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SOME INEQUALITIES FOR THE NONLINEAR MATRIX EQUATIONS

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Abstract. The nonlinear matrix equations $X \pm A^*X^{-1}A = Q$ have many applications in control theory, dynamic programming and statistics. In the present work, by using some related matrix equalities and linear algebraic techniques, we propose some trace and determinant inequalities for the solution of the above nonlinear matrix equations. Furthermore, we give some inequalities for the eigenvalues of their solutions.

1. Introduction

Let $M_{m,n}$ stand for the set of all $m \times n$ complex matrices, and $M_{n,n}$ will be abbreviated as M_n. The symbols $\mathrm{tr}(A)$, A^*, $\lambda_i(A)$ and $\det A$ denote the trace, conjugate transpose, eigenvalue and determinant of a square matrix A, respectively. The eigenvalues of a Hermitian matrix A are ordered as $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \lambda_n(A)$. Let $A \geqslant 0$ and $A > 0$ denote the Hermitian positive semi-definite and positive definite matrix, respectively.

In practice, solution bounds of the equations can give rough estimates before actually solving them and can as a check of whether the solution techniques for them actually resulted in valid solutions. Besides, solution bounds of the equations can also help us reduce the computational burdens. Therefore, it is necessary to study the bounds of the solution. In [4], lower bounds for the sum of the eigenvalues of the solution to the algebraic Riccati equation were proposed. In [5], several bounds for the traces of the solutions of the algebraic Riccati and Lyapunov matrix equations were presented, respectively. In [8], some inequalities for the extremal eigenvalues of the solution for the Lyapunov matrix equations were introduced.

In this paper, we consider some inequalities of the solutions to the following nonlinear matrix equations:

$$X + A^*X^{-1}A = Q \tag{1}$$

and

$$X - A^*X^{-1}A = Q \tag{2}$$

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where $A,Q,X \in M_n$, $Q>0$, $X>0$, and X is unknown.

Many authors have studied Eqs. (1) and (2) and have obtained some results [1–3, 6, 9–10]. However, to our knowledge, the bounds of the solution of trace and determinant for Eq. (1) and (2) have not been considered.

In this paper, we first give some trace and determinant inequalities for the solutions to Eqs. (1) and (2). Then, we present some bounds for the eigenvalues of their solutions.

2. Trace and determinant inequalities for the solutions of Eqs. (1) and (2)

Firstly, we give some lemmas which are useful for obtaining the main results.

LEMMA 1. [5] *Let $X,Y \in M_n$ and $X \geqslant 0,\ Y \geqslant 0$. Then*

$$\lambda_n(Y)tr(X) \leqslant tr(XY) \leqslant \lambda_1(Y)tr(X).$$

If $X>0,\ Y \geqslant 0$, then $tr(X^{-1}Y)tr(X) \geqslant \lambda_n(Y)n^2$.

LEMMA 2. [7] *(a) Let $X,Y \in M_n$ and $X>0,\ Y>0$. Then*

$$tr(XY) \geqslant n(detX)^{\frac{1}{n}}(detY)^{\frac{1}{n}}.$$

(b) Let $X,Y \in M_n$ and $X \geqslant 0,\ Y \geqslant 0$. Then

$$(det(X+Y))^{\frac{1}{n}} \geqslant (detX)^{\frac{1}{n}} + (detY)^{\frac{1}{n}}.$$

THEOREM 1. *The trace for the solution of the nonlinear equation $X+A^*X^{-1}A = Q$ satisfies the following inequality:*

$$\frac{tr(Q)-\sqrt{tr^2(Q)-4n^2\lambda_n(AA^*)}}{2} \leqslant tr(X) \leqslant \frac{tr(Q)+\sqrt{tr^2(Q)-4n^2\lambda_n(AA^*)}}{2}. \quad (3)$$

Proof. Taking the trace on both sides of the matrix equation $X+A^*X^{-1}A=Q$, we get

$$tr(X)+tr(A^*X^{-1}A) = tr(Q).$$

Since $AA^* \geqslant 0,\ X^{-1}>0$, from Lemma 1 and the trace property $tr(AB)=tr(BA)$, we have

$$\begin{aligned} tr(Q) &= tr(X)+tr(AA^*X^{-1}) \\ &\geqslant tr(X)+\lambda_n(AA^*)tr(X^{-1}) \\ &\geqslant tr(X)+\lambda_n(AA^*)\frac{n^2}{tr(X)}. \end{aligned}$$

The above inequality can be reduced as

$$tr^2(X)-tr(Q)tr(X)+n^2\lambda_n(AA^*) \leqslant 0.$$

Then, we get

$$\frac{tr(Q)-\sqrt{tr^2(Q)-4n^2\lambda_n(AA^*)}}{2} \leqslant tr(X) \leqslant \frac{tr(Q)+\sqrt{tr^2(Q)-4n^2\lambda_n(AA^*)}}{2}. \quad \square$$

THEOREM 2. *If A is nonsingular, then for the solution of Eq. (1), we have*

$$\frac{tr(Q)-\sqrt{tr^2(Q)-4n^2(det(AA^*))^{\frac{1}{n}}}}{2} \leqslant tr(X) \leqslant \frac{tr(Q)+\sqrt{tr^2(Q)-4n^2(det(AA^*))^{\frac{1}{n}}}}{2} \tag{4}$$

Proof. If A is nonsingular, then $AA^* > 0$. From Lemma 2 (a), we have

$$tr(AA^*X^{-1}) \geqslant n(det(AA^*))^{\frac{1}{n}}(det(X^{-1}))^{\frac{1}{n}}, \text{ and } tr(X) \geqslant n(detX)^{\frac{1}{n}}.$$

Then

$$\begin{aligned} tr(Q) &= tr(X)+tr(AA^*X^{-1}) \\ &\geqslant tr(X)+n(det(AA^*))^{\frac{1}{n}}(det(X^{-1}))^{\frac{1}{n}} \\ &\geqslant tr(X)+\frac{n^2(det(AA^*))^{\frac{1}{n}}}{tr(X)}. \end{aligned}$$

Thus

$$tr^2(X)-tr(Q)tr(X)+n^2(det(AA^*))^{\frac{1}{n}} \leqslant 0.$$

So

$$\frac{tr(Q)-\sqrt{tr^2(Q)-4n^2(det(AA^*))^{\frac{1}{n}}}}{2} \leqslant tr(X) \leqslant \frac{tr(Q)+\sqrt{tr^2(Q)-4n^2(det(AA^*))^{\frac{1}{n}}}}{2}.$$

$\square$

REMARK 1. Since $(det(AA^*))^{\frac{1}{n}} \geqslant \lambda_n(AA^*)$, then the bound (4) is better than the bound (3) in the case that A is nonsingular.

THEOREM 3. *The solution to Eq. (1) has the bound*

$$\begin{aligned} &\frac{(detQ)^{\frac{1}{n}}-\sqrt{(detQ)^{\frac{2}{n}}-4(det(AA^*))^{\frac{1}{n}}}}{2} \\ &\leqslant (detX)^{\frac{1}{n}} \leqslant \frac{(detQ)^{\frac{1}{n}}+\sqrt{(detQ)^{\frac{2}{n}}-4(det(AA^*))^{\frac{1}{n}}}}{2} \end{aligned} \tag{5}$$

Proof. Since $X > 0, A^*X^{-1}A \geqslant 0$, by Lemma 2(b), for Eq. (1), we have

$$(detQ)^{\frac{1}{n}} = (det(X + A^*X^{-1}A))^{\frac{1}{n}} \geqslant (detX)^{\frac{1}{n}} + (det(AA^*))^{\frac{1}{n}}(det(X^{-1}))^{\frac{1}{n}},$$

and then, we get

$$(detX)^{\frac{2}{n}} - (detQ)^{\frac{1}{n}}(detX)^{\frac{1}{n}} + (det(AA^*))^{\frac{1}{n}} \leqslant 0,$$

thus

$$\frac{(detQ)^{\frac{1}{n}} - \sqrt{(detQ)^{\frac{2}{n}} - 4(det(AA^*))^{\frac{1}{n}}}}{2}$$
$$\leqslant (detX)^{\frac{1}{n}} \leqslant \frac{(detQ)^{\frac{1}{n}} + \sqrt{(detQ)^{\frac{2}{n}} - 4(det(AA^*))^{\frac{1}{n}}}}{2}. \quad \square$$

Similarly, for Eq. (2), we have the following results.

THEOREM 4. *For the solution of Eq. (2), we have*

$$tr(X) \geqslant \frac{tr(Q) + \sqrt{tr^2(Q) + 4n^2\lambda_n(AA^*)}}{2}. \tag{6}$$

If A is nonsingular, then

$$tr(X) \geqslant \frac{tr(Q) + \sqrt{tr^2(Q) + 4n^2(det(AA^*))^{1/n}}}{2}. \tag{7}$$

It is easy to verify that the bound (7) is better than the bound (6), and the bound for $\det X$ of Eq. (2) satisfies

$$(detX)^{\frac{1}{n}} \geqslant \frac{(detQ)^{\frac{1}{n}} + \sqrt{(detQ)^{\frac{2}{n}} + 4(det(AA^*))^{\frac{1}{n}}}}{2}.$$

3. Eigenvalue inequalities for the solution of Eq. (1)

LEMMA 3. [4] *(a) For $n \times n$ Hermitian matrices V and W, with $1 \leqslant i,\ j,\ k \leqslant n$, the following inequalities are satisfied:*

$$\lambda_{i+j-1}(V+W) \leqslant \lambda_i(V) + \lambda_j(W), \quad i+j \leqslant n+1,$$
$$\lambda_{i+j-n}(V+W) \geqslant \lambda_i(V) + \lambda_j(W), \quad i+j \geqslant n+1,$$
$$\sum_{i=1}^{k} \lambda_{n-i+1}(V+W) \geqslant \sum_{i=1}^{k} \lambda_{n-i+1}(V) + \sum_{i=1}^{k} \lambda_{n-i+1}(W).$$

(b) For $X \geqslant 0$ and $Y \geqslant 0$ of order n, with $1 \leqslant i,\ j \leqslant n$, we have

$$\lambda_{i+j-1}(XY) \leqslant \lambda_i(X)\lambda_j(Y), \quad i+j \leqslant n+1,$$
$$\lambda_{i+j-n}(XY) \geqslant \lambda_i(X)\lambda_j(Y), \quad i+j \geqslant n+1.$$

THEOREM 5. *Let X satisfy Eq. (1). Then for any $i=1,2,\cdots,n$, when $\lambda_n^2(Q)-4\lambda_1(AA^*)\geqslant 0$, we have*

$$\frac{\lambda_1(Q)-\sqrt{\lambda_1^2(Q)-4\lambda_n(AA^*)}}{2}\leqslant\lambda_i(X)\leqslant\frac{\lambda_n(Q)-\sqrt{\lambda_n^2(Q)-4\lambda_1(AA^*)}}{2},$$

or

$$\frac{\lambda_n(Q)+\sqrt{\lambda_n^2(Q)-4\lambda_1(AA^*)}}{2}\leqslant\lambda_i(X)\leqslant\frac{\lambda_1(Q)+\sqrt{\lambda_1^2(Q)-4\lambda_n(AA^*)}}{2}.$$

Proof. When $i+j=n+1$, from Lemma 3 (a), we have

$$\begin{aligned}\lambda_n(Q)=\lambda_{i+j-1}(Q)&=\lambda_{i+j-1}(X+A^*X^{-1}A)\\&\leqslant\lambda_i(X)+\lambda_j(AA^*X^{-1}).\end{aligned}$$

Since $AA^*\geqslant 0$, $X^{-1}>0$, according to Lemma 3 (b) and the fact $\lambda_{n-i+1}(X^{-1})=\frac{1}{\lambda_i(X)}$, we have

$$\lambda_j(AA^*X^{-1})\leqslant\lambda_{n-i+1}(X^{-1})\lambda_{i+j-n}(AA^*)=\frac{\lambda_1(AA^*)}{\lambda_i(X)},$$

then

$$\lambda_n(Q)\leqslant\lambda_i(X)+\frac{\lambda_1(AA^*)}{\lambda_i(X)},$$

i.e.,

$$\lambda_i^2(X)-\lambda_n(Q)\lambda_i(X)+\lambda_1(AA^*)\geqslant 0. \tag{8}$$

On the other hand, when $i+j=n+1$, again from Lemma 3, we have

$$\begin{aligned}\lambda_1(Q)=\lambda_{i+j-n}(Q)&=\lambda_{i+j-n}(X+A^*X^{-1}A)\\&\geqslant\lambda_i(X)+\lambda_j(AA^*X^{-1})\\&\geqslant\lambda_i(X)+\lambda_{n-i+1}(X^{-1})\lambda_{i+j-1}(AA^*)\\&\geqslant\lambda_i(X)+\frac{\lambda_n(AA^*)}{\lambda_i(X)}.\end{aligned}$$

Then, we have

$$\lambda_i^2(X)-\lambda_1(Q)\lambda_i(X)+\lambda_n(AA^*)\leqslant 0. \tag{9}$$

Let

$$\alpha_1=\frac{\lambda_n(Q)-\sqrt{\lambda_n^2(Q)-4\lambda_1(AA^*)}}{2},\quad\beta_1=\frac{\lambda_n(Q)+\sqrt{\lambda_n^2(Q)-4\lambda_1(AA^*)}}{2},$$
$$\alpha_2=\frac{\lambda_1(Q)-\sqrt{\lambda_1^2(Q)-4\lambda_n(AA^*)}}{2},\quad\beta_2=\frac{\lambda_1(Q)+\sqrt{\lambda_1^2(Q)-4\lambda_n(AA^*)}}{2}.$$

When $\lambda_n^2(Q) - 4\lambda_1(AA^*) \geqslant 0$, from (8), we have

$$0 < \lambda_i(X) \leqslant \alpha_1, \ \text{ or } \ \lambda_i(X) \geqslant \beta_1.$$

And from (9), we get

$$\alpha_2 \leqslant \lambda_i(X) \leqslant \beta_2.$$

Let $a = \lambda_1(Q)$, $b = \lambda_n(Q)$, $c = \lambda_1(AA^*)$, $d = \lambda_n(AA^*)$. Then

$$\begin{aligned} 2\alpha_1 &= b - \sqrt{b^2 - 4c}, \\ 2\beta_1 &= b + \sqrt{b^2 - 4c}, \\ 2\alpha_2 &= a - \sqrt{a^2 - 4d}, \\ 2\beta_2 &= a + \sqrt{a^2 - 4d}. \end{aligned}$$

Obviously, $\alpha_1 \leqslant \beta_1$. Since $a > b$, $c > d$, then $\beta_1 \leqslant \beta_2$. Since

$$\begin{aligned} &(a - \sqrt{a^2 - 4d}) - (b - \sqrt{b^2 - 4d}) \\ &= (a-b) + \frac{(b+a)(b-a)}{\sqrt{a^2-4d} + \sqrt{b^2-4d}} \\ &= (a-b)\left(1 - \frac{a+b}{\sqrt{a^2-4d} + \sqrt{b^2-4d}}\right) \leqslant 0, \end{aligned}$$

then $a - \sqrt{a^2 - 4d} \leqslant b - \sqrt{b^2 - 4d} \leqslant b - \sqrt{b^2 - 4c}$, thus $\alpha_2 \leqslant \alpha_1$. Then, we have

$$\alpha_2 \leqslant \lambda_i(X) \leqslant \alpha_1 \ \text{ or } \ \beta_1 \leqslant \lambda_i(X) \leqslant \beta_2. \quad \square$$

REMARK 2. The result of Theorem 5 has been involved in [1] and [2], but our method is different form them.

THEOREM 6. *Let X satisfy Eq. (1). Then, for any $k = 1, 2, \cdots, n$, we have*

$$\sum_{i=1}^{k} \lambda_{n-i+1}(X) \leqslant \sum_{i=1}^{k} \lambda_{n-i+1}(Q) - \frac{1}{\lambda_1(X)} \sum_{i=1}^{k} \lambda_{n-i+1}(AA^*),$$

$$\sum_{i=1}^{k} \lambda_i(X) \geqslant \sum_{i=1}^{k} \lambda_i(Q) - \frac{1}{\lambda_n(X)} \sum_{i=1}^{k} \lambda_i(AA^*).$$

Proof. From Lemma 3 (a), we have

$$\begin{aligned} \sum_{i=1}^{k} \lambda_{n-i+1}(Q) &= \sum_{i=1}^{k} \lambda_{n-i+1}(X + A^*X^{-1}A) \\ &\geqslant \sum_{i=1}^{k} \lambda_{n-i+1}(X) + \sum_{i=1}^{k} \lambda_{n-i+1}(AA^*X^{-1}) \\ &\geqslant \sum_{i=1}^{k} \lambda_{n-i+1}(X) + \sum_{i=1}^{k} \lambda_n(X^{-1})\lambda_{n-i+1}(AA^*) \\ &= \sum_{i=1}^{k} \lambda_{n-i+1}(X) + \frac{1}{\lambda_1(X)} \sum_{i=1}^{k} \lambda_{n-i+1}(AA^*). \end{aligned}$$

Then

$$\sum_{i=1}^{k}\lambda_{n-i+1}(X)\leqslant\sum_{i=1}^{k}\lambda_{n-i+1}(Q)-\frac{\sum_{i=1}^{k}\lambda_{n-i+1}(AA^*)}{\lambda_1(X)}.$$

On the other hand, since

$$\begin{aligned}\sum_{i=1}^{k}\lambda_i(Q)&=\sum_{i=1}^{k}\lambda_i(X+A^*X^{-1}A)\\&\leqslant\sum_{i=1}^{k}\lambda_i(X)+\sum_{i=1}^{k}\lambda_i(AA^*X^{-1})\\&\leqslant\sum_{i=1}^{k}\lambda_i(X)+\sum_{i=1}^{k}\lambda_1(X^{-1})\lambda_i(AA^*),\end{aligned}$$

then

$$\sum_{i=1}^{k}\lambda_i(X)\geqslant\sum_{i=1}^{k}\lambda_i(Q)-\frac{\sum_{i=1}^{k}\lambda_i(AA^*)}{\lambda_n(X)}.\quad\square$$

REMARK 3. For Eq. (2), similar to Theorem 5 and 6, we can also obtain some bounds for the eigenvalues of its solutions.

4. Conclusion

In this paper, we have proposed some inequalities for the trace and determinant for the solution of the nonlinear matrix equations, and we also give some bounds for the eigenvalues of their solutions.

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