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OPTIMAL POLYNOMIAL BOUNDS FOR THE EXPONENTIAL FUNCTION

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Abstract. We find polynomial lower and upper bounds of e^x on some respective intervals. To be specific, for each natural number n, we construct polynomials $p_n(x)$ and $q_n(x)$ of degree n so that $p_n(x) \leqslant p_{n+1}(x) \leqslant e^x$ and $e^x \leqslant q_{n+1}(x) \leqslant q_n(x)$ on some intervals, respectively. These polynomials are optimal in the sense that if $p(x)$ (or $q(x)$) is a polynomial of degree n with $p_{n-1}(x) \leqslant p(x) \leqslant e^x$ (or $e^x \leqslant q(x) \leqslant q_{n-1}(x)$) then $p(x) \leqslant p_n(x)$ (or $q_n(x) \leqslant q(x)$). The fact that $1/p_n(-x)$ works as an upper bound of e^x on a switched interval is interesting. We also provide the size comparison between two upper bounds $q_n(x)$ and $1/p_n(-x)$.

1. Introduction

Starting from the trivial bound of the exponential function

$$1+x \leqslant e^x \quad \text{or} \quad e^x \leqslant \frac{1}{1-x} \qquad (x<1), \tag{1.1}$$

various acute bounds or generalizations of (1.1) have been researched (see [3], [5] and [6]). Most recent results are Kim [2], [4] and Bae [1]. In [1], we have constructed the following polynomial lower bounds of arbitrary degree for the exponential function.

THEOREM 1.1. *For* $n \geqslant 1$, *let*

$$p_n(x) = 1 + \alpha_1 x + \sum_{j=2}^{n} \alpha_j x^2 (x+1)^{j-2}$$

where $\alpha_1 = 1$ *and*

$$\alpha_n = \frac{n-2}{e}\left(\sum_{j=0}^{n-2}\frac{1}{j!} - e\right) + \frac{1}{(n-2)!\,e}, \quad n \geqslant 2.$$

Then $p_n(x) \leqslant p_{n+1}(x) \leqslant e^x$ *for* $x \geqslant -1$. *Furthermore, these bounds are optimal in the sense that if* $p(x)$ *is a polynomial of degree* n *with* $p_{n-1}(x) \leqslant p(x) \leqslant e^x$ *for* $x \geqslant -1$, *then* $p(x) \leqslant p_n(x)$ *for* $x \geqslant -1$.

Note the inequality holds on the ray $[-1,\infty)$. In §3, we generalize Theorem 1.1 so that the constructed polynomial lower bounds for the exponential function are valid

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on an arbitrary ray $[u,\infty)$. In Theorem 1.1, one may observe that we used $p_1(x) = 1+x$, the tangent line of e^x at $x=0$, as the base linear bound for e^x. That will be generalized to an arbitrary tangent line of e^x at $x=v>u$. So, with the base linear bound $p_1(x) = e^v + e^v(x-v)$, we construct optimal polynomials $p_n(x)$ of degree n satisfying $p_n(x) \leqslant p_{n+1}(x) \leqslant e^x$ on $[u,\infty)$.

In §4, the same idea is applied for the upper bounds of e^x on $[\mu,\nu]$. Here the base linear bound is

$$q_1(x) = e^{\mu} + \frac{e^{\nu}-e^{\mu}}{\nu-\mu}(x-\mu),$$

the secant line connecting two points (μ,e^{μ}) and (ν,e^{ν}). Then we are to find optimal polynomials $q_n(x)$ of degree n satisfying $e^x \leqslant q_{n+1}(x) \leqslant q_n(x)$ on $[\mu,\nu]$.

Changing variable to $-x$ in the polynomial lower bounds $p_n(x)$ in §3 and taking reciprocals, we have rational upper bounds $\frac{1}{p_n(-x)}$ such that

$$e^x \leqslant \frac{1}{p_{n+1}(-x)} \leqslant \frac{1}{p_n(-x)}$$

on $(-\infty,-u]$. If we set $u=-\nu, v=-\mu$, we obtain two sorts of upper bounds $q_n(x)$ and $\frac{1}{p_n(-x)}$ for e^x on $[\mu,\nu]$. In §5, we try to offer a size comparison between these bounds of which methods are quite technical.

We introduce some preliminaries in the next section. For the simplification of the argument, we assume that all functions in this paper are real valued on $\mathbf{R}$ and analytic on $\mathbf{C}$. Also, when we talk about the number of zeros of a function, that number includes all the multiplicities.

2. Preliminaries

This section is exactly the same as [1, §2]. Because it is short and simple, we include it in this paper for reader's convenience.

PROPOSITION 2.1. *Let a be a fixed real number. Suppose $f^{(j)}(a)=0$ $(0 \leqslant j \leqslant k-1)$ and $f^{(k)}(a) > 0$ for some positive integer k.*

(i) *If k is even, then there exists an interval around a on which $f(x)$ achieves the unique minimum $f(a)=0$.*

(ii) *If k is odd, then there exists an interval around a on which $f(x)$ is strictly increasing.*

Proof. Let $h(x)$ be a function defined on an open interval I containing a. We say that $h(x)$ has unique minimum (UM) property when $h(a)=0$ and it is the unique minimum on I, and $h(x)$ has strictly increasing (SI) property when $h(a)=0$ and $h(x)$ is strictly increasing on I. It is easy to see that if $h'(x)$ has UM-property and $h(a)=0$ then $h(x)$ has SI-property. On the other hand, if $h'(x)$ has SI-property and $h(a)=0$ then $h(x)$ has UM-property. Since $f^{(k)}(a)>0$, by the continuity of $f^{(k)}(x)$, there exists an open interval I containing a on which $f^{(k)}(x) > 0$.

Hense, on the interval I, $f^{(k-1)}(x)$ has SI-property, $f^{(k-2)}(x)$ has UM-property, $f^{(k-3)}(x)$ has SI-property, $\cdots$, and so on. Pursuing these alternations k times, we have the conclusions of the proposition. □

PROPOSITION 2.2. *Let* $p(x)$ *be a polynomial of degree* n. *Then* $e^x - p(x)$ *has at most* $n+1$ *zeros.*

Proof. Suppose a function $f(x)$ has k zeros. Applying Rolle's Theorem, we know that its derivative $f'(x)$ has at least $k-1$ zeros. In other words, if $f'(x)$ has $k-1$ zeros, then $f(x)$ has at most k zeros. Now the proof is straightforward by using induction on n, the degree of $p(x)$. □

DEFINITION 2.3. We say that a is a zero of $f(x)$ of multiplicity m when $f^{(j)}(a) = 0$ for $j = 0, 1, 2, \cdots, m-1$ and $f^{(m)}(a) \neq 0$. In this case, $Z(a, f) = m$ is called the multiplicity of the zero a of f.

We will introduce one more proposition for the proof of which we need the following lemma.

LEMMA 2.4. *Suppose there exists an interval* $[a, a+\delta]$ *on which* $f(x) \geqslant 0$ *and* $f(a) = 0$. *Then there exists* λ $(0 < \lambda < \delta)$ *such that* $f'(x) \geqslant 0$ *on* $[a, a+\lambda]$.

Proof. We may assume that $f(x)$ is not constant and $Z(a, f) = m \geqslant 1$. Then the Taylor series of f and its derivative at $x = a$ have the form

$$f(x) = \sum_{j=m}^{\infty} \frac{f^{(j)}(a)}{j!}(x-a)^j, \qquad f'(x) = \sum_{j=m}^{\infty} \frac{f^{(j)}(a)}{(j-1)!}(x-a)^{j-1}.$$

If x is close enough to a, then the first terms dominate both series. Therefore $f^{(m)}(a)$ has to be positive and we can find a small enough $\lambda > 0$ so that $f'(x) \geqslant 0$ on $[a, a+\lambda]$. □

PROPOSITION 2.5. *Suppose* $0 \leqslant f(x) \leqslant g(x)$ *for* $x \geqslant a$ *and* $Z(a, g) = m$. *Then* $Z(a, f) \geqslant m$ *and* $f^{(m)}(a) \leqslant g^{(m)}(a)$.

Proof. By the definition of a multiple zero, $g^{(j)}(a) = 0$ for $j = 0, 1, 2, \cdots, m-1$. Hence, for each $j \in \{0, 1, 2, \cdots, m\}$, it suffices to show that $0 \leqslant f^{(j)}(x) \leqslant g^{(j)}(x)$ on the interval $[a, a+\lambda_j]$ for some $\lambda_j > 0$. We use an induction on j. Assume that $0 \leqslant f^{(j-1)}(x) \leqslant g^{(j-1)}(x)$ on the interval $[a, a+\lambda_{j-1}]$. Then note $f^{(j-1)}(a) = g^{(j-1)}(a) = 0$. Now applying Lemma 2.4 twice, one for $f^{(j-1)}(x)$ and another for $g^{(j-1)}(x) - f^{(j-1)}(x)$, we can find an interval $[a, a+\lambda_j]$ on which $0 \leqslant f^{(j)}(x) \leqslant g^{(j)}(x)$. □

3. Polynomial lower bounds of e^x

In this section we prove the following result.

THEOREM 3.1. *For* $n \geqslant 1$, *let*

$$p_n(x) = e^v + e^u \alpha_1 (x-v) + e^u \sum_{j=2}^{n} \alpha_j (x-v)^2 (x-u)^{j-2} \tag{3.1}$$

where $\alpha_1 = e^{v-u}$ *and*

$$\alpha_n = \frac{n-1-v+u}{(v-u)^n}\left(\sum_{j=0}^{n-2}\frac{(v-u)^j}{j!} - e^{v-u}\right) + \frac{1}{(n-2)!\,(v-u)}, \quad n \geqslant 2. \tag{3.2}$$

Then $p_n(x) \leqslant p_{n+1}(x) \leqslant e^x$ *for* $x \geqslant u$. *Furthermore, these bounds are optimal in the sense that if* $p(x)$ *is a polynomial of degree* n *with* $p_{n-1}(x) \leqslant p(x) \leqslant e^x$ *for* $x \geqslant u$, *then* $p(x) \leqslant p_n(x)$ *for* $x \geqslant u$.

REMARK. Examining the proof of the theorem, one may weaken the condition for the optimality a little: *If* $p(x)$ *is a polynomial of degree* n *with* $p_{n-1}(x) \leqslant p(x) \leqslant e^x$ *for* $u \leqslant x \leqslant v$, *then* $p(x) \leqslant p_n(x)$ *for* $x \geqslant u$.

When we take $u=-1, v=0$, we have exactly Theorem 1.1. For the proof of this theorem, we need some properties of the sequence α_n defined by equation (3.2).

LEMMA 3.2. *For* $n \geqslant 1$, *we have*

(i) $\alpha_n > 0$

(ii) $(v-u)^2\alpha_{n+2} = 2(v-u)\alpha_{n+1} - \alpha_n + \dfrac{1}{n!}$

(iii) $\displaystyle\sum_{n=2}^{\infty} \alpha_n (v-u)^n = \frac{1}{2}(v-u)^2 e^{v-u}.$

Proof. For any $0 < t < n+1$, note that

$$\begin{aligned} e^t - \sum_{j=0}^{n-1}\frac{t^j}{j!} &= \frac{t^n}{n!}\left(1 + \frac{t}{n+1} + \frac{t^2}{(n+1)(n+2)} + \cdots\right) \\ &< \frac{t^n}{n!}\left(1 + \frac{t}{n+1} + \frac{t^2}{(n+1)^2} + \cdots\right) = \frac{t^n(n+1)}{n!\,(n+1-t)}. \end{aligned} \tag{3.3}$$

For $v-u \geqslant n-1$, observe that

$$\alpha_n = \frac{n-1-(v-u)}{(v-u)^n}\left(\sum_{j=0}^{n-2}\frac{(v-u)^j}{j!} - e^{v-u}\right) + \frac{1}{(n-2)!\,(v-u)} > 0.$$

And for $v-u<n-1$, (3.3) gives

$$\begin{aligned}\alpha_n &= \frac{n-1-v+u}{(v-u)^n}\left(\sum_{j=0}^{n-2}\frac{(v-u)^j}{j!}-e^{v-u}\right)+\frac{1}{(n-2)!\,(v-u)}\\ &> \frac{n-1-v+u}{(v-u)^n}\frac{-(v-u)^{n-1}n}{(n-1)!\,(n-v+u)}+\frac{1}{(n-2)!\,(v-u)}\\ &= \frac{1}{(n-1)!\,(n-v+u)}>0.\end{aligned}$$

For (ii), a direct calculation shows that

$$\begin{aligned}&2(v-u)\alpha_{n+1}-\alpha_n+\frac{1}{n!}\\ =&\ \frac{2(n-v+u)}{(v-u)^n}\left(\sum_{j=0}^{n-1}\frac{(v-u)^j}{j!}-e^{v-u}\right)+\frac{2}{(n-1)!}\\ &-\frac{n-1-v+u}{(v-u)^n}\left(\sum_{j=0}^{n-2}\frac{(v-u)^j}{j!}-e^{v-u}\right)-\frac{1}{(n-2)!\,(v-u)}+\frac{1}{n!}\\ =&\ \frac{n+1-v+u}{(v-u)^n}\left(\sum_{j=0}^{n}\frac{(v-u)^j}{j!}-e^{v-u}\right)+\frac{v-u}{n!}=(v-u)^2\alpha_{n+2}.\end{aligned}$$

For (iii), let $\sigma_n=\sum_{k=1}^n(k-v+n)=\frac{n(n+1)}{2}-n(v-u)$ with $\sigma_0=0$. Then we have

$$\begin{aligned}&\sum_{k=2}^{n+1}\alpha_k(v-u)^k\\ =&\sum_{k=2}^{n+1}(k-1-v+u)\left(\sum_{j=0}^{k-2}\frac{(v-u)^j}{j!}-e^{v-u}\right)+\sum_{k=2}^{n+1}\frac{(v-u)^{k-1}}{(k-2)!}\\ =&\sum_{k=0}^{n-1}(\sigma_n-\sigma_k)\frac{(v-u)^k}{k!}-\sigma_n e^{v-u}+\sum_{k=2}^{n+1}\frac{(v-u)^{k-1}}{(k-2)!}\\ =&\,\sigma_n\left(\sum_{k=0}^{n-1}\frac{(v-u)^k}{k!}-e^{v-u}\right)-\sum_{k=1}^{n-1}\sigma_k\frac{(v-u)^k}{k!}+\sum_{k=2}^{n+1}\frac{(v-u)^{k-1}}{(k-2)!}.\end{aligned}$$

Note that, by (3.3), the first term of the last expression tends to zero as n tends to infinity. Thus we obtain

$$\begin{aligned}&\sum_{n=2}^{\infty}\alpha_n(v-u)^n=(v-u)e^{v-u}-\sum_{n=1}^{\infty}\sigma_n\frac{(v-u)^n}{n!}\\ =&\,(v-u)e^{v-u}-\sum_{n=1}^{\infty}\left(\frac{n(n+1)}{2}-n(v-u)\right)\frac{(v-u)^n}{n!}=\frac{1}{2}(v-u)^2e^{v-u}.\end{aligned}$$

For the last equality, one may use the identity

$$(2t+t^2)e^t=\sum_{n=0}^{\infty}\frac{n+2}{n!}\,t^{n+1}$$

which follows from the differentiation of $t^2e^t = \sum_{n=0}^{\infty} \frac{1}{n!} t^{n+2}$. $\square$

LEMMA 3.3. *For* $n \geqslant 1$, *let* $g_n(x) = e^x - p_n(x)$. *Then*

(i) $g_n(v) = g_n'(v) = 0$ *and*

$$g_n''(v) = e^v - 2e^u \sum_{j=2}^n \alpha_j (v-u)^{j-2} > 0 \quad (\textit{with } g_1''(v) = e^v).$$

(ii) *For* $n \geqslant 2$, $g_n(x)$ *has zero of multiplicity* $n-1$ *at* u *and*

$$g_n^{(n-1)}(u) = e^u (n-1)!(v-u)^2 \alpha_{n+1} > 0.$$

Proof. Note that

$$p_n'(x) = e^v + 2e^u \sum_{j=2}^n \alpha_j (x-v)(x-u)^{j-2} + e^u \sum_{j=3}^n (j-2)\alpha_j (x-v)^2 (x-u)^{j-3},$$

$$\begin{aligned} p_n''(x) &= 2e^u \sum_{j=2}^n \alpha_j (x-u)^{j-2} + 4e^u \sum_{j=3}^n (j-2)\alpha_j (x-v)(x-u)^{j-3} \\ &\quad + e^u \sum_{j=4}^n (j-2)(j-3)\alpha_j (x-v)^2 (x-u)^{j-4}. \end{aligned}$$

Invoking Lemma 3.2, (i) is immediate now. For $k \geqslant 2$, let us define $f_k(x) = \alpha_k (x-v)^2 (x-u)^{k-2}$. Then we observe that

$$\begin{aligned} f_k^{(j)}(u) &= 0 \quad \text{if} \quad 0 \leqslant j < k-2 \quad \text{or} \quad j > k \\ f_k^{(k)}(x) &= k!\alpha_k \\ f_k^{(k-1)}(x) &= (k-1)!2\alpha_k (x-v) + (k-1)!(k-2)\alpha_k (x-u) \\ f_k^{(k-2)}(x) &= (k-2)!\alpha_k (x-v)^2 + (k-2)!2(k-2)\alpha_k (x-v)(x-u) \\ &\quad + \frac{1}{2}(k-2)!(k-2)(k-3)\alpha_k (x-u)^2. \end{aligned} \tag{3.4}$$

Note that by Lemma 3.2 (ii) $g_2'(u) = e^u(1 - \alpha_1 - 2\alpha_2(u-v)) = (v-u)^2\alpha_3$. Next, for $n \geqslant 3$, since

$$g_n(x) = e^x - e^v - e^u \alpha_1 (x-v) - e^u \sum_{k=2}^n f_k(x), \tag{3.5}$$

we have

$$\begin{aligned} \frac{g_n^{(n-1)}(u)}{e^u} &= 1 - f_{n-1}^{(n-1)}(u) - f_n^{(n-1)}(u) \\ &= 1 - (n-1)!\alpha_{n-1} + (n-1)!2\alpha_n (v-u) \\ &= (n-1)! \left(2\alpha_n (v-u) - \alpha_{n-1} + \frac{1}{(n-1)!} \right) \\ &= (n-1)!(v-u)^2 \alpha_{n+1} > 0 \end{aligned}$$

by (3.4) and Lemma 3.2 (ii). Finally, it remains to show $g_n^{(j)}(u) = 0$, for $0 \leqslant j \leqslant n-2$. Direct calculation gives $p_n(u) = e^u$ and by Lemma 3.2 (ii) $p_n'(u) = e^u(\alpha_1 - 2\alpha_2(v-u) + \alpha_3(v-u)^2) = e^u$. Hence $g_n(u) = g_n^{(1)}(u) = 0$. Using (3.5) and Lemma 3.2 (ii) again for $2 \leqslant j \leqslant n-2$, we obtain

$$\begin{aligned}\frac{g_n^{(j)}(u)}{e^u} &= 1 - \sum_{k=2}^{n} f_k^{(j)}(u) = 1 - f_j^{(j)}(u) - f_{j+1}^{(j)}(u) - f_{j+2}^{(j)}(u) \\ &= 1 - j!\,\alpha_j + j!\,2\,\alpha_{j+1}(v-u) - j!\,\alpha_{j+2}(v-u)^2 \\ &= j!\left(-\alpha_{j+2}(v-u)^2 + 2\alpha_{j+1}(v-u) - \alpha_j + \frac{1}{j!}\right) = 0. \quad \square\end{aligned}$$

Now we prove Theorem 3.1.

Proof of Theorem 3.1. Since by Lemma 3.2 α_n is a positive sequence and $(x-v)^2(x-u)^{j-2} \geqslant 0$ for $x \geqslant u$, $j \geqslant 2$, it is clear from equation (3.1) that $p_n(x) \leqslant p_{n+1}(x)$ for $x \geqslant u$. We are to show $g_n(x) \geqslant 0$ for $x \geqslant u$. Lemma 3.3 shows that $g_n(x)$ has double zero at $x = v$ and $(n-1)$-multiple zero at $x = u$. By Proposition 2.2, these are all the zeros of $g_n(x)$. Lemma 3.3 (i) implies that $g_n(v) = 0$ is a local minimum. Moreover, combining Lemma 3.3 (ii) with Proposition 2.1, we realize that $g_n(u) = 0$ is a local minimum for odd $n(>1)$ and $g_n(x)$ is increasing in some neighborhood of $x = u$ for even n. Based on this analysis, we may conclude that $g_n(x) \geqslant 0$ for $x \geqslant u$. Finally, suppose $p(x)$ is a polynomial of degree n with $p_{n-1}(x) \leqslant p(x) \leqslant e^x$ for $x \geqslant u$. This means $0 \leqslant p(x) - p_{n-1}(x) \leqslant g_{n-1}(x)$ for $x \geqslant u$. Let $f(x) = p(x) - p_{n-1}(x)$. Then $f(x)$ is a polynomial of degree n and by Lemma 3.3 and Proposition 2.5, it has double zero at v and $n-2$-multiple zero at u. Hence

$$f(x) = \alpha(x-v)^2(x-u)^{n-2} \tag{3.6}$$

for some $\alpha \in \mathbf{R}$. By Proposition 2.5 with Lemma 3.3 (ii), we obtain

$$(n-2)!\,\alpha(u-v)^2 = f^{(n-2)}(u) \leqslant g_{n-1}^{(n-2)}(u) = (n-2)!(v-u)^2 e^u \alpha_n$$

which implies $\alpha \leqslant e^u \alpha_n$. Therefore by equation (3.1), (3.2), and (3.6) we have

$$p_n(x) - p(x) = p_n(x) - p_{n-1}(x) - f(x) = (e^u\alpha_n - \alpha)(x-v)^2(x-u)^{n-2} \geqslant 0$$

for $x \geqslant u$. $\square$

In Proposition 5.6, we will show that $p_n(x)$ converges rapidly to e^x in the sense that $e^x - p_n(x)$ is negligible if n is large enough. Therefore for all sufficiently large n, $p_n(x)$ is positive on $[u,\infty)$. Changing variable to $-x$ and taking reciprocals, we obtain a (rational) upper bound version of Theorem 3.1.

THEOREM 3.4. *For all sufficiently large* n, $p_n(x) > 0$ *on* $[u,\infty)$ *and*

$$e^x \leqslant \frac{1}{p_{n+1}(-x)} \leqslant \frac{1}{p_n(-x)}$$

for $x \leqslant -u$. *Furthermore, these bounds are optimal in the sense that if* $p(x)$ *is a polynomial of degree* n *with* $e^x \leqslant \frac{1}{p(-x)} \leqslant \frac{1}{p_{n-1}(-x)}$ *for* $x \leqslant -u$, *then* $\frac{1}{p_n(-x)} \leqslant \frac{1}{p(-x)}$ *for* $x \leqslant -u$.

4. Polynomial upper bounds of e^x

Consider the secant line connecting two points (μ, e^{μ}) and (ν, e^{ν}). One can ask for the best quadratic polynomial approximation of e^x that is between the secant line and the curve $y = e^x$. If there exists such quadratic polynomial, then one can ask for the best cubic polynomial between the quadratic and $y = e^x$, and so on. The following theorem answers these questions. We assume that $\mu < \nu$ through the paper.

THEOREM 4.1. *For* $n \geqslant 1$, *let*

$$q_n(x) = \sum_{j=0}^{n-1} \frac{e^{\mu}}{j!}(x-\mu)^j + \beta_n (x-\mu)^n \tag{4.1}$$

where

$$\beta_n = \sum_{j=0}^{\infty} \frac{e^{\mu}}{(n+j)!}(\nu-\mu)^j.$$

Then $e^x \leqslant q_{n+1}(x) \leqslant q_n(x)$ *on* $[\mu, \nu]$. *Furthermore, these bounds are optimal in the sense that if* $q(x)$ *is a polynomial of degree* n *with* $e^x \leqslant q(x) \leqslant q_{n-1}(x)$ *on* $[\mu, \nu]$, *then* $q_n(x) \leqslant q(x)$ *on* $[\mu, \nu]$.

We need a simple observation for the proof of the theorem.

LEMMA 4.2. *For* $n \geqslant 1$,

(i) $\beta_n > 0$

(ii) $\beta_{n+1} = \dfrac{1}{\nu-\mu}\left(\beta_n - \dfrac{e^{\mu}}{n!}\right)$.

Proof. These properties are obvious from the definition. □

Proof of Theorem 4.1. Let

$$g_n(x) = q_n(x) - e^x = \sum_{j=0}^{n-1} \frac{e^{\mu}}{j!}(x-\mu)^j + \beta_n (x-\mu)^n - e^x. \tag{4.2}$$

Note $g_n^{(j)}(\mu) = 0$ for $0 \leqslant j \leqslant n-1$ and by Lemma 4.2 (ii), $g_n^{(n)}(\mu) = n!\beta_n - e^{\mu} = n!(\nu-\mu)\beta_{n+1} > 0$. Hence $g_n(x)$ has n-multiple zero at $x = \mu$ and it is increasing on a small neighborhood of $x = \mu$. Also note $g_n(\nu) = 0$ by the definition of β_n. Invoking Proposition 2.2, these are all the zeros of $g_n(x)$. Therefore we conclude that $g_n(x) \geqslant 0$ or $e^x \leqslant q_n(x)$ on $[\mu, \nu]$. Let

$$h_n(x) = \frac{q_n(x) - q_{n+1}(x)}{(x-\mu)^n} = \beta_n - \frac{e^{\mu}}{n!} - \beta_{n+1}(x-\mu).$$

Then $h_n'(x) = -\beta_{n+1} < 0$, $h_n(\mu) = \beta_n - e^{\mu}/n! = (\nu-\mu)\beta_{n+1} > 0$ and $h_n(\nu) = \beta_n - e^{\mu}/n! - \beta_{n+1}(\nu-\mu) = 0$ which means that $h_n(x)$ is decreasing to zero on $[\mu,\nu]$. So, $h_n(x) \geqslant 0$ or $q_{n+1}(x) \leqslant q_n(x)$ on $[\mu,\nu]$. Now it remains to show the optimality of the polynomials. Suppose $q(x)$ is a polynomial of degree n such that $e^x \leqslant q(x) \leqslant q_{n-1}(x)$ on $[\mu,\nu]$. Then $0 \leqslant q_{n-1}(x) - q(x) \leqslant q_{n-1}(x) - e^x = g_{n-1}(x)$ on $[\mu,\nu]$. Let $f(x) = q_{n-1}(x) - q(x)$. We know that $g_{n-1}(x)$ has $(n-1)$-multiple zero at $x=\mu$ and a simple zero at $x=\nu$. By Proposition 2.5, $f(x)$ also has $(n-1)$-multiple zero at $x=\mu$ and at least a simple zero at $x=\nu$ and $f^{(n-1)}(\mu) \leqslant g_{n-1}^{(n-1)}(\mu)$. Since $\deg(f(x)) = n$, $x=\nu$ must be a simple zero. Hence $f(x) = \beta(x-\mu)^{n-1}(x-\nu)$ for some real β and by Lemma 4.2 (ii), $(n-1)!\beta(\mu-\nu) \leqslant (n-1)!(\nu-\mu)\beta_n$ which implies $\beta+\beta_n \geqslant 0$. Then for $x \in [\mu,\nu]$,

$$\begin{aligned} q(x) - q_n(x) &= q_{n-1}(x) - q_n(x) - f(x) \\ &= (x-\mu)^{n-1}\left(\beta_{n-1} - \frac{e^{\mu}}{(n-1)!} - \beta_n(x-\mu) - \beta(x-\nu)\right) \\ &= (x-\mu)^{n-1}\left(\beta_n(\nu-x) - \beta(x-\nu)\right) \\ &= (x-\mu)^{n-1}(\nu-x)(\beta+\beta_n) \geqslant 0. \quad \square \end{aligned}$$

5. A size comparison

Let $u=-\nu$, $v=-\mu$ through this section. According to Theorem 3.4 and Theorem 4.1, $\frac{1}{p_n(-x)}$ and $q_n(x)$ are both upper bounds for e^x on $[\mu,\nu]$. Then one may ask which bound is the more precise. The following theorem tells us that there is no uniform inequality between them but shows an interesting result.

THEOREM 5.1. *For all sufficiently large integers* n,

(i) *The equation* $q_n(x) - \dfrac{1}{p_n(-x)} = 0$ *has only one root* λ_n *on* (μ,ν) *and*

$$q_n(x) - \frac{1}{p_n(-x)} \begin{cases} \leqslant 0, & \text{if} \quad x \in [\mu,\lambda_n], \\ \geqslant 0, & \text{if} \quad x \in [\lambda_n,\nu]. \end{cases}$$

(ii) $\lim\limits_{n\to\infty} \lambda_n = \dfrac{\mu+\nu}{2}$

Roughly speaking, the theorem exposes that $q_n(x)$ is the sharper bound on the first half of the interval $[\mu,\nu]$ and $\frac{1}{p_n(-x)}$ is the sharper bound on the second half. The proof of the theorem is quite complicated and technical. First, we wish to establish another expression for the number α_n defined by (3.2) whose proof needs the lemma below.

LEMMA 5.2. *For integers* n,k *with* $0 \leqslant k < n$,

(i) $\sum_{j=0}^{k}(-1)^j\binom{n}{k-j} = \binom{n-1}{k}$

(ii) $\sum_{j=0}^{k}(-1)^j(j+1)\binom{n+1}{k-j} = \binom{n-1}{k}$.

Proof. We use the induction on k. The equations are trivial for $k=0$. Next

$$\sum_{j=0}^{k}(-1)^j\binom{n}{k-j} = \binom{n}{k} - \sum_{j=0}^{k-1}(-1)^j\binom{n}{k-1-j} = \binom{n}{k} - \binom{n-1}{k-1} = \binom{n-1}{k}$$

where the second equality holds by the induction hypothesis. Similarly, by the induction hypothesis for (ii) and by (i)

$$\begin{aligned}
&\sum_{j=0}^{k}(-1)^j(j+1)\binom{n+1}{k-j} = \binom{n+1}{k} - \sum_{j=0}^{k-1}(-1)^j(j+2)\binom{n+1}{k-1-j} \\
&= \binom{n+1}{k} - \binom{n-1}{k-1} - \sum_{j=0}^{k-1}(-1)^j\binom{n+1}{k-1-j} \\
&= \binom{n+1}{k} - \binom{n-1}{k-1} - \binom{n}{k-1} = \binom{n}{k} - \binom{n-1}{k-1} = \binom{n-1}{k}. \quad \square
\end{aligned}$$

PROPOSITION 5.3. *For* $n \geqslant 2$, *the numbers* α_n *defined by (3.2) satisfy (with* $u=-\nu$ *and* $v=-\mu$*)*

$$\alpha_n = e^{\nu-\mu}\sum_{j=0}^{\infty}\frac{1}{(n+j)!}\binom{n+j-2}{j}(\mu-\nu)^j = \sum_{j=0}^{\infty}\frac{j+1}{(n+j)!}(\nu-\mu)^j.$$

Proof. Note that by (3.2)

$$\begin{aligned}
\alpha_n &= \frac{n-1+\mu-\nu}{(\nu-\mu)^n}\left(\sum_{j=0}^{n-2}\frac{(\nu-\mu)^j}{j!} - e^{\nu-\mu}\right) + \frac{1}{(n-2)!\,(\nu-\mu)} \\
&= \frac{n-1+\mu-\nu}{(\nu-\mu)^n}\left(-\sum_{j=n-1}^{\infty}\frac{(\nu-\mu)^j}{j!}\right) + \frac{1}{(n-2)!\,(\nu-\mu)} \\
&= \left(\frac{1-n}{\nu-\mu}+1\right)\sum_{j=0}^{\infty}\frac{(\nu-\mu)^j}{(n+j-1)!} + \frac{1}{(n-2)!\,(\nu-\mu)} \\
&= \left(1+\frac{n-1}{\mu-\nu}\right)e^{\nu-\mu}\sum_{j=0}^{\infty}\frac{(\mu-\nu)^j}{j!}\sum_{j=0}^{\infty}\frac{(-1)^j(\mu-\nu)^j}{(n+j-1)!} - \frac{1}{(n-2)!\,(\mu-\nu)}.
\end{aligned}$$

Using Lemma 5.2 (i), it is easy to see that

$$\sum_{j=0}^{\infty}\frac{(\mu-\nu)^j}{j!}\sum_{j=0}^{\infty}\frac{(-1)^j(\mu-\nu)^j}{(n+j-1)!}=\sum_{j=0}^{\infty}\frac{1}{(n+j-1)!}\binom{n+j-2}{j}(\mu-\nu)^j. \tag{5.1}$$

Thus we obtain

$$\begin{aligned}\frac{\alpha_n}{e^{\nu-\mu}} &= \left(1+\frac{n-1}{\mu-\nu}\right)\sum_{j=0}^{\infty}\frac{1}{(n+j-1)!}\binom{n+j-2}{j}(\mu-\nu)^j \\ &\quad -\frac{1}{(n-2)!\,(\mu-\nu)}\sum_{j=0}^{\infty}\frac{(\mu-\nu)^j}{j!} \\ &= \sum_{j=0}^{\infty}\frac{1}{(n+j-1)!}\binom{n+j-2}{j}(\mu-\nu)^j+\frac{n-1}{\mu-\nu}\frac{1}{(n-1)!} \\ &\quad +(n-1)\sum_{j=0}^{\infty}\frac{1}{(n+j)!}\binom{n+j-1}{j+1}(\mu-\nu)^j \\ &\quad -\frac{1}{(n-2)!(\mu-\nu)}-\frac{1}{(n-2)!}\sum_{j=0}^{\infty}\frac{(\mu-\nu)^j}{(j+1)!} \\ &= \sum_{j=0}^{\infty}\frac{1}{(n+j)!}\binom{n+j-2}{j}(\mu-\nu)^j.\end{aligned}$$

Also by using Lemma 5.2 (ii), one can show that

$$\begin{aligned}\alpha_n &= e^{\nu-\mu}\sum_{j=0}^{\infty}\frac{1}{(n+j)!}\binom{n+j-2}{j}(\mu-\nu)^j \\ &= \sum_{j=0}^{\infty}\frac{(\nu-\mu)^j}{j!}\sum_{j=0}^{\infty}\frac{(-1)^j}{(n+j)!}\binom{n+j-2}{j}(\nu-\mu)^j=\sum_{j=0}^{\infty}\frac{j+1}{(n+j)!}(\nu-\mu)^j. \quad \square\end{aligned}$$

DEFINITION 5.4. *For integers* $n \geqslant 2$, *we define*

$$\begin{aligned}\gamma_n &= \sum_{j=0}^{\infty}\frac{1}{(n+j)!}\binom{n+j-1}{j}(\mu-\nu)^j, \\ \delta_n &= e^{\mu-\nu}\alpha_n=\sum_{j=0}^{\infty}\frac{1}{(n+j)!}\binom{n+j-2}{j}(\mu-\nu)^j.\end{aligned}$$

LEMMA 5.5. *For integers* $n \geqslant 2$,

(i) $\gamma_n = e^{\mu-\nu}\sum_{j=0}^{\infty}\frac{(\nu-\mu)^j}{(n+j)!} > 0$

(ii) $\delta_n-\gamma_n=(\nu-\mu)\delta_{n+1}$

(iii) $\dfrac{e^{\mu-\nu}}{n!} \leqslant \delta_n \leqslant \dfrac{e^{\nu-\mu}}{n!}$.

Proof. (i) Using Lemma 5.2 again, we see that

$$e^{\mu-\nu}\sum_{j=0}^{\infty}\frac{(\nu-\mu)^j}{(n+j)!}=\sum_{j=0}^{\infty}\frac{(\mu-\nu)^j}{j!}\sum_{j=0}^{\infty}\frac{(-1)^j(\mu-\nu)^j}{(n+j)!}$$
$$=\sum_{j=0}^{\infty}\frac{1}{(n+j)!}\binom{n+j-1}{j}(\mu-\nu)^j=\gamma_n.$$

(ii) This assertion is clear because $\binom{n+j-1}{j}-\binom{n+j-2}{j}=\binom{n+j-2}{j-1}$.
(iii) Note that

$$\delta_n\leqslant\sum_{j=0}^{\infty}\frac{(\nu-\mu)^j}{(n+j)(n+j-1)j!(n-2)!}\leqslant\frac{1}{n!}\sum_{j=0}^{\infty}\frac{(\nu-\mu)^j}{j!}=\frac{e^{\nu-\mu}}{n!}$$

and using α_n defined by (3.2), we have

$$\frac{\delta_n}{e^{\mu-\nu}}=\alpha_n=\frac{n-1-\nu+\mu}{(\nu-\mu)^n}\left(\sum_{j=0}^{n-2}\frac{(\nu-\mu)^j}{j!}-e^{\nu-\mu}\right)+\frac{1}{(n-2)!(\nu-\mu)}$$
$$=\frac{1}{(n-2)!(\nu-\mu)}-\frac{n-1-\nu+\mu}{(\nu-\mu)^n}\sum_{j=n-1}^{\infty}\frac{(\nu-\mu)^j}{j!}.$$

Here $\sum_{j=n-1}^{\infty}\frac{(\nu-\mu)^j}{j!}\geqslant\frac{(\nu-\mu)^{n-1}}{(n-1)!}$ and if $n>\nu-\mu$, then

$$\sum_{j=n-1}^{\infty}\frac{(\nu-\mu)^j}{j!}=\frac{(\nu-\mu)^{n-1}}{(n-1)!}\left(1+\frac{\nu-\mu}{n}+\frac{(\nu-\mu)^2}{n(n+1)}+\cdots\right)$$
$$\leqslant\frac{(\nu-\mu)^{n-1}}{(n-1)!}\sum_{j=0}^{\infty}\left(\frac{\nu-\mu}{n}\right)^j=\frac{n(\nu-\mu)^{n-1}}{(n-1)!(n-\nu+\mu)}.$$

Thus if $n-1<\nu-\mu$ then

$$\frac{\delta_n}{e^{\mu-\nu}}\geqslant\frac{1}{(n-2)!(\nu-\mu)}+\frac{\nu-\mu-n+1}{(\nu-\mu)^n}\cdot\frac{(\nu-\mu)^{n-1}}{(n-1)!}=\frac{1}{(n-1)!}\geqslant\frac{1}{n!}$$

and if $n-1\geqslant\nu-\mu$ then

$$\frac{\delta_n}{e^{\mu-\nu}}\geqslant\frac{1}{(n-2)!(\nu-\mu)}-\frac{n-1-\nu+\mu}{(\nu-\mu)^n}\cdot\frac{n(\nu-\mu)^{n-1}}{(n-1)!(n-\nu+\mu)}$$
$$=\frac{1}{(n-1)!(n-\nu+\mu)}\geqslant\frac{1}{n!}. \quad \square \tag{5.2}$$

PROPOSITION 5.6. *For any real number* x,

$$q_n(x) - e^x = e^\nu (x-\mu)^n (\nu - x) \sum_{j=0}^{\infty} \gamma_{n+j+1} (x-\mu)^j, \tag{5.3}$$

$$e^x - p_n(x) = e^u (x-v)^2 (x-u)^{n-1} \sum_{j=0}^{\infty} \alpha_{n+j+1} (x-u)^j \tag{5.4}$$

and

$$0 < q_n(x) - e^x \leqslant \frac{e^x}{(n+1)!} (x-\mu)^n (\nu - x), \quad \textit{for} \quad \mu < x < \nu, \tag{5.5}$$

$$0 < e^x - p_n(x) \leqslant \frac{e^{x+2(v-u)}}{(n+1)!} (x-u)^{n-1} (x-v)^2, \quad \textit{for} \quad x > u. \tag{5.6}$$

Proof. First, observe that

$$\begin{aligned}
q_n(x) - e^x &= \sum_{j=0}^{n-1} \frac{e^\mu}{j!} (x-\mu)^j + \beta_n (x-\mu)^n - \sum_{j=0}^{\infty} \frac{e^\mu}{j!} (x-\mu)^j \\
&= e^\mu (x-\mu)^n \sum_{j=1}^{\infty} \frac{1}{(n+j)!} \left((\nu-\mu)^j - (x-\mu)^j \right) \\
&= e^\mu (x-\mu)^n (\nu - x) \sum_{j=1}^{\infty} \frac{1}{(n+j)!} \sum_{k=0}^{j-1} (\nu-\mu)^{j-1-k} (x-\mu)^k \\
&= e^\mu (x-\mu)^n (\nu - x) \sum_{j=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(\nu-\mu)^k}{(n+j+1+k)!} \right) (x-\mu)^j \\
&= e^\nu (x-\mu)^n (\nu - x) \sum_{j=0}^{\infty} \gamma_{n+j+1} (x-\mu)^j
\end{aligned}$$

which gives equation (5.3). Since by definition 5.4 and Lemma 5.5 (i)

$$0 < \gamma_n = \sum_{j=0}^{\infty} \frac{1}{j!(n-1)!(n+j)} (\mu - \nu)^j \leqslant \frac{1}{n!} \sum_{j=0}^{\infty} \frac{(\mu-\nu)^j}{j!} = \frac{1}{n!} e^{\mu - \nu},$$

we have

$$\begin{aligned}
0 < q_n(x) - e^x &= e^\nu (x-\mu)^n (\nu - x) \sum_{j=0}^{\infty} \gamma_{n+j+1} (x-\mu)^j \\
&\leqslant e^\mu (x-\mu)^n (\nu - x) \sum_{j=0}^{\infty} \frac{(x-\mu)^j}{(n+j+1)!} \\
&\leqslant e^\mu (x-\mu)^n (\nu - x) \frac{1}{(n+1)!} \sum_{j=0}^{\infty} \frac{(x-\mu)^j}{j!} \\
&= \frac{e^x}{(n+1)!} (x-\mu)^n (\nu - x)
\end{aligned}$$

for $\mu < x < \nu$. On the other hand, note that

$$\begin{aligned}
e^x &= e^v + e^v(x-v) + e^v(x-v)^2 \sum_{j=2}^{\infty} \frac{(x-v)^{j-2}}{j!} \\
&= e^v + e^v(x-v) + e^v(x-v)^2 \sum_{j=2}^{\infty} \frac{1}{j!} \sum_{k=0}^{j-2} \binom{j-2}{k} (u-v)^{j-2-k}(x-u)^k \\
&= e^v + e^v(x-v) \\
&\quad + e^u(x-v)^2 \sum_{j=2}^{\infty} \left(e^{v-u} \sum_{k=0}^{\infty} \frac{1}{(j+k)!} \binom{j-2+k}{k} (u-v)^k \right) (x-u)^{j-2} \\
&= e^v + e^v(x-v) + e^u(x-v)^2 \sum_{j=2}^{\infty} \alpha_j (x-u)^{j-2}
\end{aligned}$$

which implies (5.4) by definition (3.1) of $p_n(x)$. Again by definition 5.4 and Lemma 5.5 (iii) with $\nu = -u, \mu = -v$, we know that $\alpha_n \leqslant \frac{e^{2(v-u)}}{n!}$. Therefore

$$\begin{aligned}
e^x - p_n(x) &= e^u(x-v)^2(x-u)^{n-1} \sum_{j=0}^{\infty} \alpha_{n+j+1}(x-u)^j \\
&\leqslant e^u(x-v)^2(x-u)^{n-1} e^{2(v-u)} \sum_{j=0}^{\infty} \frac{(x-u)^j}{(n+j+1)!} \\
&\leqslant \frac{e^{x+2(v-u)}}{(n+1)!}(x-u)^{n-1}(x-v)^2
\end{aligned}$$

for $x > u$. $\square$

The proof of Theorem 5.1 will be presented step by step. By equation (5.4) with $-x$ and $u = -\nu, v = -\mu$,

$$\begin{aligned}
e^{-x} - p_n(-x) &= e^{-\nu}(x-\mu)^2(\nu-x)^{n-1} \sum_{j=0}^{\infty} \alpha_{n+j+1}(\nu-x)^j \\
&= e^{-\mu}(x-\mu)^2(\nu-x)^{n-1} \sum_{j=0}^{\infty} \delta_{n+j+1}(\nu-x)^j
\end{aligned}$$

on $[\mu, \nu]$. Therefore

$$\begin{aligned}
q_n(x) - \frac{1}{p_n(-x)} &= q_n(x) - e^x + e^x - \frac{1}{p_n(-x)} \\
&= q_n(x) - e^x - \frac{e^x}{p_n(-x)}(e^{-x} - p_n(-x))
\end{aligned}$$

$$
\begin{aligned}
&= e^{\nu}(x-\mu)^n(\nu-x)\sum_{j=0}^{\infty}\gamma_{n+j+1}(x-\mu)^j \\
&\quad - \frac{e^{x-\mu}}{p_n(-x)}(x-\mu)^2(\nu-x)^{n-1}\sum_{j=0}^{\infty}\delta_{n+j+1}(\nu-x)^j \\
&= (x-\mu)^2(\nu-x)T(x)
\end{aligned}
$$

where

$$
\begin{aligned}
T(x) &= (x-\mu)^{n-2}f(x) - (\nu-x)^{n-2}g(x), \\
f(x) &= e^{\nu}\sum_{j=0}^{\infty}\gamma_{n+j+1}(x-\mu)^j, \\
g(x) &= \frac{e^{x-\mu}}{p_n(-x)}\sum_{j=0}^{\infty}\delta_{n+j+1}(\nu-x)^j.
\end{aligned}
$$

And let

$$
\begin{aligned}
g_1(x) &= \sum_{j=0}^{\infty}\delta_{n+j+1}(\nu-x)^j, \\
h(x) &= (\nu-x)^{n-2}g(x) = \frac{1-e^x p_n(-x)}{(x-\mu)^2(\nu-x)\,p_n(-x)}.
\end{aligned}
$$

Then $g(x) = \frac{e^{x-\mu}}{p_n(-x)}g_1(x)$ and $e^{-x} - p_n(-x) = e^{-\mu}(x-\mu)^2(\nu-x)^{n-1}g_1(x)$.

We assume that n is sufficiently large. Then $f(x) > 0$, $g(x) > 0$ on $[\mu,\nu]$ and $T(\mu) = -(\nu-\mu)^{n-2}g(\mu) < 0$, $T(\nu) = (\nu-\mu)^{n-2}f(\nu) > 0$. Hence the first assertion of the theorem would follow if we show that $T(x)$ is increasing on $[\mu,\nu]$. But, to see that $T(x)$ is increasing, it is enough to show that $h(x) = (\nu-x)^{n-2}g(x)$ is decreasing because $(x-\mu)^{n-2}f(x)$ is clearly increasing.

LEMMA 5.7. *$g_1(x) > |g_1'(x)|$ on $[\mu,\nu]$.*

Proof. Note that $|g_1'(x)| = \sum_{j=0}^{\infty}(j+1)\delta_{n+j+2}(\nu-x)^j$. Invoking Proposition 5.3,

$$
\begin{aligned}
&e^{\nu-\mu}(\delta_{n+j+1} - (j+1)\delta_{n+j+2}) \\
&= \alpha_{n+j+1} - (j+1)\alpha_{n+j+2} \\
&= \sum_{i=0}^{\infty}\frac{i+1}{(n+i+j+1)!}(\nu-\mu)^i - \sum_{i=0}^{\infty}\frac{(i+1)(j+1)}{(n+i+j+2)!}(\nu-\mu)^i \\
&= \sum_{i=0}^{\infty}\frac{(i+1)(n+i+1)}{(n+i+j+2)!}(\nu-\mu)^i > 0.
\end{aligned}
$$

Then $g_1(x) > |g_1'(x)|$ is obvious now. $\square$

LEMMA 5.8. *If n is sufficiently large, then*

(i) $p_n'(-x)>0$ *and* $g'(x)>0$ *on* $[\mu,\nu]$

(ii) $$\frac{e^{-x}-p_n'(-x)}{e^{-\mu}(x-\mu)(\nu-x)^{n-2}g_1(x)} \leqslant (x-\mu)(\nu-x)+(n-1)(x-\mu)-2(\nu-x)$$

(iii) $$\frac{p_n'(-x)-p_n(-x)}{e^{-\mu}(x-\mu)(\nu-x)^{n-2}g_1(x)} \leqslant (x-\mu)(\nu-x)+2(\nu-x)-(n-1)(x-\mu).$$

Proof. Note that

$$\begin{aligned}\frac{e^{-x}-p_n(-x)}{e^{-\mu}} &= (x-\mu)^2(\nu-x)^{n-1}g_1(x),\\ \frac{p_n'(-x)-e^{-x}}{e^{-\mu}} &= 2(x-\mu)(\nu-x)^{n-1}g_1(x)\\ &\quad -(n-1)(x-\mu)^2(\nu-x)^{n-2}g_1(x)+(x-\mu)^2(\nu-x)^{n-1}g_1'(x).\end{aligned}$$

Since $g_1'(x)$ is negative, by deleting the last term of the second equation and adding them side by side, we obtain (iii). And (ii) follows if we replace $g_1'(x)$ by $-g_1(x)$. Finally, by Lemma 5.5 (iii),

$$\begin{aligned}g_1(x) &= \sum_{j=0}^{\infty}\delta_{n+j+1}(\nu-x)^j \leqslant \sum_{j=0}^{\infty}\frac{e^{\nu-\mu}}{(n+j+1)!}(\nu-x)^j\\ &\leqslant \sum_{j=0}^{\infty}\frac{e^{\nu-\mu}}{(n+1)!j!}(\nu-x)^j = \frac{e^{\nu-\mu}}{(n+1)!}e^{\nu-x} \leqslant \frac{e^{2(\nu-\mu)}}{(n+1)!}\end{aligned} \tag{5.7}$$

on $[\mu,\nu]$. Therefore for sufficiently large n,

$$\begin{aligned}p_n'(-x) &= e^{-x}+2e^{-\mu}(x-\mu)(\nu-x)^{n-1}g_1(x)\\ &\quad -e^{-\mu}(n-1)(x-\mu)^2(\nu-x)^{n-2}g_1(x)+e^{-\mu}(x-\mu)^2(\nu-x)^{n-1}g_1'(x)\\ &\geqslant e^{-\nu}-e^{-\mu}(n+2)(\nu-\mu)^n\frac{e^{2(\nu-\mu)}}{(n+1)!}>0.\end{aligned}$$

By this result and Lemma 5.7, we conclude that

$$g'(x)=\frac{p_n(-x)(g_1(x)+g_1'(x))+g_1(x)p_n'(-x)}{e^{\mu-x}p_n^2(-x)}$$

is positive. □

LEMMA 5.9. *If n is sufficiently large and $x\in[\mu,\nu]$, then*

$$\frac{1}{p_n(-x)} \leqslant e^x+2e^{4\nu-3\mu}\frac{(\nu-\mu)^{n+1}}{(n+1)!}.$$

Proof. Using (5.7), we see that

$$e^{-x} - p_n(-x) = e^{-\mu}(x-\mu)^2(\nu-x)^{n-1}g_1(x) \leqslant e^{-\mu}(\nu-\mu)^{n+1}\frac{e^{2(\nu-\mu)}}{(n+1)!}.$$

Therefore

$$\frac{1}{p_n(-x)} = \frac{e^x}{1-e^x(e^{-x}-p_n(-x))} \leqslant \frac{e^x}{1-e^{3(\nu-\mu)}\frac{(\nu-\mu)^{n+1}}{(n+1)!}}$$
$$= e^x(1+\varepsilon+\varepsilon^2+\cdots) \leqslant e^x(1+2\varepsilon) \leqslant e^x + 2e^\nu\varepsilon.$$

where $\varepsilon = e^{3(\nu-\mu)}\frac{(\nu-\mu)^{n+1}}{(n+1)!}$. □

Proof of Theorem 5.1. (i) It suffices to show that $h(x) = (\nu-x)^{n-2}g(x)$ is decreasing on $[\mu,\nu]$. Note that $h'(x) = A(x)/e^{-x}(x-\mu)^3(\nu-x)^2 p_n^2(-x)$ where

$$\begin{aligned} A(x) &= (3x-\mu-2\nu)p_n(-x)(e^{-x}-p_n(-x)) \\ &\quad + (x-\mu)(\nu-x)(e^{-x}p_n'(-x)-p_n^2(-x)) \\ &= (3x-\mu-2\nu)p_n(-x)(e^{-x}-p_n(-x)) \\ &\quad + (x-\mu)(\nu-x)(e^{-x}+p_n(-x))(p_n'(-x)-p_n(-x)) \\ &\quad + (x-\mu)(\nu-x)p_n(-x)(e^{-x}-p_n'(-x)). \end{aligned}$$

Applying the inequalities in Lemma 5.8, we have

$$\begin{aligned} &\frac{A(x)}{e^{-\mu}(x-\mu)^2(\nu-x)^{n-1}g_1(x)} \\ &\leqslant p_n(-x)(3x-\mu-2\nu) \\ &\quad + (e^{-x}+p_n(-x))((x-\mu)(\nu-x)+2(\nu-x)-(n-1)(x-\mu)) \\ &\quad + p_n(-x)((x-\mu)(\nu-x)+(n-1)(x-\mu)-2(\nu-x)) \\ &= p_n(-x)Q_1(x)+e^{-x}Q_2(x) =: B(x) \end{aligned}$$

where

$$\begin{aligned} Q_1(x) &= 3x-\mu-2\nu+2(x-\mu)(\nu-x) \\ Q_2(x) &= (x-\mu)(\nu-x)+2(\nu-x)-(n-1)(x-\mu). \end{aligned}$$

Because $p_n(-x) \leqslant e^{-x}$, if $Q_1(x) \geqslant 0$, then

$$B(x) \leqslant e^{-x}(Q_1(x)+Q_2(x)) = e^{-x}(x-\mu)(3(\nu-x)+2-n) < 0$$

for sufficiently large n. Now assume that $Q_1(x) < 0$. Since

$$e^{-x} - p_n(-x) = e^{-\mu}(x-\mu)^2(\nu-x)^{n-1}g_1(x) \leqslant e^{-\mu}(x-\mu)(\nu-\mu)^n\frac{e^{2(\nu-\mu)}}{(n+1)!},$$

we have

$$
\begin{aligned}
B(x) &= p_n(-x)Q_1(x) + e^{-x}Q_2(x) \\
&\leqslant \left(e^{-x} - (x-\mu)\frac{e^{2\nu}(\nu-\mu)^n}{e^{3\mu}(n+1)!} \right) Q_1(x) + e^{-x}Q_2(x) \\
&= e^{-x}(Q_1(x)+Q_2(x)) - (x-\mu)\frac{e^{2\nu}(\nu-\mu)^n}{e^{3\mu}(n+1)!}Q_1(x) \\
&= e^{-x}(x-\mu)\left(3(\nu-x)+2-n-e^x\frac{e^{2\nu}(\nu-\mu)^n}{e^{3\mu}(n+1)!}Q_1(x) \right) < 0
\end{aligned}
$$

on the open interval (μ,ν). Therefore $h'(x)<0$ and so $h(x)$ is decreasing.

(ii) Now $T(x)$ has the unique zero λ_n on (μ,ν). Let $\tau_0 = \frac{\mu+\nu}{2}$ and $\eta_0 = \frac{\nu-\mu}{2}$ so that $\tau_0-\mu = \nu-\tau_0 = \eta_0$. Then, since $\frac{1}{p_n(-x)} \geqslant e^x$,

$$
\begin{aligned}
\frac{T(\tau_0)}{\eta_0^{n-2}} &= e^\nu \sum_{j=0}^{\infty} \gamma_{n+j+1}\cdot\eta_0^j - e^{\tau_0-\mu}\frac{1}{p_n(-\tau_0)}\sum_{j=0}^{\infty}\delta_{n+j+1}\cdot\eta_0^j \\
&\leqslant e^\nu \sum_{j=0}^{\infty}(\gamma_{n+j+1}-\delta_{n+j+1})\,\eta_0^j \\
&= -e^\nu(\nu-\mu)\sum_{j=0}^{\infty}\delta_{n+j+2}\cdot\eta_0^j < 0
\end{aligned}
$$

which means that $T(\tau_0)<0$. Let

$$
\tau_1 = \tau_1(n) = \mu + \frac{\nu-\mu}{2}\left(1+\frac{\nu-\mu}{e^{2(\mu-\nu)}(n+2)-\nu+\mu}\right)^{\frac{1}{n-2}}.
$$

Note that $\tau_1 > \frac{\mu+\nu}{2} = \tau_0$ and $\lim_{n\to\infty}\tau_1(n) = \tau_0$. Thus, if $T(\tau_1)>0$, then $\tau_0 \leqslant \lambda_n \leqslant \tau_1(n)$ and we can conclude that $\lim_{n\to\infty}\lambda_n = \tau_0 = \frac{\mu+\nu}{2}$. It remains to show that $T(\tau_1)>0$. Because $T(x) = (x-\mu)^{n-2}f(x)-h(x)$ and $f(x)$ is increasing while $h(x)$ is decreasing, we know that $T(\tau_1) \geqslant (\tau_1-\mu)^{n-2}f(\tau_0)-h(\tau_0)$. Let $\eta_1 = \tau_1-\mu$ and $\overline{\eta} = \eta_1/\eta_0$ so that $\eta_0<\eta_1$. Then

$$
T(\tau_1) \geqslant \eta_1^{n-2}e^\nu\sum_{j=0}^{\infty}\gamma_{n+j+1}\cdot\eta_0^j - \eta_0^{n-2}\frac{e^{\tau_0-\mu}}{p_n(-\tau_0)}\sum_{j=0}^{\infty}\delta_{n+j+1}\cdot\eta_0^j. \tag{5.8}
$$

Combining (5.8) with Lemma 5.9, we have

$$
\begin{aligned}
T(\tau_1) \geqslant{}& \eta_1^{n-2}e^\nu\sum_{j=0}^{\infty}\gamma_{n+j+1}\cdot\eta_0^j - \eta_0^{n-2}e^\nu\sum_{j=0}^{\infty}\delta_{n+j+1}\cdot\eta_0^j \\
&-2\eta_0^{n-2}e^{\tau_0+4(\nu-\mu)}\frac{(\nu-\mu)^{n+1}}{(n+1)!}\sum_{j=0}^{\infty}\delta_{n+j+1}\cdot\eta_0^j
\end{aligned}
$$

Since (5.7) implies that $\sum_{j=0}^{\infty} \delta_{n+j+1} \cdot \eta_0^j \leqslant \frac{e^{\nu-\mu+\eta_0}}{(n+1)!}$, we obtain

$$\frac{T(\tau_1)}{\eta_0^{n-2} e^{\nu}} \geqslant \sum_{j=0}^{\infty} \left(\overline{\eta}^{n-2} \gamma_{n+j+1} - \delta_{n+j+1}\right) \eta_0^j - \frac{2e^{5(\nu-\mu)}(\nu-\mu)^{n+1}}{(n+1)!(n+1)!}.$$

Now we complete the proof of the positivity of $T(\tau_1)$ by showing that

$$\overline{\eta}^{n-2} \gamma_{n+j+1} - \delta_{n+j+1} \geqslant 0 \tag{5.9}$$

for all $j \geqslant 0$ and by showing that the first term of the infinite sum

$$\overline{\eta}^{n-2} \gamma_{n+1} - \delta_{n+1} > \frac{2e^{5(\nu-\mu)}(\nu-\mu)^{n+1}}{(n+1)!(n+1)!}. \tag{5.10}$$

By the definition of τ_1, we see that

$$\overline{\eta}^{n-2} = 1 + \frac{\nu-\mu}{e^{2(\mu-\nu)}(n+2) - \nu + \mu}$$

and by Lemma 5.5, we have

$$\begin{aligned}\frac{\delta_{n+j+1}}{\gamma_{n+j+1}} &= 1 + (\nu-\mu)\frac{\delta_{n+j+2}}{\gamma_{n+j+1}} = 1 + (\nu-\mu)\frac{\delta_{n+j+2}}{\delta_{n+j+1} - (\nu-\mu)\delta_{n+j+2}} \\ &\leqslant 1 + (\nu-\mu)\frac{e^{\nu-\mu}/(n+j+2)!}{e^{\mu-\nu}/(n+j+1)! - (\nu-\mu)e^{\nu-\mu}/(n+j+2)!} \\ &\leqslant 1 + \frac{\nu-\mu}{e^{2(\mu-\nu)}(n+2) - \nu + \mu} = \overline{\eta}^{n-2}\end{aligned}$$

from which (5.9) follows. Finally, by inequality (5.2), if $n-1 \geqslant \nu - \mu$ then $\delta_n \geqslant e^{\mu-\nu}/((n-1)!(n-\nu+\mu))$. Therefore

$$\begin{aligned}&\overline{\eta}^{n-2} \gamma_{n+1} - \delta_{n+1} \\ &= \left(1 + \frac{\nu-\mu}{e^{2(\mu-\nu)}(n+2) - \nu + \mu}\right)(\delta_{n+1} - (\nu-\mu)\delta_{n+2}) - \delta_{n+1} \\ &= \frac{\nu-\mu}{e^{2(\mu-\nu)}(n+2) - \nu + \mu}\delta_{n+1} - \left(1 + \frac{\nu-\mu}{e^{2(\mu-\nu)}(n+2) - \nu + \mu}\right)(\nu-\mu)\delta_{n+2} \\ &\geqslant \frac{\nu-\mu}{e^{2(\mu-\nu)}(n+2) - \nu + \mu} \cdot \frac{e^{\mu-\nu}}{n!(n+1-\nu+\mu)} \\ &\quad - \left(1 + \frac{\nu-\mu}{e^{2(\mu-\nu)}(n+2) - \nu + \mu}\right)(\nu-\mu)\frac{e^{\nu-\mu}}{(n+2)!} \\ &= \frac{e^{\mu-\nu}(\nu-\mu)^2}{\left(e^{2(\mu-\nu)}(n+2) - \nu + \mu\right)(n+1)!(n+1-\nu+\mu)}\end{aligned}$$

which is obviously greater than the right side of (5.10) for all sufficiently large n. □

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