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MAXIMUM PRINCIPLES FOR A CLASS OF LINEAR ELLIPTIC EQUATIONS OF EVEN ORDER

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Abstract. In this paper we define several types of functions on the solution to a class of linear elliptic equations of even order. We establish that these functions satisfy a classical maximum principle. As a consequence we obtain uniqueness results and bounds on various quantities of interest.

1. Introduction

The technique of applying classical maximum principles to certain functions defined on the solution of a differential equation of order $\geqslant 2$ is well-known (see the book of Sperb [15] or the survey paper [10] and the vast literature cited therein). Utilizing this technique one can, for example, deduce results about the solution itself or perhaps obtain apriori bounds on various gradients of the solution. An interesting boundary value problem on which this technique has been applied concerns a class of differential equations known as m-metaharmonic equations

$$\begin{cases} \Delta^m u - a_{m-1}(x)\Delta^{m-1}u + a_{m-2}(x)\Delta^{m-2}u + \cdots + (-1)^m a_0(x)u = f & \text{in } \Omega \\ u = g_1,\ \Delta u = g_2, \ldots,\ \Delta^{m-1}u = g_m & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $a_i, i = 0,\ldots,m-1$, are bounded and the domain $\Omega \subset \mathbb{R}^n, n \geqslant 2$ is bounded.

In [6], Dunninger develops maximum principle results for this boundary value problem, on an arbitrary domain Ω, in the case where $m = 2$, $n \geqslant 2$, $a_1 = 0$, $a_0 \equiv$ constant $\geqslant 0$. Schaefer [12, 13] obtains such results on domains Ω in which $\partial\Omega$ has positive curvature, and assumes that $m = 2,3$ and $n = 2$, respectively. The author requires that $a_1 = 0$ in [12] and $a_2, a_1 \geqslant 0$, $a_0 > 0$ in [13].

To contrast, Danet [4] treats the nonconstant coefficient case for an arbitrary domain Ω. Here, uniqueness results are obtained both for $m = 2$ and $m = 3$, where $n \geqslant 2$. For the case $m = 3$, $n = 2$, S. Goyal and V. Goyal in [7] also deduce uniqueness results for (1.1) for arbitrary domains. Later, in [5], the author studies (1.1) for domains with boundaries that have positive curvature in the case $m = 4$, $n = 2$. Herein, the assumptions $a_3, a_2, a_1 \geqslant 0$, $a_0 \geqslant 0$ are made on the coefficients.

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Maximum principle results for the case where the differential equation in (1.1) has order > 8 can be found in [16] and in [2], for domains which are planes or lines. Another such result appears in [1], in which the authors pose an interesting open problem: If $f = 0$ in Ω, $g_2 = \cdots = g_m = 0$ on $\partial\Omega$, $m \geqslant 3$, $n \geqslant 2$, $a_{m-1} = \cdots = a_1 = 0$ in Ω do all the solutions of (1.1) satisfy the maximum principle

$$\max_{\overline{\Omega}} |u| \leqslant C \max_{\partial\Omega} |u|, \tag{1.2}$$

where $C > 1$ is a constant? This problem, as it turns out, can be solved when Ω is a class C^2 domain ([14]).

In this paper we study the differential equation in (1.1) from two approaches. We primarily utilize functions containing the squares of terms of the form $(\Delta^i u)^2$. We show that these functions satisfy a generalized maximum principle (see next section for an explicit definition) and deduce the uniqueness of classical solutions classical solutions, (i.e., solutions in $C^{2m}(\Omega) \cap C^{2m-2}(\overline{\Omega})$, $m \geqslant 2$), of the boundary value problem (1.1). We also show that (1.2) is valid even if Ω is an arbitrary domain. We then take a new, second approach to the m-metaharmonic equation in (1.1), in that we utilize functions containing the squares of certain higher order gradient terms. These functions are, partially, generalizations of functions used by L. E. Payne in [9]. In this work the author deduces maximum principles results for the semilinear equation $\Delta^2 u = f(u)$ by employing functions containing the square of the second gradient of the solution u. In our paper, this alternative class of functions will yield integral bounds on certain gradient terms for some interesting boundary conditions, in which, for example both $\Delta^i u = 0$ and $\frac{\partial(\Delta^i u)}{\partial n} = 0$ for certain values of i. Both classes of functions are used to analyze the principle equation in (1.1) as well as the specific cases $m = 2, 3$, and $m = 4$.

2. Assumptions and notation

Throughout this paper we shall assume that $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$ is a bounded domain, $m \geqslant 2$ and the coefficients a_i, $i = 0, \ldots, m-1$ are bounded in Ω. Also we shall suppose that $a_0 \not\equiv 0$. Additionally, diamΩ will denote the diameter of Ω.

Partial differentiation is denoted by commas. Furthermore, we identify the products of the first, second, and third gradients of the functions v and w as follows

$$\nabla v \cdot \nabla w = v_{,i} w_{,i}, \qquad \nabla^2 v : \nabla^2 w = v_{,ij} w_{,ij}, \qquad \nabla^3 v \vdots \nabla^3 w = v_{,ijk} w_{,ijk}.$$

We denote the squares of the first, second, third, and fourth gradients of w by

$$|\nabla w|^2 = w_{,i} w_{,i}, \quad |\nabla^2 w|^2 = w_{,ij} w_{,ij}, \quad |\nabla^3 w|^2 = w_{ijk} w_{,ijk}, \quad |\nabla^4 w|^2 = w_{,ijkl} w_{,ijkl}.$$

For the sake of brevity, we shall say that a function Φ satisfies a generalized maximum principle in Ω, if either there exists a constant $k \in \mathbb{R}$ such that $\Phi \equiv k$ in Ω or Φ does not attain a nonnegative maximum in Ω.

In (1.1), for simplicity, we shall only consider the case when m is even, i.e., we shall study the equation

$$\Delta^m u - a_{m-1}(x)\Delta^{m-1}u + a_{m-2}(x)\Delta^{m-2}u - \cdots + a_0(x)u = 0 \quad \text{in } \Omega. \tag{2.1}$$

Similar results will hold if m is odd.

3. Some useful results

We now state two maximum principles that will be applied several times in the paper. The first result appears in [4].

THEOREM 3.1. *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy the inequality* $\mathrm{L}u \equiv \Delta u + \gamma(x)u \geqslant 0$ *in Ω, where $\gamma \geqslant 0$ in Ω. Suppose that*

$$\sup_{\Omega} \gamma < \frac{4n+4}{(\operatorname{diam}\Omega)^2} \tag{3.2}$$

holds.

Then, the function u/w_1 satisfies a generalized maximum principle in Ω. Here $w_1(x) = 1 - \alpha(x_1^2 + \cdots + x_n^2) \in C^\infty(\mathrm{I\!R}^n)$, $\alpha = \sup_\Omega \gamma/2n$.

If Ω lies in strip of width d and if we impose the restriction

$$\sup_{\Omega} \gamma < \frac{\pi^2}{d^2}, \tag{3.3}$$

we obtain that u/w_2 satisfies a generalized maximum principle in Ω. Here

$$w_2 = \cos\frac{\pi(2x_i - d)}{2(d+\varepsilon)} \prod_{j=1}^{n} \cosh(\varepsilon x_j) \in C^\infty(\overline{\Omega}),$$

for some $i \in \{1,\ldots,n\}$, where $\varepsilon > 0$ is small.

A similar maximum principle holds for more general operators

THEOREM 3.2. *Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy the inequality*

$$\mathrm{L}_1 u \equiv \Delta u + \sum_{i=1}^{n} \beta_i(x)\frac{\partial u}{\partial x_i} + \gamma(x)u \geqslant 0, \text{ where } \gamma \geqslant 0 \text{ in } \Omega.$$

(a) If there exists $i \in \{1,\ldots,n\}$ such that $\beta_i \geqslant 0$ in Ω and if

$$\sup_{\Omega} \gamma < \frac{4}{d^2 e^2}, \tag{3.4}$$

then, the function u/w_3 satisfies a generalized maximum principle in Ω. Here $w_3(x) = 1 - \beta e^{\alpha x_i} \in C^\infty(\mathrm{I\!R}^n)$ for some $i \in \{1,\ldots,n\}$, where $\beta = \sup_\Omega \gamma/\alpha^2$ and $\alpha > 0$ is a constant.

(b) Let $i_1,\ldots,i_n \in \{1,\ldots,n\}$ be distinct numbers. Suppose that one of the following conditions holds

(i) $\beta_k \geqslant 0$ *for all* $k = 1,\dots,n$ *in* Ω*; or*

(ii) there exist(s) $i_1,\dots,i_q$ $(1 \leqslant q \leqslant n-1)$ *such that* $\beta_{i_1},\dots,\beta_{i_q} \leqslant 0$ *in* Ω *and rest of coefficients* β_k *are nonnegative in* Ω*; or*

(iii) $\beta_k \leqslant 0$ *for all* $k = 1,\dots,n$ *in* Ω*;*

If in addition

$$\sup_{\Omega} \gamma \leqslant \frac{2(n-1)}{\operatorname{diam}\Omega(\operatorname{diam}\Omega+\delta)(\sqrt{n}+1)}, \tag{3.5}$$

where δ *is any positive constant, then, there exists a function* w_4 *(for more details see* [3]*) such that* u/w_4 *satisfies a generalized maximum principle in* Ω.

The proof of part (a) follows from Theorem 10, [11], p. 73 and the results on p. 73–74 in [11]. The proof of part (b) follows from Theorem 10, [11], p. 73 and Lemma 1, [3].

REMARKS.

1. A broad class of domains satisfy $\Omega \subset B_{\operatorname{diam}\Omega/2} = \{x \mid x_1^2+\cdots+x_n^2 < (\operatorname{diam}\Omega)^2/4\}$. For these domains $C(n,\operatorname{diam}\Omega) = (4n+4)/(\operatorname{diam}\Omega)^2$ (in Theorem 3.1) may be replaced by $C_1(n,\operatorname{diam}\Omega) = 8n/(\operatorname{diam}\Omega)^2$ (see [4]).

2. For some domains it is possible to improve the constant $C_1(n,\operatorname{diam}\Omega)$ (also see [4] for details).

4. Main results

THEOREM 4.1. *Let* u *be a classical solution of (2.1). Suppose that* $a_{m-1} = \ldots = a_1 = 0,\ a_0 > 0,\ \Delta a_0 \leqslant 0$ *in* Ω.

We define the functions P_1 *and* P_2

$$\mathrm{P}_1 = (\Delta^{m-1}u)^2 + (\Delta^{m-2}u)^2 + \cdots + u^2,$$

$$\mathrm{P}_2 = \frac{\mathrm{P}_1}{a_0}.$$

(a) If the inequality

$$\max\{1+\sup_{\Omega} a_0, 2\} + \sup_{\Omega} \frac{\Delta a_0}{a_0} \leqslant 0, \tag{4.6}$$

holds, then the function P_2 *attains its maximum value on* $\partial\Omega$.

(b) If we have

$$\max\{1+\sup_{\Omega} a_0, 2\} + \sup_{\Omega} \frac{\Delta a_0}{a_0} < \frac{4}{d^2 e^2} \tag{4.7}$$

and if there exists $i \in \{1,\dots,n\}$ *such that* $\frac{\partial}{\partial x_i}\left(\frac{1}{a_0}\right) \geqslant 0$ *in* Ω*, then the function* P_2/w_3 *satisfies a generalized maximum principle in* Ω.

(c) If the strict inequality

$$\max\{1+\sup_{\Omega} a_0, 2\} + \sup_{\Omega} \frac{\Delta a_0}{a_0} < \frac{2(n-1)}{\operatorname{diam}\Omega(\operatorname{diam}\Omega+\delta)(\sqrt{n}+1)}, \tag{4.8}$$

holds, where δ is any positive constant, if the coefficients $\beta_i = \frac{\partial}{\partial x_i}\left(\frac{1}{a_0}\right)$ satisfy one of conditions of part (b) of Theorem 3.2, then the function P_2/w_4 satisfies a generalized maximum principle in Ω.

Proof. (a) A computation shows that

$$\Delta \mathrm{P}_2 - 2a_0\nabla\left(\frac{1}{a_0}\right)\nabla \mathrm{P}_2 = -\frac{\mathrm{P}_1}{a_0^2}\Delta a_0 + \frac{1}{a_0}\Delta \mathrm{P}_1$$
$$\geqslant -\frac{\mathrm{P}_1}{a_0^2}\Delta a_0 - \frac{1}{a_0}\Big((1+a_0)(\Delta^{m-1}u)^2 + 2(\Delta^{m-2}u)^2$$
$$+ \ldots + 2(\Delta u)^2 + 2(1+a_0)u^2\Big).$$

Hence P_2 satisfies the differential inequality

$$\Delta \mathrm{P}_2 - 2a_0\nabla\left(\frac{1}{a_0}\right)\nabla \mathrm{P}_2 \geqslant -\frac{1}{a_0}\left(\left(1+a_0+\frac{\Delta a_0}{a_0}\right)(\Delta^{m-1}u)^2\right.$$
$$+\left(2+\frac{\Delta a_0}{a_0}\right)(\Delta^{m-2}u)^2 + \ldots$$
$$\left.+\left(2+\frac{\Delta a_0}{a_0}\right)(\Delta u)^2 + \left(1+a_0+\frac{\Delta a_0}{a_0}\right)u^2\right) \geqslant 0 \quad \text{in } \Omega.$$

As consequence of the classical maximum principle P_2 attains its maximum value on $\partial\Omega$. The proofs of (b) and (c) follow from the generalized maximum principle (Theorem 3.2). □

REMARKS.

1. A similar result (but weaker) to Theorem 4.1, case a). can be stated if we do not impose smoothness or sign conditions on the coefficient a_0. The function P_1/w_1 satisfies a generalized maximum principle in Ω if $a_{m-1} = \ldots = a_1 = 0$ in Ω (a_0 is of arbitrary sign in Ω) and if

$$\max\{1+\sup_{\Omega} a_0^2, 2\} < \frac{4n+4}{(\operatorname{diam}\Omega)^2}, \tag{4.9}$$

2. If $m = 2$ and $a_0 \geqslant 1$ the result of Theorem 4.1, case a). can be sharpened, i.e., a maximum principle for the the function $\mathrm{P}_1^* = ((\Delta u)^2 + u^2)/a_0 w_1$ holds if (4.6) is replaced by

$$\sup_{\Omega} a_0 - 1 + \sup_{\Omega}\frac{\Delta a_0}{a_0} \leqslant 0. \tag{4.10}$$

The next result allows a_0 to be large if a_{m-1} is large

THEOREM 4.2. *Let u be a classical solution of (2.1). We consider the function P_3*

$$\mathrm{P}_3 = (\Delta^{m-1}u)^2 + 2a_{m-2}(\Delta^{m-2}u)^2 + (\Delta^{m-3}u)^2 + \cdots + u^2.$$

Suppose that $a_{m-1},\ a_{m-2} > 0$ *and* $\Delta(1/a_{m-2}) \leqslant 0$ *in* Ω. *If*

$$\sup_{\Omega}\left\{\frac{a_0^2}{2a_{m-1}+1}\right\} < \frac{4n+4}{(\operatorname{diam}\Omega)^2}, \tag{4.11}$$

$$\frac{a_0^2}{2a_{m-1}} > \max\left\{1+\sup_{\Omega} a_1,\dots,1+\sup_{\Omega} a_{m-3}\right\}, \tag{4.12}$$

$$\frac{a_0^2}{2a_{m-1}+1} > \sup\left\{|a_1|+\ldots+|a_{m-3}|\right\}, \tag{4.13}$$

and

$$\left(\frac{a_0^2}{2a_{m-1}}+1\right)a_{m-2} > 1 \quad \text{in}\,\Omega \tag{4.14}$$

then, the function P_3/w_1 *satisfies a generalized maximum principle in* Ω.

The proof is similar to the proof of Theorem 4.1 and hence is omitted.

REMARK. The coefficient a_0 can be replaced by $a_{m-j},\ j=4,\dots,m-1$ if there exists a $j=4,\dots,m-1$ such that

$$\frac{a_{m-j}^2}{2a_{m-1}} > \max_{k=3,\dots,m}\left\{2+\sup_{\Omega} a_k\right\}, \tag{4.15}$$

$$\frac{a_{m-j}^2}{2a_{m-1}}+2 > \sup_{\Omega}\left\{|a_0|+\cdots|a_{m-j-1}|+|a_{m-j+1}|+\ldots+|a_{m-3}|\right\}, \tag{4.16}$$

and

$$\left(\frac{a_{m-j}^2}{2a_{m-2}}+1\right)a_{m-2} > 1 \quad \text{in}\,\Omega. \tag{4.17}$$

We now show that the uniqueness result and the maximum principle (1.2) holds.

THEOREM 4.3. *There is at most one classical solution of the boundary value problem (1.1) provided the coefficients* $a_{m-1},\dots,a_0$ *satisfy the conditions imposed in Theorem 4.1 or Theorem 4.2.*

Proof. Suppose that the hypotheses of Theorem 4.1 are satisfied. Define $u = u_1 - u_2$, where u_1 and u_2 are solutions of (1.1). Then u_1 and u_2 satisfy the equation (2.1) and

$$u = \Delta u = \cdots = \Delta^{m-1} u = 0 \quad \text{on}\ \partial\Omega. \tag{4.18}$$

Hence, by Theorem 3.1 either

(i) there exists a constant $k \in \mathbb{R}$ such that

$$\frac{\mathrm{P}_1}{w_1} \equiv k \quad \text{in}\,\Omega, \tag{4.19}$$

or

(ii)
$$\frac{\mathrm{P}_1}{w_1} \quad \text{does not attain a maximum in } \Omega.$$

Case (i) By continuity (4.19) holds in $\overline{\Omega}$. By the boundary conditions (4.18) we obtain $\mathrm{P}_1 = 0$ on $\partial\Omega$, i.e., $k = 0$. It follows that $\mathrm{P}_1 \equiv 0$ in Ω, which means $u \equiv 0$ in Ω. Hence $u_1 = u_2$ in Ω.

Case (ii) From
$$\max_{\overline{\Omega}} \frac{\mathrm{P}_1}{w_1} = \max_{\partial\Omega} \frac{\mathrm{P}_1}{w_1}$$
and (4.18) we get
$$0 \leqslant \max_{\overline{\Omega}} \frac{\mathrm{P}_1}{w_1} = 0,$$
i.e., $u_1 = u_2$ in Ω.

We can argue similarly if we are under the hypotheses of Theorem 4.2. □

THEOREM 4.4. *We consider the boundary value problem (1.1), where $f = 0$ in Ω and $g_2 = \cdots = g_m = 0$ on $\partial\Omega$. Then (1.2) holds for all solutions of (1.1) provided the coefficients $a_{m-1}, \ldots, a_0$ are subject to one of the conditions imposed in Theorem 4.1 or Theorem 4.2.*

Proof. Suppose that the hypotheses of Theorem 4.1 are satisfied. By Theorem 3.1 we see that
$$\frac{u^2}{w_1} \leqslant \max_{\partial\Omega} \frac{\mathrm{P}_1}{w_1} = \max_{\partial\Omega} \frac{u^2}{w_1} \leqslant \frac{(\max_{\partial\Omega}|u|)^2}{\min_{\partial\Omega} w_1} \quad \text{in } \Omega.$$
This inequality proves the desired result. □

REMARK. The boundary value problem
$$\begin{cases} \Delta^m u + 2^m u = 0 & \text{in } \Omega = (0,\pi)\times(0,\pi) \\ u = \Delta u = \cdots = \Delta^{m-1} u = 0 & \text{on } \partial\Omega, \end{cases}$$

has (at least) the solutions $u_1(x,y) \equiv 0$ and $u_2(x,y) = \sin x \sin y$ in Ω. This example shows that if we do not impose some restrictions on the coefficients $a_{m-1}, \ldots, a_0$, then the uniqueness result (Theorem 4.3) might be violated. To obtain a uniqueness result one must impose the restriction (4.9).

Our last result, as well as the preceeding ones, are valid for functions containing terms of the form $(\Delta^i u)^2$ and for boundary conditions of the form $\Delta^i u = 0$. Here, we state and prove a maximum principle result for the homogeneous equation $\Delta^m u = 0$ utilizing a different class of functions, in particular, functions containing the squares of certain second gradient terms. Now, we state our first result of this type

THEOREM 4.5. *Suppose that $u \in C^{2m+1}(\Omega) \cap C^{2m-1}(\overline{\Omega})$ is a solution of $\Delta^m u = 0$. Furthermore, let*
$$\mathrm{P}_4 = |\nabla^2(\Delta^{m-2}u)|^2 - \nabla(\Delta^{m-2}u)\cdot\nabla(\Delta^{m-1}u) - \frac{(4-n)}{2(n+2)}(\Delta^{m-1}u)^2. \tag{4.20}$$

Then P_4 takes its maximum value on $\partial\Omega$.

Proof. First we calculate

$$\begin{aligned}\Delta P_4 = &2|\nabla^3(\Delta^{m-2}u)|^2 + 2\nabla^2(\Delta^{m-2}u) : \nabla^2(\Delta^{m-1}u) \\ &-|\nabla(\Delta^{m-1}u)|^2 - \nabla^2(\Delta^{m-2}u) : \nabla^2(\Delta^{m-1}u) \\ &-\nabla^2(\Delta^{m-2}u) : \nabla^2(\Delta^{m-1}u) - \nabla(\Delta^{m-2}u)\cdot\nabla(\Delta^m u) \\ &-\frac{(4-n)}{(n+2)}|\nabla(\Delta^{m-1}u)|^2 - \frac{(4-n)}{(n+2)}\Delta^{m-1}u\Delta^m u.\end{aligned} \tag{4.21}$$

Upon simplifying (4.21) we see that

$$\Delta P_4 = 2|\nabla^3(\Delta^{m-2}u)|^2 - \frac{6}{(n+2)}|\nabla(\Delta^{m-1}u)|^2.$$

The proof now follows by substituting w for $\Delta^{m-2}u$ in the well-known inequality (see [9])

$$|\nabla^3 w|^2 \geqslant \frac{3}{n+2}|\nabla(\Delta w)|^2,$$

which holds for functions $w \in C^3(\Omega)$.

REMARK. We note that the smoothness conditions for u are more strict than in our previous results; this is due to the presence of a gradient terms in P_4. Secondly, when Dirchlet boundary conditions are present, one can deduce an apriori bound on $|\nabla(\Delta^{m-2}u)|^2$ using this result (see [9] for more details). Lastly, additional results using functions similiar to P_4 are proven in the next section of this paper.

5. The cases $m = 4, m = 3$ and $m = 2$

In this final section, we treat the particular cases $m = 4$, $m = 3$ and $m = 2$.

It is interesting to note that the following results cannot be deduced from the results of Section 3.

THEOREM 5.1. *Let u be a classical solution of (2.1), where $m = 4$. Assume that*

$$a_3 > 0 \quad in\ \Omega, \tag{5.22}$$

$$a_2 \geqslant 0 \quad in\ \Omega, \tag{5.23}$$

$$a_0 > 0, \quad \Delta(1/a_0) \leqslant 0 \quad in\ \Omega, \tag{5.24}$$

and

$$a_2 - a_0 - 1 > 0, \quad \Delta(1/(a_2 - a_0 - 1)) \leqslant 0 \quad in\ \Omega. \tag{5.25}$$

are satisfied.

We define the function

$$P_5 = \frac{1}{2}(\Delta^3 u + \Delta u)^2 + \frac{a_0}{2}(\Delta^2 u + u)^2 + \frac{a_2 - a_0 - 1}{2}(\Delta^2 u)^2 + \frac{a_2 - a_0 - 1}{2}(\Delta u)^2.$$

If

$$\sup_{\Omega} \frac{(a_1+a_3)^2}{a_3^2(a_2-a_0-1)} < \frac{8n+8}{(\operatorname{diam}\Omega)^2}, \tag{5.26}$$

then, the function P_5/w_1 *satisfies a generalized maximum principle in* Ω.

If $a_1 = 0$ *and* $a_3 \geqslant 0$ *in* Ω *a similar result holds for* P_5/w_1 *under the restriction*

$$\sup_{\Omega} \frac{a_3}{(a_2-a_0-1)} < \frac{8n+8}{(\operatorname{diam}\Omega)^2}. \tag{5.27}$$

If

$$4a_3^2 a_1 \geqslant (a_1+a_3)^2,\ a_1 > 0 \quad in\ \Omega, \tag{5.28}$$

(5.22), (5.24) and (5.25) are satisfied then the function P_5 *attains its maximum value on* $\partial\Omega$.

The proof is similar to the proof of Theorem 4.1 and hence is omitted.

With the aid of the above theorem we can establish a uniqueness result.

THEOREM 5.2. *Suppose that* $m = 4$ *and one of the following conditions hold:*

(a) $a_3, a_1 \geqslant 0$ *(*a_3, a_1 *constants),* $a_2 \geqslant 0$, $a_0 > 0$ *in* Ω, $n = 2$ *and the curvature of* $\partial\Omega$ *is strictly positive;*

(b) the hypotheses of Theorem 5.1.

Then there exists at most one classical solution of the boundary value problem (1.1).

Proof. (a) For a proof see Theorem 1, [5].

(b) The proof is achieved by arguing exactly as in Theorem 4.3. □

THEOREM 5.3. *Let* u *be a classical solution of (2.1), where* $m = 2$ *and* $a_1 \equiv$ *const.* > 0, $a_0 > 0$ *in* $\overline{\Omega}$.

Suppose that

$$\sup_{\Omega} \left(a_1 - \frac{1}{a_1} \left(\frac{a_0-1}{a_0} \right)^2 \right) < \frac{2n+2}{(\operatorname{diam}\Omega)^2}. \tag{5.29}$$

Let

$$P_6 = \frac{1}{2}(\Delta u - au)^2 + \frac{1}{2}(\Delta u)^2 + u^2.$$

Then the function P_6/w_1 *satisfies a generalized maximum principle in* Ω.

If

$$a_1^2 \geqslant \left(\frac{a_0-1}{a_0} \right)^2 \quad \text{in}\ \Omega, \tag{5.30}$$

then the function P_6 *attains its maximum value on* $\partial\Omega$ *(here the assumption (5.29) is not needed).*

The proof is similar to the proof of Theorem 4.1.

REMARK. A classical result ([1]) tells us that the boundary value problem

$$\begin{cases} \Delta^2 u - a_1(x)\Delta u + a_0(x)u = f & \text{in } \Omega \subset \mathbb{R}^n \\ u = g,\ \Delta u = h & \text{on } \partial\Omega, \end{cases} \tag{5.31}$$

has a unique solution if $a_1, a_0 > 0$ and if $\Delta a_0 < 0$ or $\Delta(1/a_0) < 0$ in Ω.

Theorem 5.3 tells us that if $a_1 \geqslant 1$ and $a_0 > 0$ then the boundary value problem (5.31) has a unique solution. We see that no smoothness restrictions are needed on the coefficient a_0.

To contrast the above results we now deduce some maximum principle results, using auxiliary functions containing the squares of certain gradients of order greater than two, for the cases $m = 3$ and $m = 4$.

THEOREM 5.4. *Let $u \in C^7(\Omega) \cap C^5(\overline{\Omega})$ be a solution to (1.1) for m=3. Define,*

$$\begin{aligned} \mathrm{P}_7 = & |\nabla^3 u|^2 + \frac{1}{2}\nabla u \cdot \nabla(\Delta^2 u) - \nabla^2(\Delta u) : (\nabla^2 u) + \beta(x)u^2 + (\Delta^2 u)^2 \\ & + \left[\frac{n-7}{2(n+5)} + \phi(x)\right] |\nabla(\Delta u)|^2 + |\nabla u|^2, \textit{where, } \beta(x), \phi(x) \in C^2(\bar{\Omega}). \end{aligned} \tag{5.32}$$

Furthermore, assume that

$$\beta \geqslant \frac{9n|\nabla a_1|^2}{32}, \quad a_2 \geqslant \frac{9}{2}na_1^2, \quad \frac{a_0}{4} \geqslant \max\left\{1 + \frac{a_2^2}{16}, \frac{3|\nabla a_2|^2}{16a_2}\right\}, \tag{5.33}$$

$$\frac{\Delta\beta}{4} \geqslant \max\left\{n\beta^2, \frac{3a_0^2}{a_2}, \frac{3|\nabla a_0|^2}{32\beta}, \frac{6|\nabla\beta|^2}{\beta}\right\}, \tag{5.34}$$

$$\frac{\Delta\phi}{3} \geqslant \max\left\{2, \frac{2|\nabla\phi|^2}{\phi}, \frac{[\gamma(x)]^2}{4}, \frac{3a_1^2}{8\beta}\right\}, \textit{ where } \gamma(x) = 2\phi(x) + \frac{3(n-3)}{2(n+5)}. \tag{5.35}$$

Then P_7 attains its maximum value on $\partial\Omega$.

Proof. By a straightforward calculation, we have

$$\begin{aligned} \Delta\mathrm{P}_7 = & 2|\nabla^4 u|^2 + \frac{1}{2}\nabla(\Delta u) \cdot \nabla(\Delta^2 u) + \frac{1}{2}\nabla u \cdot \nabla(\Delta^3 u) \\ & + \left[2\phi + \frac{n-7}{n+5}\right] \left[(\nabla^2(\Delta u) : \nabla^2(\Delta u) + \nabla(\Delta u) \cdot \nabla(\Delta^2 u))\right] \\ & + \Delta\phi|\nabla(\Delta u)|^2 + 2\nabla\phi \cdot \nabla(|\nabla(\Delta u)|^2) + 2\Delta^2 u\Delta^3 u + 2|\nabla^2 u|^2 \\ & + \Delta\beta u^2 + 2\nabla\beta \cdot \nabla(u^2) + 2\beta u\Delta u + 2\beta|\nabla u|^2 + 2\nabla u \cdot \nabla(\Delta u) \\ & + 2|\nabla(\Delta^2 u)|^2 - |\nabla^2(\Delta u)|^2. \end{aligned} \tag{5.36}$$

From the inequalities

$$|\nabla^4 u|^2 \geqslant \frac{6}{n+5}|\nabla^2(\Delta u)|^2, \quad |\nabla^2 u|^2 \geqslant \frac{1}{n}(\Delta u)^2 \text{ (see [8, 9])}$$

we deduce that

$$\begin{aligned}\Delta \mathrm{P}_7 \geqslant\ & \frac{1}{2}\nabla u\cdot a_0\nabla u+\gamma(x)\nabla(\Delta u)\cdot\nabla(\Delta^2 u)+2\phi|\nabla^2(\Delta u)|^2 \\ & +\frac{1}{2}\nabla u\cdot[(\nabla a_2)\Delta^2 u+a_2\nabla(\Delta^2 u)-(\nabla a_1)\Delta u-a_1\nabla(\Delta u)+(\nabla a_0)u] \\ & +\Delta\phi\nabla(\Delta u)\cdot\nabla(\Delta u)+2\nabla\phi\cdot\nabla(|\nabla(\Delta u)|^2)+\Delta\beta u^2+2\nabla\beta\cdot\nabla(u^2) \\ & +2\beta u\Delta u+2\beta|\nabla u|^2+\frac{2}{n}(\Delta u)^2+2\Delta^2 u(a_2\Delta^2 u-a_1\Delta u+a_0 u) \\ & +2|\nabla(\Delta^2 u)|^2+2\nabla u\cdot\nabla(\Delta u).\end{aligned} \tag{5.37}$$

To complete the proof, we first establish a set of useful inequalities involving the term $|\nabla(\Delta u)|^2$:

$$2\nabla u\cdot\nabla(\Delta u)\geqslant -|\nabla u|^2-|\nabla(\Delta u)|^2, \tag{5.38}$$

$$\frac{\Delta\phi}{3}|\nabla(\Delta u)|^2+\gamma(x)\nabla(\Delta u)\cdot\nabla(\Delta^2 u)+|\nabla(\Delta^2 u)|^2\geqslant|\nabla(\Delta u)|^2\left[\frac{\Delta\phi}{3}-\frac{(\gamma(x))^2}{4}\right], \tag{5.39}$$

$$2\phi|\nabla^2(\Delta u)|^2+2\nabla\phi\cdot\nabla(|\nabla(\Delta u)|^2)+\frac{\Delta\phi}{3}|\nabla(\Delta u)|^2\geqslant|\nabla(\Delta u)|^2\left[\frac{\Delta\phi}{3}-\frac{2|\nabla\phi|^2}{\phi}\right], \tag{5.40}$$

$$\frac{\beta}{3}|\nabla u|^2-\frac{a_1}{2}\nabla u\cdot\nabla(\Delta u)+\frac{\Delta\phi}{6}|\nabla(\Delta u)|^2\geqslant|\nabla(\Delta u)|^2\left[\frac{\Delta\phi}{6}-\frac{3a_1^2}{16\beta}\right]. \tag{5.41}$$

Secondly, the following inequalities involving $|\nabla u|^2$ and $(\Delta^2 u)^2$ hold:

$$|\nabla(\Delta^2 u)|^2+\frac{a_2}{2}\nabla u\cdot\nabla(\Delta^2 u)+\left(\frac{a_0}{4}-1\right)|\nabla u|^2\geqslant|\nabla u|^2\left[\frac{a_0}{4}-\left(1+\frac{a_2^2}{16}\right)\right], \tag{5.42}$$

$$\frac{a_2}{3}(\Delta^2 u)^2+\frac{1}{2}(\nabla a_2)\cdot\nabla u\Delta^2 u+\frac{a_0}{4}|\nabla u|^2\geqslant|\nabla u|^2\left[\frac{a_0}{4}-\frac{3|\nabla a_2|^2}{16a_2}\right], \tag{5.43}$$

$$\frac{a_2}{3}(\Delta^2 u)^2-2a_1\Delta^2 u\Delta u+\frac{2}{3n}(\Delta u)^2\geqslant(\Delta^2 u)^2\left[\frac{a_2}{3}-\frac{3na_1^2}{2}\right], \tag{5.44}$$

$$\frac{\beta}{3}|\nabla u|^2-\frac{1}{2}\nabla u\cdot\nabla(a_1)\Delta u+\frac{2}{3n}(\Delta u)^2\geqslant|\nabla u|^2\left[\frac{\beta}{3}-\frac{3n|\nabla a_1|^2}{32}\right]. \tag{5.45}$$

Lastly, we state several inequalities for u^2:

$$\frac{\Delta\beta}{4}u^2+2a_0u\Delta^2 u+\frac{a_2}{3}(\Delta^2 u)^2\geqslant u^2\left[\frac{\Delta\beta}{4}-3\frac{a_0^2}{a_2}\right], \tag{5.46}$$

$$\frac{\Delta\beta}{4}u^2+\frac{\nabla(a_0)\cdot(\nabla u)}{2}u+\frac{2}{3}\beta|\nabla u|^2\geqslant u^2\left[\frac{\Delta\beta}{4}-\frac{3|\nabla a_0|^2}{32\beta}\right], \tag{5.47}$$

$$\frac{\Delta\beta}{4}u^2+2\beta u\Delta u+\frac{2}{3n}(\Delta u)^2\geqslant u^2\left[\frac{\Delta\beta}{4}-\frac{3n\beta^2}{2}\right], \tag{5.48}$$

$$\frac{\Delta\beta}{4}u^2+2\nabla\beta\cdot\nabla(u^2)+\frac{2}{3}\beta|\nabla u|^2\geqslant u^2\left[\frac{\Delta\beta}{4}-\frac{6|\nabla\beta|^2}{\beta}\right]. \tag{5.49}$$

Utilizing (5.33), (5.34), (5.35) and (5.38)–(5.49) we see that P_7 is subharmonic in Ω. □

Next we obtain a maximum principle result for the equation $\Delta^4 u + a_0(x)u = 0$ utilizing an auxiliary function containing the second gradient of $\Delta^2 u$.

THEOREM 5.5. *Suppose that* $u \in C^9(\Omega) \cap C^7(\overline{\Omega})$ *be a solution to (1.1) where* $m = 4$ *and* $a_3 = a_2 = a_1 = 0$. *Let*

$$P_8 = |\nabla^2(\Delta^2 u)|^2 - \nabla(\Delta^2 u)\cdot\nabla(\Delta^3 u) - \left[\frac{4-n}{2(n+2)}\right](\Delta^3 u)^2 \tag{5.50}$$

$$+\beta(x)\left[u^2 + (\Delta u)^2 + (\Delta^2 u)^2 + (\Delta^3 u)^2\right], \text{ where } \beta(x) \in C^2(\overline{\Omega}).$$

Furthermore, we require that for $a_0 > 0$,

$$\beta^2 \geqslant \frac{3}{8}a_0^2, \tag{5.51}$$

$$(\Delta\beta)^2 \geqslant \max\left\{15\beta^2, 20\beta^2 a_0^2, \frac{15}{8}|\nabla a_0|^2, 5\left(\frac{4-n}{n+2}\right)a_0^2\right\}. \tag{5.52}$$

Then P_8 *achieves its maximum value on* $\partial\Omega$.

Proof. Applying the Laplacian to P_8 yields

$$\Delta P_8 \geqslant -\nabla(\Delta^2 u)\cdot a_0\nabla u - \nabla(\Delta^2 u)\cdot(\nabla a_0)u - \left(\frac{4-n}{n+2}\right)(\Delta^3 u)a_0 u \tag{5.53}$$

$$+\Delta\beta\left[u^2 + (\Delta u)^2 + (\Delta^2 u)^2 + (\Delta^3 u)^2\right] + 2\beta[|\nabla u|^2 + u\Delta u$$

$$+|\nabla(\Delta^2 u)|^2 + \Delta^2 u\Delta^3 u + |\nabla(\Delta^3 u)|^2 + (\Delta^3 u)a_0 u + |\nabla(\Delta u)|^2 + \Delta u\Delta^2 u]$$

$$+2\nabla\beta\cdot[\nabla(u^2) + \nabla((\Delta^2 u)^2) + \nabla((\Delta^3 u)^2) + \nabla((\Delta u)^2)].$$

Utilizing a series of inequalities similar to (5.38)–(5.49) leads to

$$\Delta P_8 \geqslant |\nabla(\Delta^2 u)|^2\left[\frac{2\beta}{3} - \frac{a_0^2}{4\beta}\right] + u^2\left[\frac{\Delta\beta}{5} - \frac{4|\nabla\beta|^2}{\beta}\right] \tag{5.54}$$

$$+u^2\left[\frac{\Delta\beta}{5} - \frac{3\beta^2}{\Delta\beta}\right] + u^2\left[\frac{\Delta\beta}{5} - \frac{3|\nabla a_0|^2}{8\beta}\right]$$

$$+(\Delta^2 u)^2\left[\frac{\Delta\beta}{3} - \frac{6|\nabla\beta|^2}{\beta}\right] + (\Delta^2 u)^2\left[\frac{\Delta\beta}{3} - \frac{4\beta^2}{\Delta\beta}\right]$$

$$+(\Delta^3 u)^2\left[\frac{\Delta\beta}{4} - \frac{6|\nabla\beta|^2}{\beta}\right] + (\Delta^3 u)^2\left[\frac{\Delta\beta}{4} - \frac{5\beta^2 a_0^2}{\Delta\beta}\right]$$

$$+(\Delta^3 u)^2\left[\frac{\Delta\beta}{4} - \frac{5\left(\frac{4-n}{n+2}\right)a_o^2}{4\Delta\beta}\right] + (\Delta u)^2\left[\frac{\Delta\beta}{2} - 4\frac{|\nabla\beta|^2}{\beta}\right]$$

$$+(\Delta u)^2\left[\frac{\Delta\beta}{2} - \frac{3\beta^2}{\Delta\beta}\right].$$

From conditions (5.51) and (5.52) it follows that $\Delta P_8 \geqslant 0$. □

Finally we state a few applications of the last two results. We impose the boundary conditions $u = \frac{\partial u}{\partial n} = \frac{\partial^2 u}{\partial n^2} = 0$ on (2.1) in the case $m = 3$. Using integration by parts (see[8]) and Theorem 5.4 we obtain the following integral bound for $|\nabla(\Delta u)|^2$:

$$\int_\Omega \left(\frac{5}{2} + \left[\frac{n-7}{2(n+5)} + \phi\right]\right) |\nabla(\Delta u)|^2 dx \leqslant \text{area}(\Omega) \max_{\partial\Omega} \left(|\nabla^3 u|^2 + \left(\frac{n-7}{2(n+5)} + \phi\right) |\nabla(\Delta u)|^2 + (\Delta^2 u)^2\right).$$

Now we consider (2.1) with $m = 4$ subject to the boundary conditions $u = \Delta u = \Delta^2 u = \frac{\partial(\Delta^2 u)}{\partial n} = 0$. From Theorem 5.5 we deduce:

$$\int_\Omega (\Delta^3 u)^2 dx \leqslant \text{area}(\Omega) \left\{\frac{n(n+2)}{3n}\right\} \max_{\partial\Omega} \left[\left(\frac{n-4}{2(n+2)} + \beta\right) (\Delta^3 u)^2 + |\nabla^2(\Delta^2 u)|^2\right].$$

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