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CARLEMAN ESTIMATES AND UNIQUE CONTINUATION PROPERTY FOR ELLIPTIC OPERATORS IN BANACH SPACES

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Abstract. The unique continuation theorems for elliptic differential-operator equations in Banach-valued L_p-space are investigated. The operator-valued multiplier theorems and the Carleman estimates for the equations are employed to obtain these results. In applications the unique continuation theorems for anisotropic elliptic differential equations and finite or infinite systems of elliptic equations are studied.

1. Introduction

The aim of this paper is to present a unique continuation result for solutions of the differential inequalities of the form:

$$\|Lu(x)\|_E \leqslant \|V(x)u(x)\|_E, \tag{1}$$

where

$$Lu = \sum_{j=1}^{n} a_j \frac{\partial^2 u}{\partial x_j^2} + Au,$$

here a_j are real numbers, A, $V(x)$ are the possible linear operators in a Banach space E.

We will prove that if $n\left(\frac{1}{p}-\frac{1}{p'}\right) \leqslant 2$, $\mu = \frac{n}{2}$, $V \in L_\mu(R^n; L(E))$, $n \geqslant 3$ and $u \in W_p^2(R^n; E(A), E)$ satisfies (1), then u is identically zero if its support is contained in a half space, where $W_p^2(R^n; E(A), E)$ is an E-valued Sobolev-Lions type space. We prove Carleman estimates in E-valued L_p spaces to obtain unique continuation. Carleman estimates initiated by the works [8] and [2]. Jerison and Kenig studied the theory of L_p Carleman estimates for Laplace operator with potential and proved unique continuation results for elliptic constant coefficient operators in [13]. This was latter generalized to elliptic variable coefficient operators by Sogge in [27, 28]. There were further improvement by Wolff [33] for elliptic operators with less regular coefficients and by Koch and Tataru [14] who considered the problem with gradients terms. A comprehensive introductions and historical references to Carleman estimates and unique continuation properties may be found e.g. in [14]. Moreover, boundary value problems for differential-operator equations (DOEs) have been studied extensively by many researchers (see [1], [9, 10], [18], [15], [23–26], [34–36] and the references therein).

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2. Notations, definitions and background

Assume $\mathbf{R}$ and $\mathbf{C}$ denote the set of all real numbers and complex numbers, respectively. Let

$$S_{\varphi}=\{\ \xi\in\mathbf{C},\ |\arg\xi|\leqslant\varphi\}\cup\{0\},\varphi\in[0\ ,\pi).$$

Let E and E_1 be two Banach spaces, and $L(E,E_1)$ denotes the spaces of all bounded linear operators from E to E_1. For $E_1=E$ we denote $L(E,E_1)$ by $L(E)$.

A linear operator A is said to be a φ-positive in a Banach space E with bound $M>0$ if $D(A)$ is dense on E and

$$\left\|(A+\xi I)^{-1}\right\|_{L(E)}\leqslant M(1+|\xi|)^{-1}$$

with $\lambda\in S_{\varphi},\varphi\in[0\ ,\pi)$, I is identity operator in E. We will sometimes use $A+\xi$ or A_{ξ} instead of $A+\xi I$ for a scalar ξ and $(A+\xi I)^{-1}$ denotes the inverse of the operator $A+\xi I$ or the resolvent of operator A. It is known $[32,\S1.15.1]$ that there exist fractional powers A^{θ} of a positive operator A and

$$E\left(A^{\theta}\right)=\left\{u\in D\left(A^{\theta}\right),\ \|u\|_{E\left(A^{\theta}\right)}=\left\|A^{\theta}u\right\|_{E}+\|u\|<\infty,\ -\infty<\theta<\infty\right\}.$$

We denote by $L_p(\Omega;E)$ the space of all strongly measurable E-valued functions on Ω with the norm

$$\|u\|_{p,E}=\|u\|_{L_p(\Omega;E)}=\left(\int_{\Omega}\|u(x)\|_E^p\ dx\right)^{1/p},\quad 1\leqslant p<\infty.$$

By $L_{p,q}(\Omega)$ and $W_{p,q}^{l}(\Omega)$ let us denote respectively the (p,q)-integrable function space and Sobolev space with mixed norms, where $1\leqslant p,q<\infty$ (see e.g. $[3,\S\ 1{,}10]$).

Let E_0 and E be two Banach spaces and E_0 is continuously and densely embedded E. Let l be a positive integer.

We introduce an E-valued function space $W_p^l(\Omega;E_0,E)$ (sometimes we call it Sobolev-Lions type space, see $[22]$) that consist of all functions $u\in L_p(\Omega;E_0)$ such that are endowed with the generalized derivatives $D_k^l u=\frac{\partial^l u}{\partial x_k^l}\in L_p(\Omega;E)$ with the norm

$$\|u\|_{W_p^l(\Omega;E_0,E)}=\|u\|_{L_p(\Omega;E_0)}+\sum_{k=1}^{n}\left\|D_k^l u\right\|_{L_p(\Omega;E)}<\infty,\quad 1\leqslant p<\infty.$$

The Banach space E is called an UMD-space if the Hilbert operator

$$(Hf)(x)=\lim_{\varepsilon\to0}\int\limits_{|x-y|>\varepsilon}\frac{f(y)}{x-y}dy$$

is bounded in $L_p(R,E)$, $p\in(1,\infty)$ (see. e.g. [5–6]). UMD spaces include e.g. L_p, l_p spaces and Lorentz spaces L_{pq}, p, $q\in(1,\infty)$.

Let E_1 and E_2 be two Banach spaces. Let $S(R^n;E)$ denotes a Schwartz class i.e. the space of all E-valued rapidly decreasing smooth functions on R^n. Let F and F^{-1} denote Fourier and inverse Fourier transformations, respectively. A function $\Psi \in C^m(R^n;L(E_1,E_2))$ is called a multiplier from $L_p(R^n;E_1)$ to $L_p(R^n;E_2)$ for $p \in (1,\infty)$ if the map $u \to Ku = F^{-1}\Psi(\xi)Fu,\ u \in S(R^n;E_1)$ is well defined and extends to a bounded linear operator

$$K:\ L_p(R^n;E_1) \to\ L_p(R^n;E_2).$$

We denote the set of all multipliers from $L_p(R^n;E_1)$ to $L_p(R^n;E_2)$ by $M_p^p(E_1,E_2)$. For $E_1 = E_2 = E$ it is denoted by $M_p(E)$. The exposition of the theory of L_p-multipliers of the Fourier transformation, and some related references, can be found in [32, §2.2.1–§2.2.4]. On the other hand, Fourier multipliers in vector-valued function spaces, have been studied e.g. by [7], [10], [12], [17–21], [31].

A set $K \subset L(E_1,E_2)$ is called R-bounded [6,7] if there is a constant C such that for all $T_1,T_2,...,T_m \in K$ and $u_1,u_2,...,u_m \in E_1,\ m \in N$

$$\int_0^1 \left\| \sum_{j=1}^m r_j(y)\, T_j u_j \right\|_{E_2} dy \leqslant C \int_0^1 \left\| \sum_{j=1}^m r_j(y)\, u_j \right\|_{E_1} dy,$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1,1\}$-valued random variables on $[0,1]$. The smallest C for which the above estimate holds is called a R-bound of the collection K and denoted by $R(K)$.

Let

$$U_n = \left\{\beta = (\beta_1,\beta_2,...\beta_n),\ \beta_j \in \{0,1\},\ j = 1,2,...,n\right\},$$
$$\xi^\beta = \xi_1^{\beta_1}\xi_2^{\beta_2}...\xi_n^{\beta_n},\ |\beta| = \sum_{j=1}^n \beta_j.$$

DEFINITION 1. The Banach space E is said to be a space satisfying a multiplier condition with respect to p when for $\Psi \in C^{(n)}(R^n;L(E_1,E_2))$ if the set

$$\left\{\xi^{|\beta|} D^\beta \Psi(\xi) : \xi \in R^n \backslash \{0\},\ \beta \in U_n\right\}$$

is R-bounded, then $\Psi \in M_p^p(E_1,E_2)$.

DEFINITION 2. The φ-positive operator A is said to be a R-positive in a Banach space E if there exists $\varphi \in [0\ ,\pi)$ such that the set

$$L_A = \left\{\xi\,(A+\xi I)^{-1} : \xi \in S_\varphi\right\}$$

is R-bounded.

REMARK 1. By virtue of [12] or [34], UMD spaces satisfy the multiplier condition with respect to $p \in (1,\infty)$. Note that, in Hilbert spaces every norm bounded set is R-bounded. Therefore, in Hilbert spaces all positive operators are R-positive. If A is a generator of a contraction semigroup on L_q, $1 \leqslant q \leqslant \infty$ 1 or A has the bounded imaginary powers with $\left\|\left(-A^{it}\right)\right\|_{L(E)} \leqslant Ce^{\nu|t|}, \nu < \frac{\pi}{2}$ or if A is a generator of a semigroup with Gaussian bound (see e.g. [10]) then those operators are R-positive. It is well known (see e.g. [17]) that any Hilbert space satisfies the multiplier condition with respect to $p \in (1,\infty)$. By virtue of [21] Mikhlin conditions are not sufficient for operator-valued multiplier theorem.

Let $H_k = \left\{\Psi_h \in M_p^p(E_1,E_2),\ h = (h_1 h_2 ..., h_n) \in K\right\}$ be a collection of multipliers in $M_p^p(E_1,E_2)$. We say that H_k is a uniform collection of multipliers if there exists a constant $M > 0$, independent on $h \in K$, such that

$$\left\|F^{-1}\Psi_h F u\right\|_{L_p(R^n;E_2)} \leqslant M \|u\|_{L_p(R^n;E_1)}$$

for all $h \in K$ and $u \in S(R^n;E_1)$.

In view of [23, Theorem A_0] we have

THEOREM A_0. *Let E_1 and E_2 be two UMD spaces and let*

$$\Psi \in C^{(n)}\left(R^n \backslash \{0\}; L(E_1,E_2)\right),\ p \in (1,\infty).$$

If

$$\sup_{h\in K} R\left\{\xi^{|\beta|} D_\xi^\beta \Psi_h(\xi) : \xi \in R^n \backslash \{0\}, \beta \in U_n\right\} \leqslant K_\beta < \infty$$

then $\Psi_h(\xi)$ is a uniformly collection of multipliers from $L_p(R^n;E_1)$ to $L_p(R^n;E_2)$.

Let $Q(\xi) = \sum_{j=1}^{n} a_j \xi_j^2$ denote the symbol of differential operator L in (1). Let

$$S_\pm = \{\xi \in R^n,\ Q(\xi) = \pm 1\}.$$

In a similar way as [29] we obtain abstract version of Stein-Tomas type restriction result

THEOREM A_1. *Let E be an UMD space. For $\frac{1}{p} - \frac{1}{p'} \geqslant \frac{2}{n+1}$ the following inequality*

$$\left\| \int_{S_\pm^{n-1}} \hat{f}(\omega) e^{i(x,\omega)} d\omega_\pm \right\|_{p',E} \leqslant C \|f\|_{p,E}$$

is satisfied.

By virtue of [9, Lemma 2.3] we have:

LEMMA A_1. *For $\lambda \in S_\varphi$ and $\xi \in S_{\varphi_1}$, $\varphi_1 + \varphi < \pi$ there is a positive constant C such that*

$$|\lambda + \xi| \geqslant C(|\lambda| + |\xi|). \tag{2}$$

3. Carleman estimates for DOE with constant coefficients

Consider at first, the following DOE

$$L(D)u=\sum_{j=1}^{n}a_jD_j^2u(x)+Au(x)=f(x),\ x\in R^n, \tag{3}$$

where $D_j=\frac{\partial}{i\partial_j}$, A is a linear operator in a Banach space E, $a=(a_1,a_2,...,a_n)$ and a_k are certain real numbers.

Let $\nu=(\nu_1,\nu_2,...,\nu_n)$ is a unique vector in R^n and t is a positive parameter. It is clear to see that

$$e^{t(x,\nu)}L\left[ue^{-t(x,\nu)}\right]=L(D+it\nu)=\sum_{j=1}^{n}a_j\left(\frac{\partial}{i\partial_j}+it\nu_j\right)^2u(x)+Au(x). \tag{4}$$

Consider the following DOE with parameters

$$L(D+it\nu)u=\sum_{j=1}^{n}a_j\left(\frac{\partial}{i\partial_j}+it\nu_j\right)^2+Au(x)=f(x),\ x\in R^n. \tag{5}$$

CONDITION 1. Assume $n\geqslant 3$, $p\in(1,\infty)$ and $\frac{2}{n+1}\leqslant\frac{1}{p}-\frac{1}{p'}\leqslant\frac{2}{n}$. Suppose there is a positive constant C such that $Q(\xi)=\sum_{j=1}^{n}a_j\xi_j^2\geqslant C\sum_{j=1}^{n}\xi_j^2$, for all $\xi\neq 0$.

First at all, we need the following lemma, which is proved in a similar way as [16, Theorem 2.3].

LEMMA 3.1. *Assume the Condition1 holds. Moreover, let E be a Banach space satisfies the multiplier condition with respect to p and A be an R-positive operator in E. Then there is a positive constant C such that for all $z\in\mathbf{C}$ with $|z|\geqslant 1$,*

$$\|u\|_{L_{p'}(R^n;E)}\leqslant C\|(L+z)u\|_{L_p(R^n;E)},\quad u\in W_p^2(R^n;E(A),E). \tag{6}$$

Proof. First of all let us notice that by a limiting argument we only have to prove (6) for $|z|\geqslant 1$ with $\operatorname{Im}z\neq 0$. Since symbol does not vanish, (6) is equivalent to following

$$\left\|F^{-1}\Phi_z^{-1}(\xi)\hat{f}(\xi)\right\|_{p',E}\leqslant C\|f\|_{p,E}, \tag{7}$$

where

$$\Phi_z(\xi)=(A+Q(\xi)+z).$$

To prove this inequality we use E. Stein's theorem on analytic interpolation [30]. Consider the family of operators

$$T_\lambda f=F^{-1}G(\lambda,z)\hat{f}(\xi),\ G(\lambda,z)=\frac{e^{\lambda^2}}{\Gamma\left(\frac{n}{2}+\lambda\right)}\Phi_z^\lambda(\xi).$$

By Stein interpolation theorem, the estimate (7) can be follow from the following two uniform estimates:

$$\left\|F^{-1}\Phi_z^{-1}(\xi)\hat{f}(\xi)\right\|_{2,E} \leqslant C\|f\|_{2,E}, \text{ for } \mathrm{Re}\lambda = 0, \tag{8}$$

$$\left\|F^{-1}\Phi_z^{-1}(\xi)\hat{f}(\xi)\right\|_{\infty,E} \leqslant C\|f\|_{1,E}, \text{ for } \mathrm{Re}\lambda \in \left[-\frac{n+1}{2}, -\frac{n}{2}\right], \tag{9}$$

where

$$|z| \geqslant 1, \ \mathrm{Im}z \neq 0.$$

The inequality (8) follows from Plancherel theorem. Also, if $\Phi_\lambda(x,z)$ denotes the Fourier transform of $G(\lambda,z)$ then (9) would follow from the following

$$\|\Phi_\lambda\|_{\infty,L(E)} \leqslant C, \ \mathrm{Re}\lambda \in \left[-\frac{n+1}{2}, -\frac{n}{2}\right], \ |z| \geqslant 1, \ \mathrm{Im}z \neq 0. \tag{10}$$

In a similar way as in [11, p. 288–289] we get that

$$\Phi_\lambda(x,z) = \frac{e^{\lambda^2}2^{\lambda+1}e^{-\pi i\eta/2}}{(2\pi)^{\frac{n}{2}}\Gamma(-\lambda)\Gamma\left(\frac{n}{2}+\lambda\right)}\left[zH^{-1}(x)\right]^{1/2(n/2+\lambda)}K_{\frac{n}{2+\lambda}}(zH)^{\frac{1}{2}},$$

where

$$H(x) = \left(A + \sum_{j=1}^{n} a_j x_j^2\right),$$

K_ν is the operator valued Bessel potential defined by

$$K_\nu u = \int_0^\infty e^{-u\cosh t}\cosh(\nu t)\,dt, \ u \in L(E)$$

and η is the signature of $Q(\xi)$. The above integral well defined. Really, due to positivity of A, the operator $H(x)$ has a bounded inverse $H^{-1}(x)$ for all $x \in R^n$, and is uniformly positive in E. Hence, there is the fractional powers of H and the operator $(zH)^{\frac{1}{2}}$ for $|z| \geqslant 1$, $\mathrm{Im}z \neq 0$ generates an anaclitic semigroup i.e. the operator function $U_z(t) = e^{-t(zH)^{\frac{1}{2}}}$ is uniformly bounded in E and has the estimate

$$\|U_z(t)\|_{L(E)} \leqslant Ce^{-\omega t}, \ \omega > 0.$$

Moreover, by reasoning as in [16, Theorem 2.3] and by using the resolvent properties of the positive operator A, the estimate (10) is obtained. □

THEOREM 3.1. *Assume the Condition1 holds. Suppose E is a Banach space satisfies the multiplier condition with respect to p and A is an R-positive operator in E. Then for $t \geqslant t_0$ there is a positive constant C (depending only on n and p) such that the estimate holds*

$$\|u\|_{L_{p'}(R^n;E)} \leqslant C\|L(D+it\nu)u\|_{L_p(R^n;E)}, \ u \in W_p^2(R^n;E(A),E).$$

Proof. By applying the Fourier transform we have from (3)

$$L(\xi+it\nu)\hat{u}(\xi)=\hat{f}(\xi),L(\xi+it\nu)=\sum_{j=1}^{n}a_j(\xi_j+it\nu_j)^2+A.$$

Without loss of generality we put $\nu=(1,0,...,0)$. Let

$$\begin{aligned}\psi_t(\xi)&=\sum_{j=1}^{n-1}a_j\xi_j^2+a_n(\xi_n+it)^2=\sum_{j=1}^{n-1}a_j\xi_j^2\ +a_n\left(\xi_n^2-t^2\right)+2a_n\xi_nti\\&=Q(\xi)+a_n\left(2\xi_nti-t^2\right).\end{aligned}$$

Since $F^{-1}B_t^{-1}(\xi)f^{\hat{}}$ is a fundamental solution for the operator $L(D+it\nu)$, hence, we obtain that the solution of the equation (3) can be represented in the form $u(x)=F^{-1}B_t^{-1}f^{\hat{}}$. Moreover the assertion of theorem can be obtained from the following uniform estimate

$$\left\|F^{-1}B_t^{-1}(\xi)\hat{f}(\xi)\right\|_{p',E}\leqslant C\|f\|_{p,E}. \tag{11}$$

It is clear to see that $\psi_t(\xi)\in S(\varphi)$ for $|\xi_n|\geqslant t$. Due to positivity of A, the operator $B_t(\xi)$ is invertible in E for $|\xi_n|\geqslant t$. Also, consider the function $\chi\in C_0^\infty(R)$ such that $\chi(y)=1$ if $|y|\in[1,2]$ and zero otherwise; moreover, $\sum\limits_{j=-\infty}^{\infty}\chi\left(2^{-j}y\right)=1$ for $y\neq 0$. Set $\chi_k(\xi_n)=\chi\left(2^k(\xi_n-t)\right)$ and let

$$\Phi_{k,t}(\xi)=\chi_k(\xi_n)B_t^{-1}(\xi),\ B_t(\xi)=A+\psi_t(\xi).$$

In order to use the localization argument, it is suffices to prove that there is a constant C, independent of k and t, for which

$$\left\|F^{-1}\Phi_{k,t}(\xi)\hat{f}(\xi)\right\|_{p',E}\leqslant C\|f\|_{p,E}. \tag{12}$$

If the above estimates hold then by using Minkowski's integral inequality and Littlewood-Paley theory (see e.g. [28]) it is not hard to see that

$$\begin{aligned}\left\|F^{-1}B_t^{-1}(\xi)\hat{f}(\xi)\right\|_{p',E}&\leqslant C\left\|\left(\sum_{k=-\infty}^{\infty}\left\|F^{-1}\Phi_{k,t}(\xi)\hat{f}(\xi)\right\|_E^2\right)^{\frac{1}{2}}\right\|_{p'}\\&\leqslant C\left(\sum_{k=-\infty}^{\infty}\left\|F^{-1}\Phi_{k,t}(\xi)\hat{f}(\xi)\right\|_{p',E}^2\right)^{\frac{1}{2}}\\&\leqslant C\left(\sum_{k=-\infty}^{\infty}\left\|F^{-1}\chi_k(\xi_n)\hat{f}(\xi)\right\|_{p,E}^2\right)^{\frac{1}{2}}\leqslant C\|f\|_{p,E}.\end{aligned}$$

Consider operator functions

$$G_{k,t}(\xi)=\chi_k(\xi_n)B_{kt}^{-1}(\xi),$$

where

$$B_{kt}(\xi)=A+Q(\xi)+a_n\left[2t\left(t+2^{-k}\right)i-t^2\right].$$

By virtue of Lemma 3.1 we have

$$\left\|F^{-1}G_{k,t}(\xi)\hat{f}(\xi)\right\|_{p,E}\leqslant C\|f\|_{p,E}. \tag{13}$$

Hence, by taking differences (13) implies that we only have to prove the estimate

$$\left\|F^{-1}\left[g_{kt}B_t^{-1}B_{kt}^{-1}\right]\hat{f}(\xi)\right\|_{p',E}\leqslant C\|f\|_{p,E}, \tag{14}$$

where

$$g_{kt}=a_n\left[2t\left(\xi_n-t-2^{-k}\right)i\right]\chi_k(\xi_n).$$

Let T_k be the operator in (14). Then if we uses the polar coordinates, $\xi=\rho\omega$, associated to Q, it is easy to see that Minkowski's integral inequality and the restriction Theorem A_1 give the following

$$\begin{aligned}\|T_kf\|_{p',E}&\leqslant\sum_{\pm}\int_0^{\infty}\left\|\int_{S_{\pm}^{n-1}}B_{t\rho}^{-1}B_{kt\rho}^{-1}g_{kt}(\xi_n)\hat{f}(\rho\omega)e^{i\rho(\omega,x)}d\omega_{\pm}\right\|_{p',E}\rho^{n-1}d\rho\\&\leqslant C\sum_{\pm}\left\|\int_0^{\infty}\rho^{n-1-\frac{2n}{p'}}F^{-1}\left[B_{t\rho}^{-1}B_{kt\rho}^{-1}g_{kt}(\xi_n)\hat{f}(\xi)\right]\right\|_{p,E}d\rho,\end{aligned}$$

where

$$B_{t\rho}=\left[A+\rho^2+a_n\left(2\xi_nti-t^2\right)\right],\quad B_{kt\rho}=\left[A+\rho^2+a_n\left(2t\left(t+2^{-k}\right)i-t^2\right)\right].$$

It is easy that $n-1-\frac{2n}{p'}=1$. Moreover, by positivity of the operator A and by definition of $\chi_k(\xi_n)$ we obtain

$$\begin{aligned}\|T_kf\|_{p',E}&\leqslant C\sum_{\pm}\left\|\int_0^{\infty}\rho^{n-1-\frac{2n}{p'}}\left\|B_{t\rho}^{-1}B_{kt\rho}^{-1}\right\|_{L(E)}F^{-1}\left[g_{kt}(\xi_n)\hat{f}(\xi)\right]\right\|_{p,E}d\rho\\&\leqslant\frac{C}{2}\|f\|_{p,E}\int_0^{\infty}\frac{t2^{-k}\rho d\rho}{\left(1+\left|\rho^2+\varphi_1(t,k)\right|\right)\left(1+\left|\rho^2+\varphi_2(t,k)\right|\right)},\end{aligned}$$

where

$$\varphi_1(t,k)=a_n\left(2i\left(t+2^{-(k-1)}\right)t-t^2\right),\ \varphi_2(t,k)=a_n\left(2i\left(t+2^{-k}\right)t-t^2\right).$$

From above and in view of (2) we get

$$\|T_k f\|_{p',E} \leqslant \frac{C}{2}\|f\|_{p,E}\int_0^\infty \frac{2^{-k}\rho d\rho}{\left(\frac{1}{t}+\left|\frac{1}{t}\rho^2+\psi_1(t,k)\right|\right)\left(\frac{1}{t}+\left|\frac{1}{t}\rho^2+\psi_2(t,k)\right|\right)}$$
$$\leqslant \frac{C}{2}\|f\|_{p,E}\int_0^\infty \frac{2^{-k}\rho d\rho}{\left(\frac{1}{t_0}+\frac{1}{t_0}\rho^2+|\psi_1(t,k)|\right)\left(\frac{1}{t_0}+\frac{1}{t_0}\rho^2+|\psi_2(t,k)|\right)} \leqslant C\|f\|_{p,E},$$

where

$$\psi_1(t,k) = a_n\left(2i\left(t+2^{-(k-1)}\right)-t\right),\ \psi_2(t,k) = a_n\left(2i\left(t+2^{-k}\right)-t\right). \quad \square$$

From the above theorem we obtain:

RESULT 3.1. Let all conditions of Theorem 3.1 hold. Then following uniform Carleman type estimate

$$\left\|e^{-t(x,\nu)}u\right\|_{L_{p'}(R^n;E)} \leqslant C\left\|e^{-t(x,\nu)}Lu\right\|_{L_p(R^n;E)} \tag{15}$$

holds for all $u \in W_p^2(R^n;E(A),E)$.

Now by using the Carleman estimate (15) we obtain:

THEOREM 3.2. *Let all conditions of Theorem 3.1 is satisfied. Then the differential operator* (3) *has a unique continuation property.*

Proof. Let $V \in L_\mu(R^n;L(E))$ and $\frac{1}{\mu} = \frac{1}{p} - \frac{1}{p'}$ and let $u \in W_p^2(R^n;E(A),E)$ is a solution of the following differential inequality

$$\|L(D)u(x)\|_E \leqslant \|V(x)u(x)\|_E. \tag{16}$$

For simplicity of notation, we assume that u is supported in the half space

$$R_{1+}^n = \left\{x = \left(x_1, x'\right) \in R^n,\ x_1 > 0\right\}$$

since the technique for other cases is similar. To prove that $u \equiv 0$ in R^n it is sufficient to show that there is $\varepsilon > 0$ so that $u \equiv 0$ in the strip

$$S_\varepsilon = \{x \in R^n,\ x_1 \leqslant \varepsilon\}.$$

Let us take ε so small that, if V is as above and the constant C in (14) such that

$$C\|V\|_{L_\mu(S_\varepsilon;B(E))} \leqslant \eta < 1. \tag{17}$$

In view of the estimates $(15),(17)$ and by using the Hölder inequality we obtain

$$\left\|e^{-tx_1}u\right\|_{L_{p'}(S_\varepsilon;E)} \leqslant C\left\|e^{-tx_1}Lu\right\|_{L_p(R^n;E)} \leqslant C\left\|e^{-tx_1}Vu\right\|_{L_p(S_\varepsilon;E)}$$
$$\leqslant C\left\|e^{-tx_1}Vu\right\|_{L_p(S_\varepsilon;E)} + C\left\|e^{-tx_1}Lu\right\|_{L_p(R^n/S_\varepsilon;E)}$$
$$\leqslant \eta\left\|e^{-tx_1}u\right\|_{L_{p'}(S_\varepsilon;E)} + C\left\|e^{-tx_1}Lu\right\|_{L_p(R^n/S_\varepsilon;E)}$$

uniformly with respect to t. Hence, we have the uniform estimate

$$\left\|e^{t(\varepsilon-x_1)}u\right\|_{L_{p'}(S_\varepsilon;E)} \leqslant \frac{C}{1-\eta}\left\|e^{-tx_1}Lu\right\|_{L_p(R^n/S_\varepsilon;E)},$$

and consequently,

$$\left\|e^{t(\varepsilon-x_1)}u\right\|_{L_{p'}(S_\varepsilon;E)} \leqslant \frac{C}{1-\eta}\left\|L_s u\right\|_{L_p(R^n;E)}.$$

Since the above inequality holds for every $t>0$ this implies $u\equiv 0$ in S_ε. □

4. Carleman estimates and unique continuation property for anisotropic elliptic PDE

Let $\Omega\subset R^l$ be an open connected set with compact C^{2m}-boundary $\partial\Omega$. Let us consider the following BVP on cylindrical domain $\tilde{\Omega}=R^n\times\Omega$ for the following PDE

$$Lu=\sum_{k=1}^{n}a_k D_k^2 u(x,y)+\sum_{|\alpha|\leqslant 2m}a_\alpha(y)D_y^\alpha u(x,y)=f(x,y), \tag{18}$$

$$x\in R^n,\ y\in\Omega,$$

$$B_j u=\sum_{|\beta|\leqslant m_j}b_{j\beta}(y)D_y^\beta u(x,y)=0,\ x\in R^n,\ y\in\partial\Omega,\ j=1,2,...,m, \tag{19}$$

where $D_j=-i\frac{\partial}{\partial y_j}$, $y=(y_1,...,y_l)$. Let $\Omega\subset R^l$ be an open connected set with compact C^{2m}-boundary $\partial\Omega$.

THEOREM 4.1. *Let the following conditions be satisfied;*

(1) $a_\alpha\in C\left(\bar{\Omega}\right)$ *for each* $|\alpha|=2m$ *and* $a_\alpha\in\left[L_\infty+L_{r_k}\right](\Omega)$ *for each* $|\alpha|=k<2m$ *with* $r_k\geqslant q$ *and* $2m-k>\frac{l}{r_k}$;

(2) $b_{j\beta}\in C^{2m-m_j}(\partial\Omega)$ *for each* j, β *and* $m_j<2m$, $\sum_{j=1}^{m}b_{j\beta}\left(y^{'}\right)\sigma_j\neq 0$, *for* $|\beta|=m_j$, $y^{'}\in\partial G$, *where* $\sigma=(\sigma_1,\sigma_2,...,\sigma_l)\in R^l$ *is a normal to* $\partial\Omega$;

(3) *for* $y\in\bar{\Omega}$, $\xi\in R^l$, $\lambda\in S(\varphi)$, $\varphi\in\left(0,\frac{\pi}{2}\right)$, $|\xi|+|\lambda|\neq 0$ *let* $\lambda+\sum_{|\alpha|=2m}a_\alpha(y)\xi^\alpha\neq 0$;

(4) *for each* $y_0\in\partial\Omega$ *local BVP in local coordinates corresponding to* y_0

$$\lambda+\sum_{|\alpha|=2m}a_\alpha(y_0)D^\alpha\vartheta(y)=0,$$

$$B_{j0}\vartheta=\sum_{|\beta|=m_j}b_{j\beta}(y_0)D^\beta u(y)=h_j,\ j=1,2,...,m$$

has a unique solution $\vartheta \in C_0(R_+)$ *for all* $h = (h_1, h_2, ..., h_l) \in R^l$, *and for* $\xi' \in R^{l-1}$ *with*

$$|\xi'| + |\lambda| \neq 0;$$

(5) $a_k > 0,\ d_k \in L_\infty(R^n),\ n \geqslant 3,\ p \in (1,\infty)$ *and* $\frac{2}{n+1} \leqslant \frac{1}{p} - \frac{1}{p'} \leqslant \frac{2}{n}$.

Then:

(a) for $\lambda \in S(\varphi_0)$, *sufficiently large* $|\lambda|$, $t \geqslant t_0$ *the Carleman type estimate*

$$\left\| e^{-t(x,\nu)} u \right\|_{L_{p'q}(\tilde{\Omega})} \leqslant C \left\| e^{-t(x,\nu)} (L+\lambda) u \right\|_{L_{pq}(\tilde{\Omega})}$$

holds for $u \in W^2_{p,q}(\tilde{\Omega})$.

(b) for $V \in L_\mu(\tilde{\Omega})$ *and* $\frac{1}{\mu} = \frac{1}{p} - \frac{1}{p'}$ *if* $u \in W^2_{p,q}(\tilde{\Omega})$ *is a solution of the differential inequality*

$$\|(L+\lambda) u(x,.)\|_{L_q(\Omega)} \leqslant \|V(x) u(x,.)\|_{L_q(\Omega)}$$

then u *is identically zero if its support is contained in a half space.*

Proof. Let $E = L_q(\Omega)$. By virtue of [5] the space $L_q(\Omega)$ is UMD for $q \in (1,\infty)$. Consider the following operator A which is defined by

$$D(A) = W^{2m}_q(\Omega; B_j u = 0),\ Au = \sum_{|\alpha| \leqslant 2m} a_\alpha(y) D^\alpha u(y).$$

The problem $(18)-(19)$ can be rewritten in the form (3), where $u(x) = u(x,.)$, $f(x) = f(x,.)$ are functions with values in $E = L_q(\Omega)$. Then by virtue of [10, Theorem 3.6 and Theorem 8.2] the operator A is R-positive in L_q. Moreover, it is known that the embedding $W^{2m}_q(\Omega) \subset L_q(\Omega)$ is compact (see e.g. [32, Theorem 3.2.5]). Then by virtue of (5) condition and by using interpolation properties of Sobolev spaces (see e.g. [32, § 4]) it is clear to see that (2) condition of the Theorem 3.1 are fulfilled too. Hence, by virtue of Theorems 3.1 and 3.2 we obtain the assertions. □

5. Carleman estimates and unique continuation property for infinite systems of elliptic equations

Consider the following infinity systems of PDE

$$Lu = \sum_{k=1}^{n} a_k D_k^2 u_m(x) + d_m u_m(x) = f_m(x),\ x \in R^n,\ m = 1, 2, ...$$

Let

$$D = \{d_m\},\ d_m > 0,\ u = \{u_m\},\ Du = \{d_m u_m\},\ m = 1, 2, ...,$$

$$l_q(D) = \left\{ u : u \in l_q, \|u\|_{l_q(D)} = \|Du\|_{l_q} = \left(\sum_{m=1}^{\infty} |d_m u_m|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$x \in R^n,\ 1 < q < \infty.$$

THEOREM 5.1. *Assume* $a_k > 0$, $n \geqslant 3$, $p \in (1,\infty)$ *and* $\frac{2}{n+1} \leqslant \frac{1}{p} - \frac{1}{p'} \leqslant \frac{2}{n}$. *Then:*

(a) $t \geqslant t_0$ *the Carleman type estimate*

$$\left\| e^{-t(x,\nu)} u \right\|_{L_{p'}\left(R^n;l_q\right)} \leqslant C \left\| e^{-t(x,\nu)} \left(L+\lambda\right) u \right\|_{L_p\left(R^n;l_q\right)}$$

holds for $u \in W_p^2\left(R^n;l_q\left(D\right),l_q\right)$.

(b) for $V \in L_\mu\left(\tilde{\Omega};L(E)\right)$ *and* $\frac{1}{\mu} = \frac{1}{p} - \frac{1}{p'}$ *if* $u \in W_p^2\left(R^n;l_q\left(D\right),l_q\right)$ *is a solution of the differential inequality*

$$\left\| Lu\left(x\right)\right\|_{l_q} \leqslant \left\| V\left(x\right) u\left(x\right)\right\|_{l_q}$$

then u *is identically zero if its support is contained in a half space.*

Proof. Let $E = l_q$ and A be infinite matrices, such that

$$A = \left[d_m \delta_{jm}\right],\ m,\ j = 1,2,\ldots\infty.$$

It is clear to see that this operator A is R- positive in l_q and all other conditions of Theorem 3.1 are hold. Therefore, by virtue of Theorems 3.1 and 3.2 we obtain the assertions. □

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