2\cosh pt} g_{p,w}(t).\end{aligned} \tag{3.2}$$

For $p = 0$, we have

$$\begin{aligned}\Delta_{0,w} &:= \frac{a^m - b^m}{m}\left(a^{1-m}\frac{\partial H_{0,w}}{\partial a} - b^{1-m}\frac{\partial H_{0,w}}{\partial b}\right)\\ &= \frac{a^m - b^m}{m}\left(\frac{a^{1-m}\sqrt{b}}{2\sqrt{a}} - \frac{b^{1-m}\sqrt{a}}{2\sqrt{b}}\right)\\ &= -\frac{a^m - b^m}{m}(ab)^{(1-m)/2}\sinh(mt).\end{aligned} \tag{3.3}$$

It is easy to verify that $\lim_{p\to 0}\Delta_{p,w} = \Delta_{0,w}$, and then (3.2) holds for all $p \in \mathbb{R}$.

Since $\Delta_{p,w}$ is symmetric with respect to a and b, without loss of generality we assume $a > b$, then $t = \ln\sqrt{a/b} > 0$. It is clear that $\frac{H_{p,w}\frac{a^m - b^m}{m}(ab)^{(p-m)/2}}{w + 2\cosh(pt)} > 0$ and therefore by Corollary 1 $H_{p,w}(a,b)$ is Schur m-power convex (Schur m-power concave) with respect to $(a,b) \in \mathbb{R}_+^2$ if and only if $\Delta_{p,w} \geqslant (\leqslant) 0$ if and only if $g(t) = g_{p,w}(t) \geqslant (\leqslant) 0$ for all $t > 0$.

It is easy to check that for $m = 0$ this lemma is also true.

This lemma is proved. □

LEMMA 3. *Let $g(t)$ be defined by (3.1). Then*

$$\lim_{t\to 0^+} \frac{g(t)}{t} = \lim_{t\to 0^+} g'(t) = 2\left(p - \frac{w+2}{2}m\right), \tag{3.4}$$

Proof. A simple calculation yields

$$g'(t) = 2(p-m)\cosh(p-m)t - wm\cosh mt. \tag{3.5}$$

Since $g(0) = 0$, applying L'Hospital's rule yields

$$\lim_{t\to 0^+} \frac{g(t)}{t} = \lim_{t\to 0^+} g'(t) = 2\left(p - \frac{w+2}{2}m\right),$$

which proves the lemma. □

LEMMA 4. *Let $\beta = \max(|p-m|, |m|)$ with $m > 0$ and let $g(t)$ be defined by (3.1). Then*

$$\lim_{t\to\infty} \frac{2\beta g(t)}{e^{\beta t}} = \begin{cases} 2(p-m) & \text{if } p > 2m \text{ or } p < 0, \\ 2(p - \frac{w+2}{2}m) & \text{if } p = 2m \text{ or } p = 0, \\ -wm & \text{if } 0 < p < 2m. \end{cases} \tag{3.6}$$

Proof. (3.6) easily follows from the limit relation

$$\lim_{t\to\infty} \frac{2\cosh\alpha t}{e^{\beta t}} = \begin{cases} 1 & \text{if } \beta = |\alpha|, \\ 0 & \text{if } \beta > |\alpha|. \end{cases} \tag{3.7}$$

We have

$$\begin{aligned} &\lim_{t\to\infty} \frac{2\beta g(t)}{e^{\beta t}} = \lim_{t\to\infty} \frac{2g'(t)}{e^{\beta t}} \\ &= \lim_{t\to\infty} 2\frac{2(p-m)\cosh((p-m)t) - wm\cosh(mt)}{e^{\beta t}} \\ &= \begin{cases} 2(p-m) & \text{if } |p-m| > |m|, \text{ i.e. } p(p-2m) > 0, \\ 2(p - \frac{w+2}{2}m) & \text{if } |p-m| = |m|, \text{ i.e. } p = 0 \text{ or } p = 2m, \\ -wm & \text{if } |p-m| < |m|, \text{ i.e. } p(p-2m) < 0, \end{cases} \end{aligned}$$

which implies (3.6). □

4. Proof of Main Results

Proof of Theorem 3. By Lemma 3.1, to prove Theorem 3, it suffices to prove that $g_{p,w}(t) \geqslant 0$ for all $t > 0$ if and only if $(p,w) \in \Omega_1$.
Necessity. We prove that $(p,w) \in \Omega_1$ is the necessary conditions for $g(t) = g_{p,w}(t) \geqslant 0$ for all $t > 0$. It follows that

$$\lim_{t\to 0^+} \frac{g(t)}{t} \geqslant 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{2\beta g(t)}{e^{\beta t}} \geqslant 0. \tag{4.1}$$

Now, we obtain the necessary conditions from (4.1) together with (3.4) and (3.6). To this aim, we divide the proof of necessity into three cases.

Case 1:

$$\begin{cases} 2(p - \frac{w+2}{2}m) \geqslant 0, \\ 2(p-m) \geqslant 0, \\ p > 2m \text{ or } \ p < 0 \end{cases} \Longrightarrow \begin{cases} p \geqslant \frac{w+2}{2}m, \\ p > 2m, \end{cases}$$

which implies $(p,w) \in \{w > -2, p \geqslant \frac{w+2}{2}m, p > 2m\} := \Omega_{11}$.

Case 2:

$$\begin{cases} 2(p - \frac{w+2}{2}m) \geqslant 0, \\ 2(p - \frac{w+2}{2}m) \geqslant 0, \\ p = 2m \text{ or } \ p = 0 \end{cases} \Longrightarrow \begin{cases} -2 < w \leqslant 2, \\ p = 2m, \end{cases}$$

which implies $(p,w) \in \{-2 < w \leqslant 2, p = 2m\} := \Omega_{12}$.

Case 3:

$$\begin{cases} 2(p - \frac{w+2}{2}m) \geqslant 0, \\ -wm \geqslant 0, \\ 0 < p < 2m \end{cases} \Longrightarrow \begin{cases} \frac{w+2}{2}m \leqslant p < 2m, \\ -2 < w \leqslant 0, \end{cases}$$

which implies $(p,w) \in \{-2 < w \leqslant 0, \frac{w+2}{2}m \leqslant p < 2m\} := \Omega_{13}$.

Summarizing the above three cases yield

$$\begin{aligned} (p,w) &\in \Omega_{11} \cup \Omega_{12} \cup \Omega_{13} \\ &= \left\{-2 < w \leqslant 0, p \geqslant \frac{w+2}{2}m\right\} \cup \left\{w > 0, p \geqslant \max\left(\frac{w+2}{2}m, 2m\right)\right\} = \Omega_1. \end{aligned}$$

Sufficiency. We prove the condition $(p,w) \in \Omega_1$ is sufficient for $g(t) = g_{p,w}(t) \geqslant 0$ for all $t > 0$. Since $g(0) = 0$, it is enough to prove $g'(t) \geqslant 0$ for all $t > 0$ if $(p,w) \in \Omega_1$. For this end, we need to write $g'(t)$ given by (3.5) in the following form, too,

$$\begin{aligned} g'(t) &= 2(p-m)\cosh((p-m)t) - wm\cosh(mt) \\ &= (2(p-m) - wm)\cosh((p-m)t) + wm(\cosh((p-m)t) - \cosh(mt)) \\ &= 2\left(p - \frac{w+2}{2}m\right)\cosh((p-m)t) + 2wm\sinh\frac{pt}{2}\sinh\left(\frac{p-2m}{2}t\right). \end{aligned} \tag{4.2}$$

Next we divide the proof of sufficiency into three cases.

Case 1: $-2 < w \leqslant 0, \frac{w+2}{2}m \leqslant p < 2m$. Since $p - \frac{w+2}{2}m \geqslant 0$, $w \leqslant 0$, $p > 0$, $p - 2m < 0$, by (4.2) we have $g'(t) \geqslant 0$ for all $t > 0$.

Case 2: $-2 < w \leqslant 0, p \geqslant 2m$. By (3.5) it is obvious that $g'(t) \geqslant 0$ for all $t > 0$.

Case 3: $w > 0, p \geqslant \max(\frac{w+2}{2}m, 2m)$. Since $p - \frac{w+2}{2}m \geqslant 0$, $w > 0$, $p > 0$, $p - 2m \geqslant 0$, by (4.2) we also have $g'(t) \geqslant 0$ for all $t > 0$.

Hence, $g'(t) \geqslant 0$ for all $t > 0$ if $(p,w) \in \Omega_1$.

Thus the proof of Theorem 3 ends. □

Proof of Theorem 4. By Lemma 3.1, it is enough to show that $g_{p,w}(t) \leqslant 0$ for all $t > 0$ if and only if $(p,w) \in \Omega_2$.

Necessity. We prove that $(p,w) \in \Omega_2$ is necessary. We start with

$$\lim_{t\to 0^+} \frac{g(t)}{t} \leqslant 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{2\beta g(t)}{e^{\beta t}} \leqslant 0 \tag{4.3}$$

and divide the proof into three cases.

Case 1:

$$\begin{cases} 2(p - \frac{w+2}{2}m) \leqslant 0, \\ 2(p-m) \leqslant 0, \\ p > 2m \text{ or } \; p < 0 \end{cases} \Longrightarrow \begin{cases} w > -2, \\ p < 0, \end{cases}$$

which implies $(p,w) \in \{w > -2, p < 0\} := \Omega_{21}$.

Case 2:

$$\begin{cases} 2(p - \frac{w+2}{2}m) \leqslant 0, \\ 2(p - \frac{w+2}{2}m) \leqslant 0, \\ p = 2m \text{ or } \; p = 0 \end{cases} \Longrightarrow \begin{cases} w \geqslant 2, \\ p = 2m \end{cases} \text{or} \begin{cases} w > -2, \\ p = 0, \end{cases}$$

which implies $(p,w) \in \{w \geqslant 2, p = 2m\} \cup \{w > -2, p = 0\} := \Omega_{22}$.

Case 3:

$$\begin{cases} 2(p - \frac{w+2}{2}m) \leqslant 0, \\ -wm \leqslant 0, \\ 0 < p < 2m \end{cases} \Longrightarrow \begin{cases} p \leqslant \frac{w+2}{2}m, \\ w \geqslant 0, \\ 0 < p < 2m, \end{cases}$$

which implies $(p,w) \in \{w \geqslant 0, 0 < p < 2m, p \leqslant \frac{w+2}{2}m\} := \Omega_{23}$.

Summarizing the above three cases yield

$$\begin{aligned} (p,w) &\in \Omega_{21} \cup \Omega_{22} \cup \Omega_{23} \\ &= \{-2 < w < 0, p < 0\} \cup \left\{ w \geqslant 0, p \leqslant \min\left(\frac{w+2}{2}m, 2m \right) \right\} = \Omega_2. \end{aligned}$$

Sufficiency. To prove the condition $(p,w) \in \Omega_2$ is sufficient, we only need to show that $g'(t) \leqslant 0$ for all $t > 0$ if $(p,w) \in \Omega_2$. We distinguish three cases.

Case 1: $-2 < w < 0, p < 0$. Since $p - \frac{w+2}{2}m < 0$, $w < 0$, $p < 0$, $p - 2m < 0$, from (4.2) we have $g'(t) < 0$ for all $t > 0$.

Case 2: $w \geqslant 0, 0 \leqslant p \leqslant \min(\frac{w+2}{2}m, 2m)$. Since $p - \frac{w+2}{2}m \leqslant 0$, $w \geqslant 0$, $p \geqslant 0$, $p - 2m \leqslant 0$, from (4.2) we also have $g'(t) < 0$ for all $t > 0$.

Case 3: $w \geqslant 0, p < 0$. By (3.5), clearly, $g'(t) < 0$ for all $t > 0$.

To sum up, $g'(t) \leqslant 0$ for all $t > 0$ if $(p,w) \in \Omega_2$.

This completes the proof of Theorem 4. $\square$

Proof of Theorem 5. Let $g_{p,w,m}(t) := g_{p,w}(t)$ defined by (3.1) and

$$p' = -p, \quad m' = -m.$$

We easily verify that, for $p, p', m, m' \in \mathbb{R}$,

$$g_{p,w,m}(t) = -g_{p\prime,w,m\prime}(t).$$

From this and Lemma 2, for $m < 0$, $H_{p,w}(a,b)$ is Schur m-power convex if and only if $H_{p\prime,w,m\prime}(a,b)$ is Schur m'-power concave with respect to $(a,b) \in \mathbb{R}_+^2$, which, by Theorem 4, if and only if

$$-2 < w < 0,\ p' < 0 \ \text{ or } \ w \geqslant 0,\ p' \leqslant \min\left(\frac{w+2}{2}m', 2m'\right),$$

which is equivalent to

$$-2 < w < 0,\ p > 0 \ \text{ or } \ w \geqslant 0,\ p \geqslant \max\left(\frac{w+2}{2}m, 2m\right),$$

that is, $(p,w) \in E_1$, which proves Theorem 5. $\square$

Proof of Theorem 6. Similarly to the proof of Theorem 5, we see that, for $m < 0$, $H_{p,w}(a,b)$ is Schur m-power concave if and only if $H_{p\prime,w}(a,b)$ is Schur m'-power convex with respect to $(a,b) \in \mathbb{R}_+^2$, which, by Theorem 3, if and only if

$$-2 < w \leqslant 0,\ p' \geqslant \frac{w+2}{2}m' \ \text{ or } \ w > 0,\ p' \geqslant \max\left(\frac{w+2}{2}m', 2m'\right),$$

which is equivalent to

$$-2 < w \leqslant 0,\ p \leqslant \frac{w+2}{2}m \ \text{ or } \ w > 0,\ p \leqslant \min\left(\frac{w+2}{2}m, 2m\right),$$

that is, $(p,w) \in E_2$. Thus the proof of Theorem 6 is finished. $\square$

Proof of Theorem 7. If $m = 0$, then

$$g(t) := g_{p,w}(t) = 2\sinh(pt).$$

Obviously, $\operatorname{sgn}(g(t)) = \operatorname{sgn}(p)$, and so $H_{p,w}(a,b)$ is Schur m-power convex (Schur m-power concave) if and only if $p \geqslant (\leqslant) 0$. $\square$

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