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PROPERTY OF A HÖLDER–TYPE INEQUALITY AND ITS APPLICATION

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Abstract. In this paper, we present a property of a Hölder-type inequality which is due to Tian, and then we obtain the similar property of Hölder's inequality. Moreover, we give a new refinement of Hölder's inequality. As application, we offer a new property of Minkowski-type inequality, and then we obtain a new refinement of Minkowski's inequality.

1. Introduction

The classical Hölder's inequality states that if $a_k \geqslant 0,\ b_k \geqslant 0\ \ (k = 1, 2, \cdots, n)$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{k=1}^{n} a_k b_k \leqslant \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} b_k^q\right)^{\frac{1}{q}}. \tag{1}$$

The inequality is reversed for $p < 1$ ($p \neq 0$). (For $p < 0$, we assume that a_k, $b_k > 0$.)

It is well known that Hölder's inequality is one of the most important inequalities in analysis. Various refinements, generalizations and applications of Hölder's inequality and its series analogues in different areas of mathematics have appeared in the literature. For example, Abramovich et al. in [1] presented a new generalization of Hölder's inequality and its reversed version in discrete and integral forms. Nikolova and Varošanec [11] obtained some new refinements of the classical Hölder inequality by using a convex function. For detailed expositions, the interested reader may consult [1–6, 9–13] and the references therein. Among various refinements of (1), Hu in [5] established the following interesting theorem.

THEOREM A. *Let* $p \geqslant q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, *let* A_n, $B_n \geqslant 0$, $\sum_n A_n^p < \infty$, $\sum_n B_n^q < \infty$, *and let* $1 - e_n + e_m \geqslant 0$, $\sum_n |e_n| < \infty$. *Then*

$$\sum_n A_n B_n \leqslant \left(\sum_n B_n^q\right)^{\frac{1}{q}-\frac{1}{p}} \left\{ \left[\left(\sum_n B_n^q\right)\left(\sum_n A_n^p\right)\right]^2 - \left[\left(\sum_n B_n^q e_n\right)\left(\sum_n A_n^p\right) - \left(\sum_n B_n^q\right)\left(\sum_n A_n^p e_n\right)\right]^2 \right\}^{\frac{1}{2p}}. \tag{2}$$

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In 2011, Tian [13] presented a Hölder-type inequality, which is the reversed version of Hu's result, as follows:

THEOREM B. *Let* $A_r \geqslant 0,\ B_r > 0,\ 1-e_r+e_s \geqslant 0\ (r,\ s=1,2,\cdots,n)$, *and let* $q<0,\ \frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\sum_{r=1}^{n} A_rB_r \geqslant \left(\sum_{r=1}^{n} A_r^p\right)^{\frac{1}{p}-\frac{1}{q}} \left\{\left[\left(\sum_{r=1}^{n} A_r^p\right)\left(\sum_{r=1}^{n} B_r^q\right)\right]^2 - \left[\left(\sum_{r=1}^{n} A_r^p e_r\right)\left(\sum_{r=1}^{n} B_r^q\right) - \left(\sum_{r=1}^{n} A_r^p\right)\left(\sum_{r=1}^{n} B_r^q e_r\right)\right]^2\right\}^{\frac{1}{2q}}. \tag{3}$$

THEOREM C. *Let* $f(x),\ g(x),\ e(x)$ *be integrable functions defined on* $[a,b]$ *and* $f(x)\geqslant 0,\ g(x)>0,\ 1-e(x)+e(y)\geqslant 0$ *for all* $x,y\in[a,b]$, *and let* $q<0,\ \frac{1}{p}+\frac{1}{q}=1$. *Then*

$$\int_a^b f(x)g(x)\mathrm{d}x \geqslant \left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{p}-\frac{1}{q}} \left[\left(\int_a^b f^p(x)\mathrm{d}x\int_a^b g^q(x)\mathrm{d}x\right)^2 - \left(\int_a^b f^p(x)e(x)\mathrm{d}x\int_a^b g^q(x)\mathrm{d}x - \int_a^b f^p(x)\mathrm{d}x\int_a^b g^q(x)e(x)\mathrm{d}x\right)^2\right]^{\frac{1}{2q}}. \tag{4}$$

REMARK 1.1. Let $A_{2k}=B_{2k-1}=1,\ A_{2k-1}=B_{2k}=0,\ k=1,2,\cdots,N,\ n=2N$, and let $e_n=\begin{cases}1 & \text{if } n \text{ even}\\ 0 & \text{if } n \text{ odd}\end{cases}$. Then from the classical Hölder's inequality (1), we obtain $0\leqslant N$ for $p>1,\ \frac{1}{p}+\frac{1}{q}=1$. However, from (2), we obtain $0\leqslant 0$.

The purpose of this work is to give some properties of (3), (4), (1) and their applications. The rest of this paper is organized as follows. In Section 2, we present properties of (3) and (4), and then we obtain the similar property of Hölder's inequality (1). Moreover, we give a new refinement of Hölder's inequality. In Section 3, a new property of Minkowski-type inequality is given. Furthermore, the result is used to improve Minkowski's inequality.

2. Property of a Hölder-type inequality

In order to prove the main results, we need the following lemmas.

LEMMA 2.1. [3] *If* $x,y>0,\ \alpha>1$ *or* $\alpha<0$, *then*

$$x^{\alpha}y^{1-\alpha} \geqslant \alpha x+(1-\alpha)y. \tag{5}$$

The inequality is reversed for $0\leqslant\alpha\leqslant 1$.

Next, we present some new properties of (3) and (4).

THEOREM 2.2. *Let* $A_r \geqslant 0,\ B_r > 0,\ 1 - e_r + e_k \geqslant 0\ \ (r,\ k = 1,2,\cdots)$, *let* $q < 0$, $\frac{1}{p} + \frac{1}{q} = 1$, *and let*

$$F(n) = \left(\sum_{r=1}^{n} A_r B_r\right)^2 - \left(\sum_{r=1}^{n} A_r^p\right)^{\frac{2}{p}-\frac{2}{q}} \left\{\left[\left(\sum_{r=1}^{n} A_r^p\right)\left(\sum_{r=1}^{n} B_r^q\right)\right]^2 - \left[\left(\sum_{r=1}^{n} A_r^p e_r\right)\left(\sum_{r=1}^{n} B_r^q\right) - \left(\sum_{r=1}^{n} A_r^p\right)\left(\sum_{r=1}^{n} B_r^q e_r\right)\right]^2\right\}^{\frac{1}{q}}. \tag{6}$$

Then

$$F(n) \leqslant F(n+1). \tag{7}$$

The integral form is as follows:

THEOREM 2.3. *Let* $f(x),\ g(x),\ e(x)$ *be integrable functions defined on* $[a,b]$ *and* $f(x) \geqslant 0,\ g(x) > 0,\ 1 - e(x) + e(y) \geqslant 0$ *for all* $x,y \in [a,b]$, *let* $q < 0,\ \frac{1}{p} + \frac{1}{q} = 1$, *and let*

$$G(t) = \left(\int_a^t f(x)g(x)\mathrm{d}x\right)^2 - \left(\int_a^t f^p(x)\mathrm{d}x\right)^{\frac{2}{p}-\frac{2}{q}} \left[\left(\int_a^t f^p(x)\mathrm{d}x \int_a^t g^q(x)\mathrm{d}x\right)^2 - \left(\int_a^t f^p(x)e(x)\mathrm{d}x \int_a^t g^q(x)\mathrm{d}x - \int_a^t f^p(x)\mathrm{d}x \int_a^t g^q(x)e(x)\mathrm{d}x\right)^2\right]^{\frac{1}{q}}. \tag{8}$$

Then we have

$$G(t_1) \leqslant G(t_2), \quad a \leqslant t_1 \leqslant t_2 \leqslant b. \tag{9}$$

Proof. Here we only need to prove the inequality (7). The proof of (9) is similar. A simple calculation gives

$$\begin{aligned} &\sum_{r=1}^{N} A_r B_r \sum_{k=1}^{N} A_k B_k (1 - e_r + e_k) \\ &= \sum_{r=1}^{N}\sum_{k=1}^{N} A_r B_r A_k B_k - \sum_{r=1}^{N}\sum_{k=1}^{N} A_r B_r A_k B_k e_r + \sum_{r=1}^{N}\sum_{k=1}^{N} A_r B_r A_k B_k e_k \\ &= \left(\sum_{r=1}^{N} A_r B_r\right)^2. \end{aligned} \tag{10}$$

For any $C_{kr} > 0,\ D_{kr} > 0,\ k,\ r = 1,2,\cdots$, we write

$$X_{nr} = \sum_{k=1}^{n} C_{kr}^p, \quad Y_{nr} = \sum_{k=1}^{n} D_{kr}^q.$$

On the one hand, from Lemma 2.1 we have for $t,s>0$

$$
\begin{aligned}
A_rB_rX_{nr}^{\frac{1}{p}}Y_{nr}^{\frac{1}{q}} &= tB_rX_{nr}^{\frac{1}{q}}t^{-1}\left(A_r^{\frac{p}{q}}Y_{nr}^{\frac{1}{q}}A_r^{1-\frac{p}{q}}X_{nr}^{\frac{1}{p}-\frac{1}{q}}\right)\\
&\geqslant \frac{1}{q}t^qB_r^qX_{nr}+\frac{1}{p}t^{-p}\left(A_r^{\frac{p}{q}}Y_{nr}^{\frac{1}{q}}A_r^{1-\frac{p}{q}}X_{nr}^{\frac{1}{p}-\frac{1}{q}}\right)^p\\
&= \frac{1}{q}t^qB_r^qX_{nr}+\frac{1}{p}t^{-p}sA_r^{\frac{p^2}{q}}Y_{nr}^{\frac{p}{q}}s^{-1}A_r^{p-\frac{p^2}{q}}X_{nr}^{1-\frac{p}{q}}\\
&\geqslant \frac{1}{q}t^qB_r^qX_{nr}+\frac{1}{p}t^{-p}\left[\frac{p}{q}s^{\frac{q}{p}}A_r^pY_{nr}+\left(1-\frac{p}{q}\right)s^{\frac{q}{p-q}}A_r^pX_{nr}\right]\\
&= \frac{1}{q}t^qB_r^qX_{nr}+\frac{1}{q}t^{-p}s^{\frac{q}{p}}A_r^pY_{nr}+\left(\frac{1}{p}-\frac{1}{q}\right)t^{-p}s^{\frac{q}{p-q}}A_r^pX_{nr}.
\end{aligned}
\tag{11}
$$

On the other hand, by Hölder's inequality we obtain

$$
\sum_{k=1}^{n}C_{kr}D_{kr}\geqslant\left(\sum_{k=1}^{n}C_{kr}^p\right)^{\frac{1}{p}}\left(\sum_{k=1}^{n}D_{kr}^q\right)^{\frac{1}{q}}=X_{nr}^{\frac{1}{p}}Y_{nr}^{\frac{1}{q}}.
\tag{12}
$$

Consequently, for any $t>0,\ s>0$, if we set

$$
\begin{aligned}
F(m,n;s,t) &= \sum_{r=1}^{m}A_rB_r\left(\sum_{k=1}^{n}C_{kr}D_{kr}\right)-\frac{1}{q}t^q\left(\sum_{r=1}^{m}B_r^qX_{nr}\right)\\
&\quad -\frac{1}{q}t^{-p}s^{\frac{q}{p}}\left(\sum_{r=1}^{m}A_r^pY_{nr}\right)-\left(\frac{1}{p}-\frac{1}{q}\right)t^{-p}s^{\frac{q}{p-q}}\left(\sum_{r=1}^{m}A_r^pX_{nr}\right),
\end{aligned}
\tag{13}
$$

then by using the inequalities (12) and (11), we have

$$
\begin{aligned}
&F(m+1,n;s,t)-F(m,n;s,t)\\
&= A_{m+1}B_{m+1}\sum_{k=1}^{n}C_{k(m+1)}D_{k(m+1)}-\frac{1}{q}t^qB_{m+1}^qX_{n(m+1)}\\
&\quad -\frac{1}{q}t^{-p}s^{\frac{q}{p}}A_{m+1}^pY_{n(m+1)}-\left(\frac{1}{p}-\frac{1}{q}\right)t^{-p}s^{\frac{q}{p-q}}A_{m+1}^pX_{n(m+1)}\\
&\geqslant A_{m+1}B_{m+1}X_{n(m+1)}^{\frac{1}{p}}Y_{n(m+1)}^{\frac{1}{q}}-\frac{1}{q}t^qB_{m+1}^qX_{n(m+1)}\\
&\quad -\frac{1}{q}t^{-p}s^{\frac{q}{p}}A_{m+1}^pY_{n(m+1)}-\left(\frac{1}{p}-\frac{1}{q}\right)t^{-p}s^{\frac{q}{p-q}}A_{m+1}^pX_{n(m+1)}\\
&\geqslant 0,
\end{aligned}
\tag{14}
$$

and

$$\begin{aligned} &F(m,n+1;s,t)-F(m,n;s,t) \\ &=\sum_{r=1}^{m} A_r B_r C_{(n+1)r} D_{(n+1)r} - \frac{1}{q} t^q \sum_{r=1}^{m} B_r^q C_{(n+1)r}^p \\ &\quad - \frac{1}{q} t^{-p} s^{\frac{q}{p}} \sum_{r=1}^{m} A_r^p D_{(n+1)r}^q - \left(\frac{1}{p}-\frac{1}{q}\right) t^{-p} s^{\frac{q}{p-q}} \sum_{r=1}^{m} A_r^p C_{(n+1)r}^p \\ &\geqslant 0. \end{aligned} \tag{15}$$

Consequently, from the inequalities (14) and (15) we obtain

$$F(n,n;s,t) \leqslant F(n+1,n;s,t) \leqslant F(n+1,n+1;s,t). \tag{16}$$

Moreover,

$$\frac{\partial F(n,n;s,t)}{\partial s} = -\frac{1}{p} t^{-p} s^{\frac{q-p}{p}} \left(\sum_{r=1}^{n} A_r^p Y_{nr}\right) + \frac{1}{p} t^{-p} s^{\frac{2q-p}{p-q}} \left(\sum_{r=1}^{n} A_r^p X_{nr}\right), \tag{17}$$

$$\begin{aligned} \frac{\partial F(n,n;s,t)}{\partial t} &= -t^{q-1}\left(\sum_{r=1}^{n} B_r^q X_{nr}\right) + \frac{p}{q} t^{-p-1} s^{\frac{q}{p}} \left(\sum_{r=1}^{n} A_r^p Y_{nr}\right) \\ &\quad + \left(1-\frac{p}{q}\right) t^{-p-1} s^{\frac{q}{p-q}} \left(\sum_{r=1}^{n} A_r^p X_{nr}\right). \end{aligned} \tag{18}$$

From the following equations

$$\frac{\partial F(n,n;s,t)}{\partial s} = 0,$$

$$\frac{\partial F(n,n;s,t)}{\partial t} = 0,$$

we obtain

$$s_0 = \left(\frac{\sum_{r=1}^{n} A_r^p X_{nr}}{\sum_{r=1}^{n} A_r^p Y_{nr}}\right)^{\frac{p(q-p)}{q^2}},$$

$$t_0 = \frac{1}{\left(\sum_{r=1}^{n} B_r^q X_{nr}\right)^{\frac{1}{pq}}} \left[\left(\sum_{r=1}^{n} A_r^p Y_{nr}\right)^{\frac{1}{q^2}} \left(\sum_{r=1}^{n} A_r^p X_{nr}\right)^{\frac{1}{pq}-\frac{1}{q^2}}\right].$$

Additionally, from (17) and (18), we have

$$\frac{\partial^2 F(n,n;s,t)}{\partial s^2} = \frac{p-q}{p^2} t^{-p} s^{\frac{q-2p}{p}} \left(\sum_{r=1}^{n} A_r^p Y_{nr}\right) + \frac{2q-p}{p(p-q)} t^{-p} s^{\frac{3q-2p}{p-q}} \left(\sum_{r=1}^{n} A_r^p X_{nr}\right), \tag{19}$$

$$\frac{\partial^2 F(n,n;s,t)}{\partial s \partial t} = t^{-p-1} s^{\frac{q-p}{p}} \left(\sum_{r=1}^{n} A_r^p Y_{nr}\right) - t^{-p-1} s^{\frac{2q-p}{p-q}} \left(\sum_{r=1}^{n} A_r^p X_{nr}\right), \tag{20}$$

$$\frac{\partial^2 F(n,n;s,t)}{\partial t^2} = (1-q)t^{q-2}\left(\sum_{r=1}^{n} B_r^q X_{nr}\right) - \frac{p(p+1)}{q} t^{-p-2} s^{\frac{q}{p}} \left(\sum_{r=1}^{n} A_r^p Y_{nr}\right) + \left(\frac{(p-q)(p+1)}{q}\right) t^{-p-2} s^{\frac{q}{p-q}} \left(\sum_{r=1}^{n} A_r^p X_{nr}\right). \tag{21}$$

Therefore, preforming some simple calculations, we obtain

$$\left.\frac{\partial^2 F(n,n;s,t)}{\partial s^2}\right|_{(s_0,t_0)} = \frac{q^2}{p^2(p-q)} \left(\sum_{r=1}^{n} B_r^q X_{nr}\right)^{\frac{1}{q}} \left(\sum_{r=1}^{n} A_r^p Y_{nr}\right)^{\frac{3pq-2p^2-p}{q^2}} \times \left(\sum_{r=1}^{n} A_r^p X_{nr}\right)^{\frac{q^2-3pq+2p^2+p-q}{q^2}} > 0, \tag{22}$$

$$\left.\frac{\partial^2 F(n,n;s,t)}{\partial s \partial t}\right|_{(s_0,t_0)} = 0, \tag{23}$$

$$\left.\frac{\partial^2 F(n,n;s,t)}{\partial t^2}\right|_{(s_0,t_0)} = -pq \left(\sum_{r=1}^{n} B_r^q X_{nr}\right)^{\frac{p+2}{pq}} \left(\sum_{r=1}^{n} A_r^p Y_{nr}\right)^{\frac{q-2}{q^2}} \times \left(\sum_{r=1}^{n} A_r^p X_{nr}\right)^{\frac{q^2-pq-2q+2p}{pq^2}} > 0. \tag{24}$$

If we set $A = \left.\frac{\partial^2 F(n,n;s,t)}{\partial s^2}\right|_{(s_0,t_0)}$, $B = \left.\frac{\partial^2 F(n,n;s,t)}{\partial s \partial t}\right|_{(s_0,t_0)}$, and $C = \left.\frac{\partial^2 F(n,n;s,t)}{\partial t^2}\right|_{(s_0,t_0)}$, then from (22), (23) and (24) we obtain

$$AC - B^2 > 0. \tag{25}$$

Hence, from the above inequality (25) and (22) we have

$$\min_{t,s>0}\{F(n,n;s,t)\} = F(n,n;s_0,t_0), \tag{26}$$

and therefore

$$\min_{t,s>0}\{F(n,n;s,t)\} = F(n,n;s_0,t_0) = \sum_{r=1}^{n} A_r B_r \left(\sum_{k=1}^{n} C_{kr} D_{kr}\right) - \frac{\left(\sum_{r=1}^{n} A_r^p X_{nr}\right)^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{r=1}^{n} A_r^p Y_{nr}\right)^{\frac{1}{q}} \left(\sum_{r=1}^{n} B_r^q X_{nr}\right)}{q\left(\sum_{r=1}^{n} B_r^q X_{nr}\right)^{\frac{1}{p}}} - \frac{\left(\sum_{r=1}^{n} B_r^q X_{nr}\right)^{\frac{1}{q}} \left(\sum_{r=1}^{n} A_r^p X_{nr}\right)^{1-\frac{p}{q}} \left(\sum_{r=1}^{n} A_r^p Y_{nr}\right)}{q\left(\sum_{r=1}^{n} A_r^p Y_{nr}\right)^{\frac{p}{q^2}} \left(\sum_{r=1}^{n} A_r^p X_{nr}\right)^{\frac{1}{q}-\frac{p}{q^2}} \left(\sum_{r=1}^{n} A_r^p Y_{nr}\right)^{1-\frac{p}{q}}} - \frac{\left(\frac{1}{p}-\frac{1}{q}\right) \left(\sum_{r=1}^{n} B_r^q X_{nr}\right)^{\frac{1}{q}} \left(\sum_{r=1}^{n} A_r^p Y_{nr}\right)^{\frac{p}{q}} \left(\sum_{r=1}^{n} A_r^p X_{nr}\right)}{\left(\sum_{r=1}^{n} A_r^q Y_{nr}\right)^{\frac{p}{q^2}} \left(\sum_{r=1}^{n} A_r^p X_{nr}\right)^{\frac{1}{q}-\frac{p}{q^2}} \left(\sum_{r=1}^{n} A_r^p X_{nr}\right)^{\frac{p}{q}}}$$

$$
\begin{aligned}
&= \sum_{r=1}^{n} A_r B_r \left(\sum_{k=1}^{n} C_{kr} D_{kr} \right) \\
&\quad - \frac{1}{q} \left(\sum_{r=1}^{n} A_r^p X_{nr} \right)^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{r=1}^{n} A_r^p Y_{nr} \right)^{\frac{1}{q}} \left(\sum_{r=1}^{n} B_r^q X_{nr} \right)^{\frac{1}{q}} \\
&\quad - \frac{1}{p} \left(\sum_{r=1}^{n} B_r^q X_{nr} \right)^{\frac{1}{q}} \left(\sum_{r=1}^{n} A_r^p Y_{nr} \right)^{\frac{p}{q}-\frac{p}{q^2}} \left(\sum_{r=1}^{n} A_r^p X_{nr} \right)^{1-\frac{1}{q}-\frac{p}{q}+\frac{p}{q^2}} \\
&= \sum_{r=1}^{n} A_r B_r \left(\sum_{k=1}^{n} C_{kr} D_{kr} \right) \\
&\quad - \left(\sum_{r=1}^{n} A_r^p X_{nr} \right)^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{r=1}^{n} A_r^p Y_{nr} \right)^{\frac{1}{q}} \left(\sum_{r=1}^{n} B_r^q X_{nr} \right)^{\frac{1}{q}}. \qquad (27)
\end{aligned}
$$

Similarly, if we take

$$
s_1 = \left(\frac{\sum_{r=1}^{n+1} A_r^p X_{(n+1)r}}{\sum_{r=1}^{n+1} A_r^p Y_{(n+1)r}} \right)^{\frac{p(q-p)}{q^2}},
$$

$$
t_1 = \frac{1}{\left(\sum_{r=1}^{n+1} B_r^q X_{(n+1)r} \right)^{\frac{1}{pq}}} \left[\left(\sum_{r=1}^{n+1} A_r^p Y_{(n+1)r} \right)^{\frac{1}{q^2}} \left(\sum_{r=1}^{n+1} A_r^p X_{(n+1)r} \right)^{\frac{1}{pq}-\frac{1}{q^2}} \right],
$$

then, we have

$$
\begin{aligned}
F(n+1, n+1; s_1, t_1) &= \sum_{r=1}^{n+1} A_r B_r \left(\sum_{k=1}^{n+1} C_{kr} D_{kr} \right) \\
&\quad - \left(\sum_{r=1}^{n+1} A_r^p X_{(n+1)r} \right)^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{r=1}^{n+1} A_r^p Y_{(n+1)r} \right)^{\frac{1}{q}} \left(\sum_{r=1}^{n+1} B_r^q X_{(n+1)r} \right)^{\frac{1}{q}}. \qquad (28)
\end{aligned}
$$

From (16), we have

$$
F(n, n; s_1, t_1) \leqslant F(n+1, n+1; s_1, t_1). \qquad (29)
$$

Consequently, from (26) and (29) we obtain

$$
F(n, n; s_0, t_0) \leqslant F(n+1, n+1; s_1, t_1). \qquad (30)
$$

Let

$$
C_{kr} = A_k (1 - e_r + e_k)^{\frac{1}{p}}, \qquad D_{kr} = B_k (1 - e_r + e_k)^{\frac{1}{q}}.
$$

Then

$$
\begin{aligned}
F(n, n; s_0, t_0) &= \sum_{r=1}^{n} A_r B_r \sum_{k=1}^{n} A_k B_k (1 - e_r + e_k) - \left[\sum_{r=1}^{n} A_r^p \sum_{k=1}^{n} A_k^p (1 - e_r + e_k) \right]^{\frac{1}{p}-\frac{1}{q}} \\
&\quad \times \left[\sum_{r=1}^{n} A_r^p \sum_{k=1}^{n} B_k^q (1 - e_r + e_k) \right]^{\frac{1}{q}} \left[\sum_{r=1}^{n} B_r^q \sum_{k=1}^{n} A_k^p (1 - e_r + e_k) \right]^{\frac{1}{q}}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(\sum_{r=1}^{n} A_rB_r\Big)^2 - \Big[\Big(\sum_{r=1}^{n} A_r^p\Big)^2\Big]^{\frac{1}{p}-\frac{1}{q}} \\
&\quad \times \Big(\sum_{r=1}^{n} A_r^p \sum_{k=1}^{n} B_k^q - \sum_{r=1}^{n} A_r^p e_r \sum_{k=1}^{n} B_k^q + \sum_{r=1}^{n} A_r^p \sum_{k=1}^{n} B_k^q e_k\Big)^{\frac{1}{q}} \\
&\quad \times \Big(\sum_{r=1}^{n} B_r^q \sum_{k=1}^{n} A_k^p - \sum_{r=1}^{n} B_r^q e_r \sum_{k=1}^{n} A_k^p + \sum_{r=1}^{n} B_r^q \sum_{k=1}^{n} A_k^p e_k\Big)^{\frac{1}{q}} \\
&= \Big(\sum_{r=1}^{n} A_rB_r\Big)^2 - \Big(\sum_{r=1}^{n} A_r^p\Big)^{\frac{2}{p}-\frac{2}{q}} \Big\{\Big[\Big(\sum_{r=1}^{n} A_r^p\Big)\Big(\sum_{r=1}^{n} B_r^q\Big)\Big]^2 \\
&\quad - \Big[\Big(\sum_{r=1}^{n} A_r^p e_r\Big)\Big(\sum_{r=1}^{n} B_r^q\Big) - \Big(\sum_{r=1}^{n} B_r^q e_r\Big)\Big(\sum_{r=1}^{n} A_r^p\Big)\Big]^2\Big\}^{\frac{1}{q}}. \qquad (31)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&F(n+1,n+1;s_1,t_1) \\
&= \Big(\sum_{r=1}^{n+1} A_rB_r\Big)^2 - \Big(\sum_{r=1}^{n+1} A_r^p\Big)^{\frac{2}{p}-\frac{2}{q}} \Big\{\Big[\Big(\sum_{r=1}^{n+1} A_r^p\Big)\Big(\sum_{r=1}^{n+1} B_r^q\Big)\Big]^2 \\
&\quad - \Big[\Big(\sum_{r=1}^{n+1} A_r^p e_r\Big)\Big(\sum_{r=1}^{n+1} B_r^q\Big) - \Big(\sum_{r=1}^{n+1} B_r^q e_r\Big)\Big(\sum_{r=1}^{n+1} A_r^p\Big)\Big]^2\Big\}^{\frac{1}{q}}. \qquad (32)
\end{aligned}
$$

Combining (30), (31) and (32) leads to inequality (7). The proof of Theorem 2.2 is completed. □

REMARK 2.4. For $q > 0$ the inequality (7) is reversed. This result has appeared in the work of Hu [6].

REMARK 2.5. Note that if we set $n = 1$ in (6), then $F(1) = 0$. Thus inequality (3) can be obtained by Theorem 2.2. Similarly, if we set $t_1 = a$, $t_2 = b$ in (9), then we obtain inequality (4) by using Theorem 2.3.

In (6), taking $\big(\sum_{r=1}^{n} A_r^p e_r\big)\big(\sum_{r=1}^{n} B_r^q\big) = \big(\sum_{r=1}^{n} B_r^q e_r\big)\big(\sum_{r=1}^{n} A_r^p\big)$, from Theorem 2.2 and Remark 2.5, we get the following interesting property of Hölder's inequality.

COROLLARY 2.6. *Let* $A_r \geqslant 0$, $B_r > 0$ $(r = 1,2,\cdots)$, *let* $q < 0$, $\frac{1}{p} + \frac{1}{q} = 1$, *and let*

$$F(n) = \Big(\sum_{r=1}^{n} A_rB_r\Big)^2 - \Big(\sum_{r=1}^{n} A_r^p\Big)^{\frac{2}{p}} \Big(\sum_{r=1}^{n} B_r^q\Big)^{\frac{2}{q}}. \qquad (33)$$

Then

$$0 \leqslant F(n) \leqslant F(n+1). \qquad (34)$$

Similarly, taking $\int_a^t f^p(x)e(x)dx \int_a^t g^q(x)dx \equiv \int_a^t g^q(x)e(x)dx \int_a^t f^p(x)dx$ in (8), from Theorem 2.3, we get the integral form as follows:

COROLLARY 2.7. *Let $f(x)$, $g(x)$ be integrable functions defined on $[a,b]$ and $f(x) \geqslant 0$, $g(x) > 0$ for all $x \in [a,b]$, let $q < 0, \frac{1}{p} + \frac{1}{q} = 1$, and let*

$$G(t) = \left(\int_a^t f(x)g(x)\mathrm{d}x \right)^2 - \left(\int_a^t f^p(x)\mathrm{d}x \right)^{\frac{2}{p}} \left(\int_a^t g^q(x)\mathrm{d}x \right)^{\frac{2}{q}}. \tag{35}$$

Then we have

$$0 \leqslant G(t_1) \leqslant G(t_2), \quad a \leqslant t_1 \leqslant t_2 \leqslant b. \tag{36}$$

From Corollary 2.6 and Corollary 2.7, we obtain the refinement of Hölder's inequality (1) as follows.

COROLLARY 2.8. *Let $A_r \geqslant 0$, $B_r > 0$ $(r = 1,2,\cdots,n)$, let $q < 0$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\sum_{r=1}^n A_r B_r \geqslant \left(\sum_{r=1}^n A_r^p \right)^{\frac{1}{p}} \left(\sum_{r=1}^n B_r^q \right)^{\frac{1}{q}} (1+\rho)^{\frac{1}{2}}, \tag{37}$$

where $\rho = \frac{(A_1B_1+A_2B_2)^2-(A_1^p+A_2^p)^{\frac{2}{p}}(B_1^q+B_2^q)^{\frac{2}{q}}}{(\sum_{r=1}^n A_r^p)^{\frac{2}{p}}(\sum_{r=1}^n B_r^q)^{\frac{2}{q}}} \geqslant 0.$

COROLLARY 2.9. *Let $f(x)$, $g(x)$ be integrable functions defined on $[a,b]$ and $f(x) \geqslant 0$, $g(x) > 0$ for all $x \in [a,b]$, and let $q < 0$, $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_a^b f(x)g(x)\mathrm{d}x \geqslant \left(\int_a^b f^p(x)\mathrm{d}x \right)^{\frac{1}{p}} \left(\int_a^b g^q(x)\mathrm{d}x \right)^{\frac{1}{q}} (1+\theta)^{\frac{1}{2}}, \tag{38}$$

where $\theta = \frac{\left(\int_a^{\frac{a+b}{2}} f(x)g(x)\mathrm{d}x\right)^2 - \left(\int_a^{\frac{a+b}{2}} f^p(x)\mathrm{d}x\right)^{\frac{2}{p}} \left(\int_a^{\frac{a+b}{2}} g^q(x)\mathrm{d}x\right)^{\frac{2}{q}}}{\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{2}{p}} \left(\int_a^b g^q(x)\mathrm{d}x\right)^{\frac{2}{q}}} \geqslant 0.$

3. Application

In this section, we present a new property of Minkowski-type inequality from inequality (36). Furthermore, we give a new refinement of Minkowski's inequality by using the result.

The classical Minkowski inequality is an important result in theoretical and applied fields. Since Minkowski discovered this inequality, it has motivated a large number of research papers giving its successively simpler proofs, providing various refinements and generalizations, and finding its series analogues in different areas of mathematics. The interested reader may refer to [2], [3], [7], and [12] and references therein. Applications of Minkowski's inequality have been studied by many authors. For example, Lu et al. [8] used Minkowski's inequality for fast full search in motion estimation. So it is of interest to develop its refinement.

THEOREM 3.1. *Let $f(x)$, $g(x)$ be integrable functions defined on $[a,b]$ and $f(x) > 0$, $g(x) > 0$ for all $x,y \in [a,b]$, let $0 < p < 1$, and let*

$$M(t) = \left[\int_a^t (f(x)+g(x))^{p-1} f(x)\mathrm{d}x\right]^2 + \left[\int_a^t (f(x)+g(x))^{p-1} g(x)\mathrm{d}x\right]^2 - \left[\left(\int_a^t f^p(x)\mathrm{d}x\right)^{\frac{2}{p}} + \left(\int_a^t g^p(x)\mathrm{d}x\right)^{\frac{2}{p}}\right]\left[\int_a^t (f(x)+g(x))^p \mathrm{d}x\right]^{2-\frac{2}{p}}. \tag{39}$$

Then we have

$$0 \leqslant M(t_1) \leqslant M(t_2), \quad a \leqslant t_1 \leqslant t_2 \leqslant b. \tag{40}$$

Proof. Denote

$$M_f(t) = \left[\int_a^t (f(x)+g(x))^{p-1} f(x)\mathrm{d}x\right]^2 - \left[\int_a^t f^p(x)\mathrm{d}x\right]^{\frac{2}{p}}\left[\int_a^t (f(x)+g(x))^p \mathrm{d}x\right]^{2-\frac{2}{p}},$$

$$M_g(t) = \left[\int_a^t (f(x)+g(x))^{p-1} g(x)\mathrm{d}x\right]^2 - \left[\int_a^t g^p(x)\mathrm{d}x\right]^{\frac{2}{p}}\left[\int_a^t (f(x)+g(x))^p \mathrm{d}x\right]^{2-\frac{2}{p}}.$$

From Corollary 2.7, we obtain that $M_f(t)$ and $M_g(t)$ are monotone increasing functions of t on $[a,b]$. Since

$$M(t) = M_f(t) + M_g(t),$$

we have

$$0 \leqslant M(t_1) \leqslant M(t_2), \quad a \leqslant t_1 \leqslant t_2 \leqslant b.$$

The proof of Theorem 3.1 is completed.

REMARK 3.2. On the one hand, from Theorem 3.1, we have $M(b) \geqslant 0$. On the other hand, from Hölder's inequality (1) we obtain

$$\left[\int_a^b (f(x)+g(x))^{p-1} f(x)\mathrm{d}x\right]\left[\int_a^b (f(x)+g(x))^{p-1} g(x)\mathrm{d}x\right] \geqslant \left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_a^t g^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\left[\int_a^b (f(x)+g(x))^p \mathrm{d}x\right]^{2-\frac{2}{p}}. \tag{41}$$

Therefore, from the above inequality and $M(b) \geqslant 0$, we obtain Minkowski's inequality for $0 < p < 1$.

From Theorem 3.1 and Remark 3.2, we get the following refinement of Minkowski's inequality:

COROLLARY 3.3. *Let* $f(x)$, $g(x)$, $M(t)$, p *be as in Theorem* 3.1. *Then*

$$\left[\int_a^b \big(f(x)+g(x)\big)^p f(x)\mathrm{d}x\right]^2$$
$$\geqslant \left[\left(\int_a^b f^p(x)\mathrm{d}x\right)^{\frac{1}{p}} + \left(\int_a^t g^p(x)\mathrm{d}x\right)^{\frac{1}{p}}\right]^2 \left[\int_a^b \big(f(x)+g(x)\big)^p \mathrm{d}x\right]^{2-\frac{2}{p}}$$
$$+M\left(\frac{a+b}{2}\right). \tag{42}$$

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