Mathematical
Inequalities
& Applications
Volume 16, Number 3 (2013), 887–893
doi:10.7153/mia-16-68

SOME LYAPUNOV–TYPE INEQUALITIES FOR A CLASS OF NONLINEAR SYSTEMS

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(*Communicated by D. Hinton*)

Abstract. In this paper, we present some new Lyapunov-type inequalities for a class of nonlinear differential systems, which are natural generalizations of the well-known Lyapunov inequality for linear second order differential equations. The results of this paper generalize some previous results on this topic.

1. Introduction

The well-known Lyapunov inequality [5] for second-order linear differential equation

$$x''(t) + q(t)x(t) = 0 \tag{1}$$

states that if $q \in C[a,b]$ and $x(t)$ is a solution of (1) such that $x(a) = x(b) = 0$ and $x(t) \neq 0$ for $t \in (a,b)$, then the following inequality holds:

$$(b-a)\int_a^b q^+(t)dt > 4, \tag{2}$$

where $q^+(t) = \max\{q(t), 0\}$, and the constant 4 is sharp, which means that it can not be replaced by a larger number.

Since this result has found many applications in the study of various properties of solutions of differential equation (1) such as oscillation theory, disconjugacy and eigenvalue problems, there have been many proofs and generalizations of (2), for example, to nonlinear second order equations, to delay differential equations, to higher order differential equations, to discrete differential equations and to linear Hamiltonian systems etc. For further examples, we recommend readers to refer to the references [1-15] and the references therein. In [5], Hartman obtained the following inequality:

$$\int_a^b q^+(t)(t-a)(b-t)dt > (b-a), \tag{3}$$

which implies (2) since it follows from (3) that the following refinement of (2) holds:

$$\int_a^b q^+(t)dt > \frac{4}{(b-a)^2}\int_a^b q^+(t)(t-a)(b-t)dt > \frac{4}{b-a}. \tag{4}$$

Mathematics subject classification (2010): 34C11.
Keywords and phrases: Lyapunov-type inequality, nonlinear differential system.

In this paper, inspired by the work of [5] and [12], we obtain some new Lyapunov-type inequalities which are natural generalizations of (4) and some previous results in the literature.

2. Main results

Consider the following nonlinear differential system:

$$\begin{aligned} x' &= a_1(t)x + a_2(t)\phi_{p'}(y), \\ y' &= -a_3(t)\phi_p(x) - a_1(t)y, \end{aligned} \tag{5}$$

where $a_k \in C([a,b],\mathbb{R})$ for $1 \leqslant k \leqslant 3$, $a_2(t) > 0 \forall t \in [a,b]$, $\phi_p(u) = |u|^{p-2}u$ with $p > 1$ and $p' = \frac{p}{p-1} > 1$ is the exponent conjugate to p.

Note that if $a_1(t) \equiv 0$, then system (5) reduces to the following nonlinear second order equation:

$$\big(r(t)\phi_p(x')\big)' + q(t)\phi_p(x) = 0, \tag{6}$$

where $r(t) = \left(\frac{1}{a_2(t)}\right)^{p-1}$, $q(t) = a_3(t)$. In [13], the first author of this paper obtained the following generalization of inequality (2):

Let $p > 1, r \in C([a,b],(0,+\infty))$ and $q \in C([a,b],\mathbb{R})$ in (6). If $x(t)$ is a solution of (6) such that $x(a) = x(b) = 0$ and $x(t) \neq 0 \forall\, t \in (a,b)$, then we have

$$\int_a^b q^+(t)dt \left(\int_a^b \frac{dt}{r^{\frac{1}{p-1}}(t)}\right)^{p-1} > 2^p. \tag{7}$$

Now the main results of this paper are the following theorem and corollaries:

THEOREM 1. *If $x(t)$ is a nonzero solution of (5) which satisfies $x(a) = x(b) = 0$, $x(t) \neq 0, t \in (a,b)$, then the following inequalities holds:*

$$\begin{aligned} &\left(\int_a^b a_3^+(t)e^{p\int_a^t a_1(s)ds}dt\right)\left(\int_a^b e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{2(p-1)} \\ &> 4^{p-1}\int_a^b a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1}dt \\ &> D_p\left(\int_a^b e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{p-1}, \end{aligned} \tag{8}$$

where

$$D_p = \begin{cases} 4^{p-1}, & 1 < p \leqslant 2; \\ 2^p, & 2 < p. \end{cases}$$

COROLLARY 1. *If $x(t)$ is a solution of (6) satisfying $x(a)=x(b)=0$ and $x(t)\neq 0$ for $t\in(a,b)$, then we have*

$$\begin{aligned}
&\int_a^b q^+(t)dt\left(\int_a^b \frac{dt}{r^{\frac{1}{p-1}}(t)}\right)^{2(p-1)}\\
&> 4^{p-1}\int_a^b q^+(t)\left(\int_a^t \frac{ds}{r^{\frac{1}{p-1}}(s)}\int_t^b \frac{ds}{r^{\frac{1}{p-1}}(s)}\right)^{p-1} dt \\
&> D_p\left(\int_a^b \frac{dt}{r^{\frac{1}{p-1}}(t)}\right)^{p-1}.
\end{aligned} \tag{9}$$

Moreover, if $r(t)\equiv 1$ in (6), then (6) reduces to

$$(\phi_p(x'))' + q(t)\phi_p(x) = 0, \tag{10}$$

and Corollary 1 reduces to the following Corollary 2.

COROLLARY 2. *If $x(t)$ is a solution of (10) satisfying $x(a)=x(b)=0$ and $x(t)\neq 0$ for $t\in(a,b)$, then we have*

$$\begin{aligned}
&(b-a)^{2(p-1)}\int_a^b q^+(t)dt\\
&> 4^{p-1}\int_a^b q^+(t)\left[(t-a)(b-t)\right]^{p-1} dt \\
&> D_p(b-a)^{p-1},
\end{aligned} \tag{11}$$

which is equivalent to

$$\int_a^b q^+(t)dt > \frac{4^{p-1}}{(b-a)^{2(p-1)}}\int_a^b q^+(t)\left[(t-a)(b-t)\right]^{p-1} dt > \frac{D_p}{(b-a)^{p-1}}. \tag{12}$$

REMARK 1. Note that if $p=2$, then equation (10) reduces to the equation (1). Thus the inequalities (8), (9) and (12) are natural generalizations of inequalities (2) and (3).

Before we prove Theorem 1, we first show an inequality, which is crucial in the proof of Theorem 1.

LEMMA 1. *If A, B, k are positive numbers, then the following inequality holds:*

$$\frac{A^k + B^k}{(A+B)^k} \geqslant C_k,$$

where

$$C_k = \begin{cases} 1, & 0<k\leqslant 1; \\ 2^{1-k}, & 1<k. \end{cases}$$

Proof. Let $B = \lambda A$, $\lambda > 0$. Then we have

$$\frac{A^k + B^k}{(A+B)^k} = f(\lambda) := \frac{1+\lambda^k}{(1+\lambda)^k},$$

and

$$f'(\lambda) = \frac{k(\lambda^{k-1} - 1)}{(1+\lambda)^{k+1}}.$$

This implies that for $0 < k \leqslant 1$, $f'(\lambda) \geqslant 0$ if $0 < \lambda \leqslant 1$ and $f'(\lambda) \leqslant 0$ if $\lambda \geqslant 1$, and hence, we have $\inf_{\lambda>0} f(\lambda) = f(0+) = f(+\infty) = 1$. Similarly, we can show that for $k \geqslant 1$, $\min_{\lambda>0} f(\lambda) = f(1) = 2^{1-k}$. $\square$

Proof of Theorem 1. Multiplying the first equation of (5) by $y(t)$ and the second one by $x(t)$, and adding the results, we get

$$(x(t)y(t))' = a_2(t)|y(t)|^{p'} - a_3(t)|x(t)|^p.$$

Integrating the above equation from a to b and using $x(a) = x(b) = 0$, we obtain

$$\int_a^b a_2(t)|y(t)|^{p'} dt = \int_a^b a_3(t)|x(t)|^p dt. \tag{13}$$

Recall that the solution for $t \in \mathbb{R}$ of the following initial value problem of the first order linear differential equation:

$$u' = p(t)u + q(t), \quad u(t_0) = 0$$

is given by

$$u(t) = e^{\int_{t_0}^t p(s)ds} \int_{t_0}^t q(s) e^{-\int_{t_0}^s p(\tau)d\tau} ds.$$

For any $t \in (a, b)$, using the assumption $x(a) = x(b) = 0$, and applying the above result to the first equation of (5), we obtain

$$\begin{aligned} x(t) &= e^{\int_a^t a_1(s)ds} \int_a^t e^{-\int_a^s a_1(\tau)d\tau} a_2(s)\phi_{p'}(y(s))ds \\ &= -e^{\int_a^t a_1(s)ds} \int_t^b e^{-\int_a^s a_1(\tau)d\tau} a_2(s)\phi_{p'}(y(s))ds. \end{aligned} \tag{14}$$

Now, applying Hölder's inequality to (14), we have

$$|x(t)| \leqslant e^{\int_a^t a_1(s)ds} \left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau} a_2(s)ds\right)^{\frac{1}{p'}} \left(\int_a^t a_2(s)|y(s)|^{p'} ds\right)^{\frac{1}{p}}, \tag{15}$$

and

$$|x(t)| \leqslant e^{\int_a^t a_1(s)ds} \left(\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau} a_2(s)ds\right)^{\frac{1}{p'}} \left(\int_t^b a_2(s)|y(s)|^{p'} ds\right)^{\frac{1}{p}}. \tag{16}$$

Let $d \in (a,b)$ be any fixed number. Then from (15), we obtain

$$\begin{aligned} &\int_a^d a_3^+(t)|x(t)|^p dt \\ &\leqslant \int_a^d a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau} a_2(s)ds\right)^{p-1} \int_a^t a_2(s)|y(s)|^{p'} dsdt \\ &\leqslant \int_a^d a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau} a_2(s)ds\right)^{p-1} dt \int_a^d a_2(t)|y(t)|^{p'} dt. \end{aligned} \tag{17}$$

Similarly, from (16), we obtain

$$\begin{aligned} &\int_d^b a_3^+(t)|x(t)|^p dt \\ &\leqslant \int_d^b a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau} a_2(s)ds\right)^{p-1} \int_t^b a_2(s)|y(s)|^{p'} dsdt \\ &\leqslant \int_d^b a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau} a_2(s)ds\right)^{p-1} dt \int_d^b a_2(t)|y(t)|^{p'} dt. \end{aligned} \tag{18}$$

Note that the function

$$h_1(x) := \int_a^x a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau} a_2(s)ds\right)^{p-1} dt$$

is nondecreasing for $x \in (a,b)$ and the function

$$h_2(x) := \int_x^b a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau} a_2(s)ds\right)^{p-1} dt$$

is nonincreasing for $x \in (a,b)$. Now it follows from $h_1(a) = h_2(b) = 0$ and $h_1(b) > 0$ and $h_2(a) > 0$ that there exists at least one $c \in (a,b)$ such that $h_1(c) = h_2(c) > 0$, that is,

$$\begin{aligned} &\int_a^c a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau} a_2(s)ds\right)^{p-1} dt \\ &= \int_c^b a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau} a_2(s)ds\right)^{p-1} dt. \end{aligned} \tag{19}$$

Let $d = c \in (a,b)$ in (17) and (18). Since (19) holds, adding (17) and (18), and using (13), we have

$$\begin{aligned} &\int_a^b a_3^+(t)|x(t)|^p dt \\ &< \int_a^c a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau} a_2(s)ds\right)^{p-1} dt \int_a^c a_2(t)|y(t)|^{p'} dt \\ &\quad + \int_c^b a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau} a_2(s)ds\right)^{p-1} dt \int_c^b a_2(t)|y(t)|^{p'} dt \\ &= \int_a^c a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau} a_2(s)ds\right)^{p-1} dt \int_a^b a_2(t)|y(t)|^{p'} dt \\ &= \int_a^c a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau} a_2(s)ds\right)^{p-1} dt \int_a^b a_3(t)|x(t)|^p dt \\ &\leqslant \int_a^c a_3^+(t) e^{p\int_a^t a_1(s)ds} \left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau} a_2(s)ds\right)^{p-1} dt \int_a^b a_3^+(t)|x(t)|^p dt. \end{aligned} \tag{20}$$

The first strict inequality in the above inequalities holds since $x(t)$ is not a constant solution (zero solution) of (5), and hence at least one inequalities in (17) and (18) is strict. From (13), we have

$$\int_a^b a_3^+(t)|x(t)|^p dt > 0,$$

which, together with (20), yields

$$\begin{aligned} 1 &< \int_a^c a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1}dt \\ &= \int_c^b a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1}dt. \end{aligned} \tag{21}$$

It follows from the first inequality of (21) that

$$\begin{aligned} &\int_a^c a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1}dt \\ &\geqslant \int_a^c a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\int_c^b e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{p-1}dt \\ &> \left(\int_c^b e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{p-1}. \end{aligned} \tag{22}$$

Similarly, from the second inequality of (21), we have

$$\begin{aligned} &\int_c^b a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1}dt \\ &\geqslant \int_a^c a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^c e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1}dt \\ &> \left(\int_a^c e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{p-1}. \end{aligned} \tag{23}$$

Adding (22) and (23) and applying Lemma 1, we obtain

$$\begin{aligned} &\int_a^b a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1}dt \\ &> \left(\int_c^b e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{p-1} + \left(\int_a^c e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{p-1} \\ &\geqslant C_{p-1}\left(\int_a^b e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{p-1}. \end{aligned} \tag{24}$$

Since $AB \leqslant \frac{(A+B)^2}{4}$ for any real number A, B, we obtain

$$\begin{aligned} &\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds \\ &\leqslant \frac{1}{4}\left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds + \int_t^b e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^2 \\ &= \frac{1}{4}\left(\int_a^b e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^2. \end{aligned} \tag{25}$$

Substituting (25) into (24), finally we get

$$
\begin{aligned}
&\left(\int_a^b a_3^+(t)e^{p\int_a^t a_1(s)ds}dt\right)\left(\int_a^b e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{2(p-1)}\\
&> 4^{p-1}\int_a^b a_3^+(t)e^{p\int_a^t a_1(s)ds}\left(\int_a^t e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\int_t^b e^{-p'\int_a^s a_1(\tau)d\tau}a_2(s)ds\right)^{p-1}dt\\
&> 4^{p-1}C_{p-1}\left(\int_a^b e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{p-1} = D_p\left(\int_a^b e^{-p'\int_a^t a_1(s)ds}a_2(t)dt\right)^{p-1},
\end{aligned}
$$

which is the result of Theorem 1. □

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(Received February 9, 2012)

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