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MATRIX YOUNG NUMERICAL RADIUS INEQUALITIES

ABBAS SALEMI AND ALEMEH SHEIKHHOSSEINI

Dedicated to Professor Rajendra Bhatia
on the occasion of his sixtieth birthday

(*Communicated by I. Perić*)

Abstract. In the present paper, we show that if $A \in M_n(\mathbb{C})$ is a non scalar strictly positive matrix such that $1 \in \sigma(A)$, and $p > q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then there exists $X \in M_n(\mathbb{C})$ such that $\omega(AXA) > \omega(\frac{1}{p}A^pX + \frac{1}{q}XA^q)$. Moreover, several numerical radius inequalities are presented for Hilbert space operators. In particular, we prove that if $p \geqslant q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then $\omega^r(A^*XB) \leqslant \left\| \frac{1}{p}(A^*|X^*|A)^{\frac{rp}{2}} + \frac{1}{q}(B^*|X|B)^{\frac{rq}{2}} \right\|$, for all $A, B, X \in \mathscr{B}(\mathbf{H})$ and $r \geqslant \frac{2}{q}$.

1. Introduction

Let $\mathscr{B}(\mathbf{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $\mathbf{H}$ with inner product $\langle .,. \rangle$. For $A \in \mathscr{B}(\mathbf{H})$, the usual operator norm of A is defined by

$$\|A\| = \sup\{\|Ax\| : x \in \mathbf{H}, \|x\| = 1\},$$

where $\|x\| = \langle x, x \rangle^{1/2}$.

The numerical range of A is defined as the set of complex numbers given by

$$W(A) = \{\langle Ax, x \rangle : x \in \mathbf{H}, \|x\| = 1\},$$

and the numerical radius of A is given by

$$\omega(A) = \sup\{|\langle Ax, x \rangle| : x \in \mathbf{H}, \|x\| = 1\}.$$

We recall the following results that were proved in [7].

LEMMA 1. *Let $A \in \mathscr{B}(\mathbf{H})$ and let $\omega(.)$ be the numerical radius. Then*

(i) *$\omega(.)$ is a norm on $\mathscr{B}(\mathbf{H})$,*

(ii) *$\omega(.)$ is not a unitarily invariant norm,*

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(iii) $\omega(.)$ *is not submultiplicative,*

(iv) $\omega(UAU^*) = \omega(A)$, *for all unitary operators* U,

(v) $\omega(A^k) \leqslant \omega(A)^k,\ k = 1,2,3,\dots$ *(power inequality)*

(vi) $\frac{1}{2}\|A\| \leqslant \omega(A) \leqslant \|A\|$.

For positive real numbers a,b, the classical Young inequality says that if $p,q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}. \tag{1}$$

Replacing a,b by their squares, we could write (1) in the form

$$(ab)^2 \leqslant \frac{a^{2p}}{p} + \frac{b^{2q}}{q}. \tag{2}$$

The authors interested to replace the numbers a,b by positive operators A,B. But there are some difficulties, for example if A and B are positive operators, the operator AB is not positive in general. One way to get around this is to compare not the operators themselves but to the singular values or norms of them.

In section 2, we establish that, if $p > q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $A \in M_n(\mathbb{C})$ be a non scalar strictly positive matrix such that $1 \in \sigma(A)$, then there exists $X \in M_n(\mathbb{C})$ such that $\omega(AXA) > \omega(\frac{1}{p}A^pX + \frac{1}{q}XA^q)$.

In section 3, by using the Young inequality we shall extend some known inequalities.

Throughout the paper we use the term positive for a positive semidefinite matrix, and strictly positive for a positive definite matrix. Also we use the notation $A \geqslant 0$ to mean that A is positive, $A > 0$ to mean it is strictly positive, $|||A|||$ to denote an arbitrary unitarily invariant norm of A and let J be an square matrix with entries equal to 1.

2. Matrix Young inequality

Bhatia and Kittaneh in 1990 [5] established a matrix mean inequality as follows:

$$|||A^*B||| \leqslant \frac{1}{2}|||A^*A + B^*B|||, \tag{3}$$

for matrices $A,B \in M_n(\mathbb{C})$.

In [4] a generalization of (3) was proved, for all $X \in M_n(\mathbb{C})$,

$$|||A^*XB||| \leqslant \frac{1}{2}|||AA^*X + XBB^*|||. \tag{4}$$

Ando in 1995 [2] obtained a matrix Young inequality:

$$|||AB||| \leqslant \left|\left|\left|\frac{A^p}{p} + \frac{B^q}{q}\right|\right|\right|, \tag{5}$$

for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and positive matrices A,B. Also, in [1], the author shows that $|||AXB|||\leqslant|||\frac{1}{p}A^pX+\frac{1}{q}XB^q|||$ does not hold in general.

Now, we consider the inequalities (3) and (5) with the numerical radius norm. We know that $\omega(A)=\|A\|$ if (but not only if) A is normal and by Lemma 1(vi), we obtain the following:

PROPOSITION 1. *If A,B are $n\times n$ matrices, then*

$$\omega(A^*B)\leqslant\frac{1}{2}\omega(A^*A+B^*B). \tag{6}$$

Moreover, if A and B are positive matrices, then

$$\omega(AB)\leqslant\omega\left(\frac{A^p}{p}+\frac{B^q}{q}\right). \tag{7}$$

For $A,B\in M_n(\mathbb{C})$, denoted by $A\circ B$ the Schur (Hadamard) product of A and B, that is, the entrywise product. The linear operator S_A on $M_n(\mathbb{C})$, called the Schur multiplier operator, defined by $S_A(X):=A\circ X$, $\forall X\in M_n(\mathbb{C})$. The induced norm of S_A with respect to numerical radius norm will be denoted by

$$\|S_A\|_\omega=\sup_{X\neq 0}\frac{\omega(S_A(X))}{\omega(X)}=\sup_{X\neq 0}\frac{\omega(A\circ X)}{\omega(X)}.$$

Ando and Okubo in 1991, [3, Theorem 2 and Corollary 4], proved the following theorem:

THEOREM 1. *For $A\in M_n(\mathbb{C})$ the following assertions are equivalent:*

(i) $\|S_A\|_\omega\leqslant 1$.

(ii) *There is $0\leqslant R\in M_n(\mathbb{C})$ such that $\begin{pmatrix} R & A\\ A^* & R\end{pmatrix}\geqslant 0$ and $R\circ I\leqslant I$.*

Moreover, if $A=(a_{ij})$ be an $n\times n$ positive matrix,

(iii) $\|S_A\|_\omega=\max a_{ii}$.

Now, in the following theorem, we show that, if $A,B\geqslant 0$, there exists $X\in M_n(\mathbb{C})$ such that $\omega(AXB)\nleqslant\omega(\frac{1}{p}A^pX+\frac{1}{q}XB^q)$.

THEOREM 2. *Let $p>q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$ and let $A\in M_n(\mathbb{C})$ be a non scalar strictly positive matrix such that $1\in\sigma(A)$. Then there exists $X\in M_n(\mathbb{C})$ such that*

$$\omega(AXA)>\omega\left(\frac{1}{p}A^pX+\frac{1}{q}XA^q\right). \tag{8}$$

Proof. Without loss of generality, assume that $A = \operatorname{diag}(a_1, a_2, a_3, \ldots, a_n)$ where $a_1 = 1$, $a_2 \neq 1$.

It is easy to show that

$$\frac{1}{p} + \frac{a_2^q}{q} \neq \frac{1}{q} + \frac{a_2^p}{p}. \tag{9}$$

Assume if possible that

$$\omega(AXA) \leqslant \omega\left(\frac{1}{p}A^pX + \frac{1}{q}XA^q\right), \quad \forall X \in M_n(\mathbb{C}). \tag{10}$$

Now, let $C = (c_{ij})$ and $E = (e_{ij})$ be $n \times n$ matrices, where $c_{ij} = \frac{a_i^p}{p} + \frac{a_j^q}{q}$, and $e_{ij} = a_i a_j$. Then we rewrite (10) in the following form

$$\omega(E \circ X) \leqslant \omega(C \circ X), \quad \forall X \in M_n(\mathbb{C}). \tag{11}$$

Let D be the entry wise inverse of C $(C \circ D = J)$. We replace X by $(D \circ X)$ in (11), then

$$\omega((E \circ D) \circ X) \leqslant \omega(X), \quad \forall X \in M_n(\mathbb{C}). \tag{12}$$

Let $F := (E \circ D) = (f_{ij})$. Then by (12), we obtain that $\omega(F \circ X) \leqslant \omega(X)$ for all $X \in M_n(\mathbb{C})$ and hence,

$$\|S_F\|_\omega = \sup_{X \neq 0} \frac{\omega(F \circ X)}{\omega(X)} \leqslant 1. \tag{13}$$

By Theorem 1, there exists an $n \times n$ matrix $X = (x_{ij}) \geqslant 0$ with $0 \leqslant x_{ii} \leqslant 1$, $(1 \leqslant i \leqslant n)$, such that $\begin{pmatrix} X & F \\ F^* & X \end{pmatrix} \geqslant 0$. By considering $\tilde{X} := (\tilde{x}_{ij})$ such that $\tilde{x}_{ij} = x_{ij}$ if $i \neq j$ and $\tilde{x}_{ii} = 1$, we obtain that

$$\begin{pmatrix} \tilde{X} & F \\ F^* & \tilde{X} \end{pmatrix} \geqslant 0.$$

Since, any principal submatrix of the above matrix is positive, we have

$$\begin{pmatrix} 1 & x & 1 & f_{12} \\ \bar{x} & 1 & f_{21} & f_{22} \\ 1 & f_{21} & 1 & x \\ f_{12} & f_{22} & \bar{x} & 1 \end{pmatrix} \geqslant 0, \quad \textit{where } x := \tilde{x}_{12} = x_{12}.$$

By using the Schur complement Theorem [4, Theorem 1.3.3], we obtain that

$$\begin{pmatrix} 1 & f_{21} & f_{22} \\ f_{21} & 1 & x \\ f_{22} & \bar{x} & 1 \end{pmatrix} - \begin{pmatrix} \bar{x} \\ 1 \\ f_{12} \end{pmatrix} \begin{pmatrix} x & 1 & f_{12} \end{pmatrix} = \begin{pmatrix} 1-|x|^2 & f_{21}-\bar{x} & f_{22}-\bar{x}f_{12} \\ f_{21}-x & 0 & x-f_{12} \\ f_{22}-xf_{12} & \bar{x}-f_{12} & 1-f_{12}^2 \end{pmatrix} \geqslant 0.$$

Since the leading principle submatrices of the above matrix is positive, we have $f_{21} - x = x - f_{12} = 0$ and hence $f_{12} = f_{21}$. But by (9) we know that $f_{12} \neq f_{21}$, a contradiction. $\square$

As in the proof of Theorem 2, the relations (10), (11), (12), and (13) are equivalent, so we state the following corollary.

COROLLARY 1. *Let $p \geqslant q > 1$ such that $\frac{1}{p}+\frac{1}{q}=1$ and $A \in M_n(\mathbb{C})$ be a strictly positive matrix, and let F be as in the proof of Theorem 2. Then the following are equivalent:*

(i) $\|S_F\|_\omega \leqslant 1$.

(ii) $\omega(AXA) \leqslant \omega(\frac{1}{p}A^pX+\frac{1}{q}XA^q) \quad (X \in M_n(\mathbb{C}))$.

The following corollary shows that, we cannot remove the condition, $p \neq q$, in Theorem 2.

COROLLARY 2. *Let $A \in M_2(\mathbb{C})$ be a non scalar strictly positive matrix such that $1 \in \sigma(A)$. Then for all $X \in M_2(\mathbb{C})$*

$$\omega(AXA) \leqslant \omega\left(\frac{1}{2}A^2X+\frac{1}{2}XA^2\right).$$

Proof. Without loss of generality, assume that $A=\operatorname{diag}(1,a)$ where $a \neq 1$, $a>0$. Let F be as in the proof of Theorem 2. Then $F=\begin{pmatrix} 1 & \frac{2a}{1+a^2} \\ \frac{2a}{1+a^2} & 1 \end{pmatrix} \geqslant 0$ and hence, by Theorem 1 (iii), we have $\|S_F\|_\omega = 1$. Now by Corollary 1, we obtain that $\omega(AXA) \leqslant \omega(\frac{1}{2}A^2X+\frac{1}{2}XA^2)$. □

The following example shows that, we cannot remove the condition $1 \in \sigma(A)$, in the Theorem 2.

EXAMPLE 1. Let $a=1.2255$, $b=0.7$, $p=5$, and let $A=\operatorname{diag}(a,b)$. Then we have $\frac{a^p}{p}+\frac{b^q}{q}=\frac{a^q}{q}+\frac{b^p}{p}$.

Now, let $F=(f_{ij})=\begin{pmatrix} \frac{a^2}{\frac{a^p}{p}+\frac{a^q}{q}} & \frac{ab}{\frac{a^p}{p}+\frac{b^q}{q}} \\ \frac{ab}{\frac{a^p}{p}+\frac{b^q}{q}} & \frac{b^2}{\frac{b^p}{p}+\frac{b^q}{q}} \end{pmatrix}$ and $X=\begin{pmatrix} 1 & f_{12} \\ f_{21} & 1 \end{pmatrix}$. It is readily seen that $X \geqslant 0$ and $\begin{pmatrix} X & F \\ F^* & X \end{pmatrix} \geqslant 0$. By Theorem 1 we obtain that $\|S_F\|_\omega \leqslant 1$. By using Corollary 1, $\omega(AXA) \leqslant \omega(\frac{1}{p}A^pX+\frac{1}{q}XA^q)$.

3. Numerical radius inequalities

Let $A \in \mathscr{B}(\mathbf{H})$. we know that $\frac{1}{2}\|A\| \leqslant \omega(A) \leqslant \|A\|$ (see Lemma 1(vi)). These inequalities was improved in [8, 10] as follows:

$$\omega(A) \leqslant \frac{1}{2}\||A|+|A^*|\| \leqslant \frac{1}{2}(\|A\|+\|A^2\|^{\frac{1}{2}}), \tag{14}$$

$$\frac{1}{4}\|A^*A+AA^*\| \leqslant \omega^2(A) \leqslant \frac{1}{2}\|A^*A+AA^*\|, \tag{15}$$

where, $|A| := (A^*A)^{\frac{1}{2}}$ is the absolute value of A.

The second inequality in (15) have been established in [6] for the numerical radius norm of operators. It has been shown that if $A,B \in \mathscr{B}(\mathbf{H})$, for $0<\alpha<1$ and $r \geqslant 1$, then

$$\omega^r(A) \leqslant \frac{1}{2}\left\| |A|^{2r\alpha} + |A^*|^{2r(1-\alpha)} \right\|, \tag{16}$$

$$\omega^r(A+B) \leqslant 2^{r-2}\left\| |A|^{2r\alpha} + |A^*|^{2r(1-\alpha)} + |B|^{2r\alpha} + |B^*|^{2r(1-\alpha)} \right\|. \tag{17}$$

In 2005, Kittaneh extended the above inequalities as follows:

THEOREM 3. [10, Theorem 2] *If* $A,B,C,D,S,T \in \mathscr{B}(\mathbf{H})$, *then for all* $\alpha \in (0,1)$,

$$\omega(ATB+CSD) \leqslant \frac{1}{2}\left(\left\| A|T^*|^{2(1-\alpha)}A^* + B^*|T|^{2(\alpha)}B + C|S^*|^{2(1-\alpha)}C^* + D^*|S|^{2(\alpha)}D \right\|\right). \tag{18}$$

In particular,

$$\omega(AB \pm BA) \leqslant \frac{1}{2}\|A^*A + AA^* + B^*B + BB^*\|. \tag{19}$$

In 2009, Shebrawi and Albadawi extended the inequality (18), in the following form:

THEOREM 4. [11, Theorem 2.5] *Let* $A_i, B_i, X_i \in \mathscr{B}(\mathbf{H})(i=1,2,\ldots,n)$, *and let* f *and* g *be nonnegative functions on* $[0,\infty)$ *which are continuous and satisfy the relation* $f(t)g(t)=t$ *for all* $t \in [0,\infty)$. *Then for all* $r \geqslant 1$,

$$\omega^r\left(\sum_{i=1}^n A_i^* X_i B_i\right) \leqslant \frac{n^{r-1}}{2}\left(\left\| \sum_{i=1}^n \left([A_i^* g^2(|X_i^*|)A_i]^r + [B_i^* f^2(|X_i|)B_i]^r\right) \right\|\right). \tag{20}$$

We shall establish a numerical radius inequality that generalizes (20) and consequently, generalize (16), (17), (18), (19). To prove our results, we need the following basic lemmas.

LEMMA 2. [9, Theorem 1] *Let* A *be an operator in* $\mathscr{B}(\mathbf{H})$, *and let* f *and* g *be nonnegative functions on* $[0,\infty)$ *which are continuous and satisfy the relation* $f(t)g(t)=t$ *for all* $t \in [0,\infty)$. *Then for all* x *and* y *in* $\mathbf{H}$.

$$|\langle Ax,y\rangle| \leqslant \|f(|A|)x\|\|g(|A^*|)y\|. \tag{21}$$

The following lemma is a consequence of the spectral theorem for positive operators and Jensen's inequality (see, e.g., [9]).

LEMMA 3. *Let* A *be a positive operator in* $\mathscr{B}(\mathbf{H})$ *and let* $x \in \mathbf{H}$ *be any unit vector. Then for all* $r \geqslant 1$,

$$\langle Ax,x\rangle^r \leqslant \langle A^r x,x\rangle. \tag{22}$$

Now, we state the following theorem which generalize (20).

THEOREM 5. *Let* $A_i, B_i, X_i \in \mathscr{B}(\mathbf{H})$ $(i = 1, 2, \dots, n)$, *and let* f *and* g *be nonnegative functions on* $[0, \infty)$ *which are continuous and satisfy the relation* $f(t)g(t) = t$ *for all* $t \in [0, \infty)$. *If* $p \geqslant q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then for all* $r \geqslant \frac{2}{q}$

$$\omega^r(\sum_{i=1}^{n} A_i^* X_i B_i) \leqslant n^{r-1} \left\| \sum_{i=1}^{n} \frac{1}{p}(B_i^* f^2(|X_i|)B_i)^{\frac{rp}{2}} + \frac{1}{q}(A_i^* g^2(|X_i^*|)A_i)^{\frac{rq}{2}} \right\|. \tag{23}$$

Proof. For every unit vector $x \in \mathbf{H}$, we have

$$\begin{aligned}
|\langle (\sum_{i=1}^{n} A_i^* X_i B_i)x, x\rangle|^r &\leqslant (\sum_{i=1}^{n} |\langle X_i B_i x, A_i x\rangle|)^r \\
&\leqslant (\sum_{i=1}^{n} \langle f^2(|X_i|)B_i x, B_i x\rangle^{\frac{1}{2}} \langle g^2(|X_i^*|)A_i x, A_i x\rangle^{\frac{1}{2}})^r \quad \text{(by (21))} \\
&\leqslant n^{r-1} \sum_{i=1}^{n} \langle f^2(|X_i|)B_i x, B_i x\rangle^{\frac{r}{2}} \langle g^2(|X_i^*|)A_i x, A_i x\rangle^{\frac{r}{2}} \\
&= n^{r-1} \sum_{i=1}^{n} \langle B_i^* f^2(|X_i|)B_i x, x\rangle^{\frac{r}{2}} \langle A_i^* g^2(|X_i^*|)A_i x, x\rangle^{\frac{r}{2}} \\
&\leqslant n^{r-1} \sum_{i=1}^{n} \left(\frac{1}{p} \langle (B_i^* f^2(|X_i|)B_i)^{\frac{rp}{2}} x, x\rangle + \frac{1}{q} \langle (A_i^* g^2(|X_i^*|)A_i)^{\frac{rq}{2}} x, x\rangle \right) \quad \text{(by (1) and (22))} \\
&= n^{r-1} \left\langle \sum_{i=1}^{n} \left(\frac{1}{p}(B_i^* f^2(|X_i|)B_i)^{\frac{rp}{2}} + \frac{1}{q}(A_i^* g^2(|X_i^*|)A_i)^{\frac{rq}{2}} \right) x, x \right\rangle.
\end{aligned}$$

Now, the result follows by taking the supremum over all unit vectors in $\mathbf{H}$. □

The inequality (23) includes several numerical radius inequalities as special cases. Samples of inequalities are demonstrated in the following.

COROLLARY 3. *Let* $A_i \in \mathscr{B}(\mathbf{H})$ $(i = 1, 2, \dots, n)$. *If* $p \geqslant q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, $r \geqslant \frac{2}{q}$ *and* $0 < \alpha < 1$, *then*

$$\omega^r(\sum_{i=1}^{n} A_i) \leqslant n^{r-1} \left\| \sum_{i=1}^{n} \left(\frac{1}{p}|A_i|^{rp\alpha} + \frac{1}{q}|A_i^*|^{rq(1-\alpha)} \right) \right\|. \tag{24}$$

Moreover, if $A_1 = A_2 = \cdots = A_n = A$, *then*

$$\omega^r(A) \leqslant \left\| \frac{1}{p}|A|^{rp\alpha} + \frac{1}{q}|A^*|^{rq(1-\alpha)} \right\|. \tag{25}$$

The inequalities (24) and (25) are generalizations of the inequalities (17) and (16), respectively. Now, we state the following numerical radius inequalities for products of operators.

COROLLARY 4. *Let* $A_i, B_i, \in \mathscr{B}(\mathbf{H})$ $(i = 1, 2, \ldots, n)$. *If* $p \geqslant q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $r \geqslant \frac{2}{q}$, *then*

$$\omega^r(\sum_{i=1}^{n} A_i^* B_i) \leqslant n^{r-1} \left\| \sum_{i=1}^{n} \left(\frac{1}{p} |B_i|^{rp} + \frac{1}{q} |A_i|^{rq} \right) \right\|. \tag{26}$$

In particular, if $n = 1$, *then*

$$\omega^r(A^* B) \leqslant \left\| \frac{1}{p} |B|^{rp} + \frac{1}{q} |A|^{rq} \right\|. \tag{27}$$

REMARK 1. By considering $n = 1$ in Theorem 5, we obtain the following

$$\omega^r(A^* X B) \leqslant \left\| \frac{1}{p} (B^* |X| B)^{\frac{rp}{2}} + \frac{1}{q} (A^* |X^*| A)^{\frac{rq}{2}} \right\|. \tag{28}$$

Also, by Lemma 1, for all $A, B, X \in \mathscr{B}(\mathbf{H})$, we obtain the following inequalities:

$$\omega(A^* X B) \leqslant \frac{1}{2} \omega(A^* |X| A + B^* |X| B), \tag{29}$$

$$\omega((A^* X B)^2) \leqslant \omega \left(\frac{1}{p} (A^* |X^*| A)^p + \frac{1}{q} (B^* |X| B)^q \right). \tag{30}$$

The inequalities (29) and in (30) are generalizations of the inequalities (6) and (2), respectively.

Finally, by using the inequality (28), we obtain an upper bound for the numerical radius of A^k. For all $k \geqslant 2$ and $r \geqslant \frac{2}{q}$

$$\omega^r(A^k) \leqslant \left\| \frac{1}{p} (A^* |A^{k-2}| A)^{\frac{rp}{2}} + \frac{1}{q} (A |(A^*)^{k-2}| A^*)^{\frac{rq}{2}} \right\|. \tag{31}$$

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Abbas Salemi
Department of Mathematics
Shahid Bahonar University of Kerman
Kerman, Iran
and
The SBUK Center of Excellence in Linear Algebra and Optimization
Kerman, Iran
e-mail: salemi@mail.uk.ac.ir

Alemeh Sheikhhosseini
Department of Mathematics
Shahid Bahonar University of Kerman
Kerman, Iran
and
The SBUK Center of Excellence in Linear Algebra and Optimization
Kerman, Iran
e-mail: hosseini8560@gmail.com