Mathematical
Inequalities
& Applications
Volume 16, Number 3 (2013), 823–829 doi:10.7153/mia-16-62

OPTIMAL HÖLDER MEAN INEQUALITY FOR THE COMPLETE ELLIPTIC INTEGRALS

YUN HUA

(*Communicated by J. Pečarić*)

Abstract. In this paper, we prove that $H_p(\mathscr{K}(r),\mathscr{K}(r')) \geqslant \mathscr{K}(\sqrt{2}/2)$ and $H_q(\mathscr{K}(r),\mathscr{K}(r')) \leqslant \mathscr{K}(\sqrt{2}/2)$ for all $r \in (0,1)$ if and only if $p \geqslant 1-4[\mathscr{K}(\sqrt{2}/2)]^4/\pi^2 = -3.789\cdots$ and $q \leqslant (\log 2)/[\log(\pi/2)-\log\mathscr{K}(\sqrt{2}/2)] = -4.1805\cdots$, where $H_p(x,y)$ denotes the Hölder mean of order p of two positive numbers x and y, $r'=\sqrt{1-r^2}$, and $\mathscr{K}(r)$ denotes the complete elliptic integral of the first kind, respectively.

1. Introduction

Throughout this paper, we denote $r'=\sqrt{1-r^2}$ for $0<r<1$. Legendre's complete elliptic integrals of the first and second kinds [13, 14] are defined by

$$\begin{cases} \mathscr{K}=\mathscr{K}(r)=\int_0^{\pi/2}(1-r^2\sin^2\theta)^{-1/2}d\theta, \\ \mathscr{K}'=\mathscr{K}'(r)=\mathscr{K}(r'), \\ \mathscr{K}(0)=\pi/2, \qquad \mathscr{K}(1)=\infty \end{cases}$$

and

$$\begin{cases} \mathscr{E}=\mathscr{E}(r)=\int_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}d\theta, \\ \mathscr{E}'=\mathscr{E}'(r)=\mathscr{E}(r'), \\ \mathscr{E}(0)=\pi/2, \qquad \mathscr{E}(1)=1, \end{cases}$$

respectively.

Recently, the complete elliptic integrals have been the subject of intensive research. In particular, many remarkable properties and inequalities can be found in the literature [1–7, 9–12].

For $p\in\mathbb{R}$ the Hölder mean $H_p(x,y)$ of order p of two positive numbers x and y is defined by

$$H_p(x,y)=\begin{cases} \left(\frac{x^p+y^p}{2}\right)^{1/p}, & p\neq 0, \\ \sqrt{xy}, & p=0. \end{cases} \tag{1.1}$$

Mathematics subject classification (2010): Primary 26E60.
Keywords and phrases: Complete elliptic integrals, Hölder mean, inequality.
This research was supported by the Project of Shandong Province Higher Educational Science and Technology Program under grant No. J11LA57.

As is known to all, $H_p(x,y)$ is continuous and strictly increasing with respect to $p \in \mathbb{R}$ for fixed $x,y>0$ with $x \neq y$. The main properties of the Hölder mean are given in [15].

In [8, Theorem 3.31], Anderson et al. studied the monotonicity and convexity of $\mathscr{K}(r)\mathscr{E}(r)$ in $(0,1)$, and obtained the following inequality:

$$H_0(\mathscr{K}(r),\mathscr{E}(r) > \frac{\pi}{2} \tag{1.2}$$

for all $r \in (0,1)$.

In a recent paper [9] Wang et al. generalized (1.2) and proved the following optimal power mean inequality for the complete elliptic integrals:

$$H_p(\mathscr{K}(r),\mathscr{E}(r)) > \frac{\pi}{2}, \tag{1.3}$$

for all $r \in (0,1)$ if and only if $p \geqslant -1/2$.

Notice that in [8, Lemma 3.32(1), (3)], Anderson, Vamanamurthy and Vuorinen studied the monotonicity of $\mathscr{K}(r)\mathscr{K}(r')$ and $\mathscr{K}(r)^p+\mathscr{K}(r')^p$ for $p \in [-3,0)$ and $r \in (0,1)$, and established the following inequalities:

$$H_0(\mathscr{K}(r),\mathscr{K}(r')) \geqslant \mathscr{K}\left(\sqrt{2}/2\right) \tag{1.4}$$

and

$$\mathscr{K}\left(\sqrt{2}/2\right) \leqslant H_p(\mathscr{K}(r),\mathscr{K}(r')) < \pi/2^{1+1/p} \tag{1.5}$$

for all $r \in (0,1)$ and $p \in [-3,0)$.

It is natural to ask what are the least value p and the greatest value q such that $H_p(\mathscr{K}(r),\mathscr{K}(r')) \geqslant \mathscr{K}(\sqrt{2}/2)$ and $H_q(\mathscr{K}(r),\mathscr{K}(r')) \leqslant \mathscr{K}(\sqrt{2}/2)$ for all $r \in (0,1)$. The main purpose of this paper is to answer this question. Our main result is the following Theorem 1.1.

THEOREM 1.1. *Inequalities*

$$H_p(\mathscr{K}(r),\mathscr{K}(r')) \geqslant \mathscr{K}\left(\sqrt{2}/2\right), \tag{1.6}$$

and

$$H_q(\mathscr{K}(r),\mathscr{K}(r')) \leqslant \mathscr{K}\left(\sqrt{2}/2\right). \tag{1.7}$$

hold for all $r \in (0,1)$ *if and only if* $p \geqslant 1-4[\mathscr{K}(\sqrt{2}/2)]^4/\pi^2 = -3.789\cdots$ *and* $q \leqslant (\ln 2)/\left[\ln(\pi/2) - \ln \mathscr{K}(\sqrt{2}/2)\right] = -4.1805\cdots$.

2. Some Lemmas

In order to establish our main result we need several lemmas, which we present in this section.

For $0 < r < 1$, the following formulas were presented in [8, Appendix E, pp. 474–475]:

$$\frac{d\mathcal{K}}{dr} = \frac{\mathcal{E} - r'^2\mathcal{K}}{rr'^2}, \qquad \frac{d\mathcal{E}}{dr} = \frac{\mathcal{E} - \mathcal{K}}{r},$$
$$\frac{d(\mathcal{E} - r'^2\mathcal{K})}{dr} = r\mathcal{K}, \qquad \frac{d(\mathcal{K} - \mathcal{E})}{dr} = \frac{r\mathcal{E}}{r'^2},$$

$$\mathcal{K}\mathcal{E}' + \mathcal{K}'\mathcal{E} - \mathcal{K}\mathcal{K}' = \frac{\pi}{2}. \tag{2.1}$$

The following Lemma 2.1 can be found in [8, Theorem 3.21 (7), and Exercise 3.43 (16) and (46)].

LEMMA 2.1. *(1) For* $c \in [1/2, \infty)$, $r'^c\mathcal{K}$ *is strictly decreasing from* $[0,1)$ *onto* $(0, \pi/2]$*;*

(2) $\left[\mathcal{E}^2 - (r'\mathcal{K})^2\right]/r^4$ *is strictly increasing from* $(0,1)$ *onto* $(\pi^2/32, 1)$*;*

(3) $(\mathcal{E} - r'^2\mathcal{K})/(r^2\mathcal{K})$ *is strictly decreasing from* $(0,1)$ *onto* $(0, 1/2)$.

LEMMA 2.2. *Let* $r \in (0,1)$. *Then the function* $f(r) = (\mathcal{E} - r'^2\mathcal{K})(\mathcal{E}' - r^2\mathcal{K}')/(r^2r'^2\mathcal{K}\mathcal{K}')$ *is strictly increasing from* $(0, \sqrt{2}/2)$ *(or strictly decreasing from* $(\sqrt{2}/2, 1)$*, respectively) onto* $(0, \pi^2/\{4[\mathcal{K}(\sqrt{2}/2)]^4\})$.

Proof. By differentiation, we have

$$\begin{aligned} f'(r) = & \frac{r\mathcal{K}(r^2\mathcal{K}) - (\mathcal{E} - r'^2\mathcal{K})[2r\mathcal{K} + r^2(\mathcal{E} - r'^2\mathcal{K})/(rr'^2)]}{r^4\mathcal{K}^2} \\ & \times \left(\frac{\mathcal{E}' - r^2\mathcal{K}'}{r'^2\mathcal{K}'}\right) + \left(\frac{\mathcal{E} - r'^2\mathcal{K}}{r^2\mathcal{K}}\right) \\ & \times \frac{-r\mathcal{K}'(r'^2\mathcal{K}') - (\mathcal{E}' - r^2\mathcal{K}')[-2r\mathcal{K}' - r'^2(\mathcal{E}' - r^2\mathcal{K}')/(rr'^2)]}{r'^4\mathcal{K}'^2}, \\ = & r[f_1(r) - f_1(r')], \end{aligned} \tag{2.2}$$

where

$$f_1(r) = \frac{(\mathcal{E} - r'^2\mathcal{K})}{r^2\mathcal{K}} \cdot \frac{(\mathcal{E}')^2 - (r\mathcal{K}')^2}{r'^4} \cdot \frac{1}{(r\mathcal{K}')^2}. \tag{2.3}$$

From (2.3) and Lemma 2.1 (1)–(3) we know that $f_1(r)$ is strictly decreasing in $(0,1)$. Then (2.2) leads to the conclusion that $f'(r)>0$ for $r\in(0,\sqrt{2}/2)$ and $f'(r)<0$ for $r\in(\sqrt{2}/2,1)$. Hence $f(r)$ is strictly increasing in $(0,\sqrt{2}/2)$ and strictly decreasing in $(\sqrt{2}/2,1)$. Moreover, making use of Lemma 2.1 (3) and (2.1) we clearly see that $f(0^+)=f(1^-)=0$ and

$$f(\sqrt{2}/2)=\frac{4\left[\mathscr{E}(\sqrt{2}/2)-(1/2)\mathscr{K}(\sqrt{2}/2)\right]^2}{\mathscr{K}(\sqrt{2}/2)^2}=\frac{\pi^2}{4[\mathscr{K}(\sqrt{2}/2)]^4}. \quad \square$$

LEMMA 2.3. *Let* $p\in\mathbb{R}$ *and* $g(r)=(\mathscr{K}/\mathscr{K}')^{p-1}(\mathscr{E}-r'^2\mathscr{K})/(\mathscr{E}'-r^2\mathscr{K}')$. *Then* $g(r)$ *is strictly increasing in* $(0,1)$ *if and only if* $p\geqslant 1-4[\mathscr{K}(\sqrt{2}/2)]^4/\pi^2=-3.789\cdots$, *and* $g(r)<1$ *for* $r\in(0,\sqrt{2}/2)$ *and* $g(r)>1$ *for* $r\in(\sqrt{2}/2,1)$ *if* $p\geqslant 1-4[\mathscr{K}(\sqrt{2}/2)]^4/\pi^2$. *Moreover, if* $p<1-4\mathscr{K}(\sqrt{2}/2)^4/\pi^2$, *then there exists* $r_0=r_0(p)\in(0,\sqrt{2}/2)$, *such that* $g(r_0)=g(r_0{}')=1$, $g(r)<1$ *for* $r\in(0,r_0)\cup(\sqrt{2}/2,r_0{}')$, *and* $g(r)>1$ *for* $r\in(r_0,\sqrt{2}/2)\cup(r_0{}',1)$

Proof. Simple computations lead to

$$g\left(\sqrt{2}/2\right)=1 \tag{2.4}$$

and

$$\begin{aligned}\frac{g'(r)}{g(r)}&=(p-1)\left(\frac{\mathscr{E}-r'^2\mathscr{K}}{rr'^2\mathscr{K}}+\frac{\mathscr{E}'-r^2\mathscr{K}'}{rr'^2\mathscr{K}'}\right)+\frac{r\mathscr{K}}{\mathscr{E}-r'^2\mathscr{K}}+\frac{r\mathscr{K}'}{\mathscr{E}'-r^2\mathscr{K}'}\\&=(p-1)\frac{\mathscr{K}\mathscr{E}'+\mathscr{K}'\mathscr{E}-\mathscr{K}\mathscr{K}'}{rr'^2\mathscr{K}\mathscr{K}'}+\frac{r(\mathscr{K}\mathscr{E}'+\mathscr{K}'\mathscr{E}-\mathscr{K}\mathscr{K}')}{(\mathscr{E}-r'^2\mathscr{K})(\mathscr{E}'-r^2\mathscr{K}')}\\&=\frac{\pi}{2rr'^2\mathscr{K}\mathscr{K}'}\left[p-1+\frac{r^2r'^2\mathscr{K}\mathscr{K}'}{(\mathscr{E}-r'^2\mathscr{K})(\mathscr{E}'-r^2\mathscr{K}')}\right].\end{aligned} \tag{2.5}$$

From Lemma 2.2 one can obtain that $r^2r'^2\mathscr{K}\mathscr{K}'/\left[(\mathscr{E}-r'^2\mathscr{K})(\mathscr{E}'-r^2\mathscr{K}')\right]$ is strictly decreasing from $(0,\sqrt{2}/2)$ (or strictly increasing from $(\sqrt{2}/2,1)$, respectively) onto $(4[\mathscr{K}(\sqrt{2}/2)]^4/\pi^2,\infty)$. Then (2.4) and (2.5) lead to the conclusion that $g(r)$ is strictly increasing in $(0,1)$ if and only if $p\geqslant 1-4[\mathscr{K}(\sqrt{2}/2)]^4/\pi^2=-3.789\cdots$, and $g(r)<1$ for $r\in(0,\sqrt{2}/2)$ and $g(r)>1$ for $r\in(\sqrt{2}/2,1)$ if $p\geqslant 1-4[\mathscr{K}(\sqrt{2}/2)]^4/\pi^2$. Moreover, if $p<1-4[\mathscr{K}(\sqrt{2}/2)]^4/\pi^2$, then from (2.5) we know that there exists $r_1\in(0,\sqrt{2}/2)$, such that $g'(r_1)=g'(r_1{}')=0$, $g'(r)>0$ for $r\in(0,r_1)\cup(r_1{}',1)$ and $g'(r)<0$ for $r\in(r_1,r_1{}')$. Hence $g(r)$ is strictly increasing in $(0,r_1)\cup(r_1{}',1)$ and strictly decreasing in $(r_1,r_1{}')$. Therefore, Lemma 2.3 follows from (2.4) and the monotonicity of $g(r)$. $\square$

3. Proof of Theorem 1.1

Proof. If $p=0$, then inequality (1.6) reduces to inequality (1.4). Thus, we only need to prove inequality (1.6) for $p\neq 0$. Let

$$F(r)=\frac{1}{s}\ln\frac{\mathscr{K}(r)^s+\mathscr{K}(r')^s}{2}\quad (s\neq 0). \tag{3.1}$$

Then simple computation leads to

$$\begin{aligned}F'(r)&=\frac{1}{s}\frac{s\mathscr{K}^{s-1}(\mathscr{E}-r'^2\mathscr{K})/(rr'^2)-s\mathscr{K}'^{s-1}(\mathscr{E}'-r^2\mathscr{K}')/(rr'^2)}{\mathscr{K}^s+\mathscr{K}'^s}\\&=\frac{\mathscr{K}^{s-1}(\mathscr{E}-r'^2\mathscr{K})-\mathscr{K}'^{s-1}(\mathscr{E}'-r^2\mathscr{K}')}{rr'^2(\mathscr{K}^s+\mathscr{K}'^s)}\\&=\frac{\mathscr{K}'^{s-1}(\mathscr{E}'-r^2\mathscr{K}')}{rr'^2(\mathscr{K}^s+\mathscr{K}'^s)}\left[\left(\frac{\mathscr{K}}{\mathscr{K}'}\right)^{s-1}\frac{\mathscr{E}-r'^2\mathscr{K}}{\mathscr{E}'-r^2\mathscr{K}'}-1\right].\end{aligned} \tag{3.2}$$

We divide the proof into two cases.

Case 1. $s\geqslant 1-4[\mathscr{K}(\sqrt{2}/2)]^4/\pi^2$. Then from (3.2) and Lemma 2.3 we know that $F'(r)<0$ for $r\in(0,\sqrt{2}/2)$ and $F'(r)>0$ for $r\in(\sqrt{2}/2,1)$. Hence, $F(r)$ is strictly decreasing in $(0,\sqrt{2}/2)$ and strictly increasing in $(\sqrt{2}/2,1)$. Then (3.1) leads to the conclusion that

$$\frac{1}{s}\ln\frac{\mathscr{K}(r)^s+\mathscr{K}(r')^s}{2}\geqslant \ln\mathscr{K}\left(\frac{\sqrt{2}}{2}\right) \tag{3.3}$$

for all $r\in(0,1)$.

Therefore, inequality (1.6) follows from (3.3).

Case 2. $s<1-4[\mathscr{K}(\sqrt{2}/2)]^4/\pi^2$. Then from (3.2) and Lemma 2.3 we clearly see that $F'(r)<0$ for $r\in(0,r_0)\cup(\sqrt{2}/2,r_0')$ and $F'(r)>0$ for $r\in(r_0,\sqrt{2}/2)\cup(r_0',1)$. Hence, $F(r)$ is strictly decreasing in $(0,r_0)\cup(\sqrt{2}/2,r_0')$, strictly increasing in $(r_0,\sqrt{2}/2)\cup(r_0',1)$, and

$$\begin{aligned}\sup_{r\in(0,1)}F(r)&=\max\left\{\lim_{r\to 0}F(r),F(\sqrt{2}/2),\lim_{r\to 1}F(r)\right\}\\&=\max\left\{\ln(\pi/2)-\frac{1}{s}\ln 2,\ln\mathscr{K}\left(\sqrt{2}/2\right)\right\}.\end{aligned} \tag{3.4}$$

Further, if $(\ln 2)/\left[\ln(\pi/2)-\ln\mathscr{K}(\sqrt{2}/2)\right]<s<1-4[\mathscr{K}(\sqrt{2}/2)]^4/\pi^2$, then from (3.4) we have $\sup_{r\in(0,1)}F(r)=\ln(\pi/2)-(\ln 2)/s$ and

$$\frac{1}{s}\ln\frac{\mathscr{K}(r)^s+\mathscr{K}(r')^s}{2}<\ln\left(\frac{\pi}{2}\right)-\frac{1}{s}\ln 2 \tag{3.5}$$

for all $r \in (0,1)$; if $s \leqslant (\ln 2)/\left[\ln(\pi/2) - \ln \mathscr{K}(\sqrt{2}/2)\right]$, then from (3.4) we get $\sup_{r\in(0,1)} F(r) = \ln \mathscr{K}(\sqrt{2}/2)$ and

$$\frac{1}{s}\ln\frac{\mathscr{K}(r)^s + \mathscr{K}(r')^s}{2} \leqslant \ln \mathscr{K}\left(\frac{\sqrt{2}}{2}\right) \tag{3.6}$$

for all $r \in (0,1)$.

Therefore, inequality (1.7) follows from (3.6).

Finally, we prove that the parameters

$$p = 1 - \frac{4[\mathscr{K}(\frac{\sqrt{2}}{2})]^4}{\pi^2}$$

and

$$q = \frac{\ln 2}{\ln(\frac{\pi}{2}) - \ln \mathscr{K}(\frac{\sqrt{2}}{2})}$$

are the best possible parameters such that inequalities (1.6) and (1.7) hold for all $r \in (0,1)$, respectively.

If $q < s < p$, then from the monotonicity of $F(r)$ we know that there exists $r \in (\sqrt{2}/2, r_0')$, such that $F(r) < F(\sqrt{2}/2)$ and $H_s(\mathscr{K}(r), \mathscr{K}(r')) < \mathscr{K}(\sqrt{2}/2)$. Moreover, equation (3.4) and inequality (3.5) imply that there exists $\delta = \delta(s) \in (0,1)$, such that $F(r) > \ln \mathscr{K}(\sqrt{2}/2)$ and $H_s(\mathscr{K}(r), \mathscr{K}(r')) > \mathscr{K}(\sqrt{2}/2)$ for $r \in (0,\delta)$. □

Acknowledgement. The author is thankful to the anonymous referees for their valuable comments on and suggestions to the original version of this paper.

REFERENCES

[1] H. ALZER AND S. L. QIU, *Monotonicity theorems and inequalities for the complete elliptic integrals*, J. Comput. Appl. Math. **172** (2004), 289–312.

[2] G. D. ANDERSON, S.-L. QIU, M. K. VAMANAMURTHY, *Elliptic integral inequalities, with applications*, Constr. Approx. **14**, 2 (1998), 195–207.

[3] M. K. VAMANAMURTHY AND M. VUORINEN, *Inequalities for means*, J. Math. Anal. Appl. **183** (1994), 155–166.

[4] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Functional inequalities for complete elliptic integrals and their ratios*, SIAM J. Math. Anal. **21** (1990), 536–549.

[5] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Functional inequalities for hypergeometric functions and complete elliptic integrals*, SIAM J. Math. Anal. **23** (1992), 512–524.

[6] G. D. ANDERSON, S. L. QIU, M. K. VAMANAMURTHY AND M. VUORINEN, *Generalized elliptic integrals and modular equations*, Pacific J. Math. **192** (2000), 1–37.

[7] S. ANDRÁS AND Á. BARICZ, *Bounds for complete elliptic integral of the first kind*, Expo. Math. **28** (2010), 357–364.

[8] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, New York, 1997.

[9] MIAO-KUN WANG, YU-MING CHU, YE-FANG QIU, SONG-LIANG QIU, *An optimal power mean inequality for the complete elliptic integrals*, Applied Mathematics Letters **24** (2011), 887–890.

[10] YU-MING CHU, YE-FANG QIU AND MIAO-KUN WANG, *Hölder mean inequalities for the complete elliptic integrals*, Integral Transforms and Special Functions, **23**,7 (2012), 521–527.

[11] MIAO-KUN WANG, YU-MING CHU, SONG-LIANG QIU, AND YUE-PING JIANG, *Convexity of the complete elliptic integrals of the first kind with respect to Hölder means*, J. Math. Anal. Appl. **388**, 2 (2011), 1141–1146.

[12] LIMING ZHOU, SONGLIANG QIU, FEI WANG, *Inequalities for the generalized elliptic integrals with respect to Hölder means*, J. Math. Anal. Appl. **386**, 2 (2012), 641–646.

[13] F. BOWMAN, *Introduction to Elliptic Functions with Applications*, Dover Publications, New York, 1961.

[14] P. F. BYRD, M. D. FRIEDMAN, *Handbook of Elliptic Integrals for Engineers and Scientists*, Springer-Verlag, New York, 1971.

[15] P. S. BULLEN, *Handbook of Means and their Inequalities*, Kluwer Academic Publishers Group, Dordrecht, 2003.

(Received December 26, 2011)

Yun Hua
Department of Information Engineering
Weihai Vocational College
Weihai City 264210, ShanDong province
P. R. China
e-mail: xxgcxhy@163.com