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NONLINEAR GRONWALL–BELLMAN TYPE INTEGRAL INEQUALITIES WITH MAXIMA

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Abstract. Integral inequalities with maxima of the unknown function are useful in the study of control theory. Known results were given for Gronwall-Bellman type integral inequalities with the maxima in the form of linear dependence on the unknown function with a single delay term. In this paper we consider a general form of nonlinear integral inequalities with the maxima and more than one delay terms. Requiring neither monotonicity nor separability of given functions, we apply monotonization to estimate the unknown function. Our result can be used to weaken conditions for some known results. We apply our result to prove boundedness of solutions for a differential equation with the maxima and an integral equation with maxima separately.

1. Introduction

Gronwall-Bellman inequality [1, 2] is an important tool in the study of existence, uniqueness, boundedness, stability, invariant manifolds and other qualitative properties of solutions of differential equations and integral equations. There can be found a lot of its generalization in various cases from literatures (e.g. [3, 4, 6, 5, 7]). A significant work was made by Bihari [8] for the integral inequality

$$u(t) \leqslant c + \int_0^t g(s)\omega(u(s))ds, \quad t \geqslant 0, \tag{1.1}$$

where $c > 0$ is a constant, g is continuous and nonnegative function and ω is continuous and nondecreasing positive function. Replacing t with a function $\alpha(t)$ in (1.1), Lipovan [9] investigated the retarded integral inequality

$$u(t) \leqslant c + \int_{t_0}^t f(s)\omega(u(s))ds + \int_{\alpha(t_0)}^{\alpha(t)} g(s)\omega(u(s))ds, \quad t_0 \leqslant t < t_1.$$

Their results were further generalized by Agarwal, Deng and Zhang [10] in 2005 to the inequality

$$u(t) \leqslant a(t) + \sum_{i=1}^n \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(t,s)\omega_i(u(s))ds, \quad t_0 \leqslant t < t_1,$$

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where the constant c is replaced with a function $a(t)$ and f_i's are continuous functions, ω_i's are continuous and nondecreasing positive functions, α_i's are continuously differentiable and nondecreasing functions. Another aspect of integral inequalities is to consider the unknown u composited with a given function on the left hand side, which has been developed (see [11, 12, 13, 14, 15]) since Ou-Yang (called Ou-Iang in some references) [16] discussed the inequality

$$u^2(t) \leqslant c^2 + 2\int_0^t f(s)u(s)ds, \quad t \geqslant 0, \tag{1.2}$$

where f is a nonnegative continuous function and c is a nonnegative constant. On the basis of discussion (see in [3, 17, 11, 19, 18]) on integral inequalities in multi-variables, for example,

$$u(x,y) \leqslant c + \int_0^x \int_0^y b(s,t)\omega(u(s,t))dtds,$$

Wang [19] generalized the idea of [10] to the inequality

$$u^p(x,y) \leqslant c(x,y) + \sum_{i=1}^n \int_{b_i(x_0)}^{b_i(x)} \int_{c_i(y_0)}^{c_i(y)} f_i(x,y,s,t)\omega_i(u(s,t))dtds,$$

where $c(x,y)$ is a function, b_i's and c_i's are continuously differentiable and nondecreasing functions, all f_i's are continuous and nonnegative functions.

Along with the development of automatic control theory and its applications to computational mathematics and modeling, attentions were also attracted to integral inequalities with the maxima of the unknown function. Actually, many problems in the control theory can be modeled in the form of differential equations with the maxima of the unknown function ([22, 20, 21]). For example, the equation describing the work of the regulator ([23]) can be presented as

$$T_0u'(t) + u(t) + q \max_{s\in[t-h,t]} u(s) = f(t), \tag{1.3}$$

where T_0 and q are constants. Equations involving maxima of unknown function are called differential equations with maxima [22, 20, 21]. In 2010 Golev [20] considered the following initial value problem

$$\begin{cases} x'(t) = f(t,x(t), \max\limits_{s\in[t-h,t]} x(s)), & t \in [0,t_1), \\ x(t) = \mu(t), & t \in [-h,0]. \end{cases} \tag{1.4}$$

Such a problem again requires a new type of integral inequalities as a tool to investigate its qualitative properties. There have been given some results for integral inequalities containing the maxima of the unknown function ([25, 24, 26, 27]). Concretely, in 2010 Hristova and Stefanova [25] discussed the following system of integral inequalities

$$\begin{aligned} u(t) \leqslant{} & a(t) + q_1(t)\int_{t_0}^t \left[p_1(s)u(s) + p_2(s)\max_{\xi\in[s-h,s]} u(\xi)\right]ds \\ & + q_2(t)\int_{\alpha(t_0)}^{\alpha(t)} \left[p_3(s)u(s) + p_4(s)\max_{\xi\in[s-h,s]} u(\xi)\right]ds, \quad t \in [t_0,t_1), \\ u(t) \leqslant{} & \psi(t), \qquad t \in [\alpha(t_0) - h, t_0]. \end{aligned} \tag{1.5}$$

where a is a continuous and nondecreasing positive function, p_i's and ψ are nonnegative continuous functions, α is a nondecreasing function, $q_i(t) \geqslant 1$ are continuous functions. Recently, Henderson and Hristova [26] considered the following system of integral inequalities

$$\begin{aligned} \varphi(u(t)) &\leqslant a(t) + \int_{t_0}^{t} \Big[p_1(s)\omega(u(s)) + p_2(s)\omega(\max_{\xi\in[s-h,s]} u(\xi)) \Big] ds \\ &\quad + \int_{\alpha(t_0)}^{\alpha(t)} \Big[p_3(s)\omega(u(s)) + p_4(s)\omega(\max_{\xi\in[s-h,s]} u(\xi)) \Big] ds,\ t\in[t_0,t_1), \\ u(t) &\leqslant \psi(t), \qquad\qquad t\in[\alpha(t_0)-h,t_0], \end{aligned} \tag{1.6}$$

where a, p_i's, ω, φ and ψ are nonnegative continuous functions and α is a nonnegative continuously differentiable and nondecreasing function. They require that both $a(t) \geqslant 1$ and α are nondecreasing, φ is strictly increasing such that $\lim_{t\to\infty} \varphi(t) = \infty$, and ω satisfies the following: (i) ω is nondecreasing on $[0,\infty)$ and positive on $(0,\infty)$, (ii) $\omega(tx) \geqslant t\omega(x)$ for all $0 \leqslant t \leqslant 1$ and all $x > 0$, and (iii) $\int_1^\infty dx/\omega(x) = \infty$.

In this paper we generally consider the system of integral inequalities

$$\begin{aligned} \varphi(u(t)) &\leqslant a(t) + \sum_{i=1}^{m} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(t,s)\omega_i(u(s))ds \\ &\quad + \sum_{j=m+1}^{m+n} \int_{\alpha_j(t_0)}^{\alpha_j(t)} f_j(t,s)\omega_j\left(\max_{\xi\in[s-h,s]} g(u(\xi))\right) ds, \quad t\in[t_0,t_1), \\ u(t) &\leqslant \psi(t), \qquad\qquad t\in[J(t_0)-h,t_0], \end{aligned} \tag{1.7}$$

where a, f_i's, ω_i's and g are nonnegative continuous functions, α_i's are nonnegative continuously differentiable and nondecreasing functions and $J(t_0) := \min_{1\leqslant i\leqslant m+n} \alpha_i(t_0)$. As required in previous works ([25, 24, 26]), we suppose that $0 \leqslant \alpha_i(t) \leqslant t$, $h > 0$ is a constant and ω_i's are definitely positive, i.e., $\omega_i(s) > 0$ for $s > 0$. In this paper we require neither monotonicity of a, ω_i's, f_i's and g nor $a(t) \geqslant 1$. We monotonize those ω_i's to make a sequence of functions in which each possesses stronger monotonicity than previous one so as to give an estimation for the unknown function. We can use our result to discuss inequalities (1.5) and (1.6), giving the stronger results under weakder conditions. Finally, we apply our result to prove boundedness of solutions for a differential equation with maxima and an integral equation with maxima separately.

2. Main result

Consider the system (1.7) of integral inequalities with $t_0 < t_1$ in $\mathbb{R}_+ := [0,\infty)$. Suppose that

(H1) all $\alpha_i : [t_0,t_1) \to \mathbb{R}_+$ $(i = 1,2,\dots,m+n)$ are continuously differentiable and nondecreasing such that $\alpha_i(t) \leqslant t$ on $[t_0,t_1)$;

(H2) $g,\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\psi : [J(t_0)-h,t_0] \to \mathbb{R}_+$ are continuous functions, φ is strictly increasing such that $\lim_{t\to\infty} \varphi(t) = \infty$;

(H3) all $f_i(t,s)\,(i=1,2,\ldots,m+n)$ are continuous and nonnegative functions on $[t_0,t_1)\times[J(t_0),t_1)$;

(H4) all $\omega_i\,(i=1,2,\ldots,m+n)$ are continuous and positive on $\mathbb{R}_+$;

(H5) $a(t)$ is continuous and nonnegative function on $[t_0,t_1)$.

THEOREM 2.1. *Suppose that $(H_1$-$H_5)$ hold, $\max_{s\in[J(t_0)-h,t_0]}\psi(s)\leqslant\varphi^{-1}(a(t_0))$, and $u\in C([J(t_0)-h,t_1),\mathbb{R}_+)$ satisfies system (1.7) of integral inequalities. Then*

$$u(t)\leqslant\varphi^{-1}\left\{W_{m+n}^{-1}\left(W_{m+n}(r_{m+n}(t))+\int_{\alpha_{m+n}(t_0)}^{\alpha_{m+n}(t)}\max_{\iota\in[t_0,t]}f_{m+n}(\iota,s)ds\right)\right\}\tag{2.1}$$

for all $t\in[t_0,T]$, where W_i^{-1} is the inverse of the function

$$W_i(u):=\int_{u_i}^{u}\frac{dx}{\tilde{\omega}_i(\varphi^{-1}(x))},\ \ u\geqslant u_i,\ \ i=1,\ldots,m+n,\tag{2.2}$$

$u_i>0$ is a given constant, $\tilde{\omega}_i$ is defined recursively by

$$\begin{aligned}
\tilde{\omega}_1(t)&:=\max_{\tau\in[0,t]}\{\omega_1(\tau)\},\\
\tilde{\omega}_{i+1}(t)&:=\max_{\tau\in[0,t]}\left\{\frac{\omega_{i+1}(\tau)}{\tilde{\omega}_i(\tau)}\right\}\tilde{\omega}_i(t),\quad i=1,2,\ldots,m-1,\\
\tilde{\omega}_{m+1}(t)&:=\max_{\tau\in[0,t]}\left\{\frac{\hat{\omega}_{m+1}(\max_{s\in[0,\tau]}\{g(s)\})}{\tilde{\omega}_m(\tau)}\right\}\tilde{\omega}_m(t),\\
\tilde{\omega}_{j+1}(t)&:=\max_{\tau\in[0,t]}\left\{\frac{\hat{\omega}_{j+1}(\max_{s\in[0,\tau]}\{g(s)\})}{\tilde{\omega}_j(\tau)}\right\}\tilde{\omega}_j(t),\quad j=m+1,\ldots,m+n-1,\\
\hat{\omega}_j(t)&:=\max_{\tau\in[0,t]}\{\omega_j(\tau)\},\quad j=m+1,\ldots,m+n,
\end{aligned}$$

$r_i(t)$ is defined by $r_1(t):=\max_{\tau\in[t_0,t]}\{a(\tau)\}$ and

$$r_{i+1}(t):=W_i^{-1}\left(W_i(r_i(t))+\int_{\alpha_i(t_0)}^{\alpha_i(t)}\max_{\iota\in[t_0,t]}f_i(\iota,s)ds\right),\quad i=1,2,\ldots,m+n-1,\tag{2.3}$$

and $T<t_1$ is the largest number such that

$$W_i(r_i(T))+\int_{\alpha_i(t_0)}^{\alpha_i(T)}\max_{\iota\in[t_0,T]}f(\iota,s)ds\leqslant\int_{u_i}^{\infty}\frac{dz}{\tilde{\omega}_i(\varphi^{-1}(z))},\quad i=1,2,3,\ldots,m+n.\tag{2.4}$$

For the special choice that $n=m=2$, $\omega_i(s)=s$, $i=1,2,3,4$, $\varphi(s)=s$, $f_1(t,s)=q_1(t)p_1(s)$, $f_2(t,s)=q_2(t)p_3(s)$, $f_3(t,s)=q_1(t)p_2(s)$, $f_4(t,s)=q_2(t)p_4(s)$, $\alpha_1(s)=\alpha_3(s)=s$ and $\alpha_2(s)=\alpha_4(t)=\alpha(s)$, where p_i, q_i are continuous and nonnegative functions and α is nonnegative continuously differentiable and nondecreasing functions, our Theorem 2.1 gives an estimate for the unknown u in system (1.5). Unlike [25] we require neither the monotonicity of a nor the condition $q_i(t)\geqslant 1$, $i=1,2$. In the special case that $n=m=2$, $\omega_i(s)=\omega(s)$, $f_i(t,s)=p_i(s)$, $i=1,2,3,4$, $g(s)=s$, $\alpha_1(t)=\alpha_3(t)=t$, $\alpha_2(t)=\alpha_4(t)=\alpha(t)$, system (1.7) is in the form of (1.6). Obviously, our Theorem 2.1 is applicable to more general forms than Theorem 6 in [26]. Even if $\omega_i(s)$ is enlarged to $\max\limits_{1\leqslant i\leqslant m+n}\omega_i(s)$ such that (1.7) is changed into the form of (1.6) where $m=n=2$, our theorem gives a finer estimate. For example, the system of inequalities

$$\begin{aligned} &u(t)\leqslant 3+2\int_1^t s\sqrt{u(s)+1}ds+2\int_1^{\sqrt{t}}\tfrac{ts}{t+1}(\max_{\xi\in[s-h,s]}u(\xi)+1)ds, && t\in[1,t_1),\\ &u(t)\leqslant t+2, && t\in[1-h,1], \end{aligned}\tag{2.5}$$

implies that

$$\begin{aligned} &u(t)\leqslant 3+2\int_1^t s(u(s)+1)ds+2\int_1^{\sqrt{t}}s(\max_{\xi\in[s-h,s]}u(\xi)+1)ds, && t\in[1,t_1),\\ &u(t)\leqslant t+2, && t\in[1-h,1]. \end{aligned}\tag{2.6}$$

by enlarging $\sqrt{s+1}$ and $t/(t+1)$ to $s+1$ and 1 respectively. Applying our Theorem 2.1, we obtain

$$u(t)\leqslant\frac{t^4+6t^2+9}{4}e^t,\quad t\in[1,t_1).\tag{2.7}$$

On the other hand, Theorem 6 of [26] gives from (2.6) that

$$u(t)\leqslant 4e^{t^2+t-2},\quad t\in[1,t_1).\tag{2.8}$$

Clearly, (2.7) is sharper than (2.8) for large t.

As in [28], we say $\mu_1\propto\mu_2$ for $\mu_1,\mu_2\colon A\subset\mathbb{R}\to\mathbb{R}\setminus\{0\}$ if $\mu_2(s)/\mu_1(s)$ is nondecreasing on A. In order to prove the theorem, we need the following Lemma.

LEMMA 1. *Suppose that*

(C1) *all* $\alpha_i:[t_0,t_1)\to\mathbb{R}_+$ $(i=1,2,\dots,m+n)$ *are continuously differentiable and nondecreasing such that* $\alpha_i(t)\leqslant t$ *on* $[t_0,t_1)$*;*

(C2) $\psi\in C([J(t_0)-h,t_0],\mathbb{R}_+)$, $p_i\in C([t_0,t_1),\mathbb{R}_+)$ *for* $i=1,2,\dots,m+n$*;*

(C3) *all* h_i $(i=1,2,\dots,m+n)$ *are continuous and nondecreasing on* $\mathbb{R}_+$ *and are positive on* $(0,\infty)$ *such that* $h_1\propto h_2\propto\dots\propto h_{m+n}$*;*

(C4) $a(t)$ *is continuously differentiable in* t *and nonnegative on* $[t_0,t_1)$, $\max\limits_{s\in[J(t_0)-h,t_0]}\psi(s)\leqslant a(t_0)$.

If $u \in C([J(t_0)-h,t_1),\mathbb{R}_+)$ *satisfies the system of inequalities*

$$\begin{aligned} u(t) &\leqslant a(t)+\sum_{i=1}^{m}\int_{\alpha_i(t_0)}^{\alpha_i(t)} p_i(s)h_i(u(s))ds \\ &\quad +\sum_{j=m+1}^{m+n}\int_{\alpha_j(t_0)}^{\alpha_j(t)} p_j(s)h_j\left(\max_{\xi\in[s-h,s]} u(\xi)\right)ds, \quad t\in[t_0,t_1), \\ u(t) &\leqslant \psi(t), \qquad t\in[J(t_0)-h,t_0], \end{aligned} \tag{2.9}$$

then

$$u(t) \leqslant H_{m+n}^{-1}\left(H_{m+n}(\hat{r}_{m+n}(t))+\int_{\alpha_{m+n}(t_0)}^{\alpha_{m+n}(t)} p_{m+n}(s)ds\right) \tag{2.10}$$

for all $t\in[t_0,T_1]$, *where* H_i^{-1} *is the inverse of the function*

$$H_i(u) := \int_{u_i}^{u}\frac{dx}{h_i(x)}, \ \ u\geqslant u_i>0, \quad i=1,2,\dots,m+n, \tag{2.11}$$

$\hat{r}_{m+n}(t)$ *is defined by* $\hat{r}_1(t) := a(t_0)+\int_{t_0}^{t}|a'(s)|ds$ *and*

$$\hat{r}_{i+1}(t) := H_i^{-1}\left(H_i(\hat{r}_i(t))+\int_{\alpha_{i+1}(t_0)}^{\alpha_{i+1}(t)} p_i(s)ds\right), \quad i=1,2,\dots,m+n-1, \tag{2.12}$$

and $T_1<t_1$ *is the largest number such that*

$$H_i(\hat{r}_i(T_1))+\int_{\alpha_i(t_0)}^{\alpha_i(T_1)} p_i(s)ds \leqslant \int_{u_i}^{\infty}\frac{dz}{h_i(z)}, \quad i=1,2,3,\dots,m+n. \tag{2.13}$$

Proof. From (2.12) we see that $\hat{r}_1(t)$ is differentiable and nondecreasing on $[t_0,t_1)$ and $\hat{r}_1(t)=a(t_0)+\int_{t_0}^{t}|a'(s)|ds \geqslant a(t)$, $\hat{r}_1(t_0)=a(t_0)$. From (2.9) we get

$$\begin{aligned} u(t) &\leqslant \hat{r}_1(t)+\sum_{i=1}^{m}\int_{\alpha_i(t_0)}^{\alpha_i(t)} p_i(s)h_i(u(s))ds \\ &\quad +\sum_{j=m+1}^{m+n}\int_{\alpha_j(t_0)}^{\alpha_j(t)} p_j(s)h_j\left(\max_{\xi\in[s-h,s]} u(\xi)\right)ds, \quad t\in[t_0,t_1), \\ u(t) &\leqslant \psi(t), \qquad t\in[J(t_0)-h,t_0]. \end{aligned} \tag{2.14}$$

Define a function $z(t):[J(t_0)-h,t_1)\to\mathbb{R}_+$ by

$$z(t)=\begin{cases} \hat{r}_1(t)+\sum_{i=1}^{m}\int_{\alpha_i(t_0)}^{\alpha_i(t)} p_i(s)h_i(u(s))ds \\ +\sum_{j=m+1}^{m+n}\int_{\alpha_j(t_0)}^{\alpha_j(t)} p_j(s)h_j\left(\max_{\xi\in[s-h,s]} u(\xi)\right)ds, & t\in[t_0,t_1), \\ \hat{r}_1(t_0), & t\in[J(t_0)-h,t_0]. \end{cases}$$

The function $z(t)$ is nondecreasing and the inequality

$$u(t)\leqslant z(t) \tag{2.15}$$

holds for $t \in [J(t_0)-h, t_1)$. Note that $\max_{s\in[t-h,t]} z(s) = z(t)$ for $t \in [J(t_0), t_1)$. Then from (2.14), (2.15) and the definition of $z(t)$, we get for $t \in [t_0, t_1)$

$$
\begin{aligned}
z(t) &\leqslant \hat{r}_1(t) + \sum_{i=1}^{m} \int_{\alpha_i(t_0)}^{\alpha_i(t)} p_i(s) h_i(u(s)) ds + \sum_{j=m+1}^{m+n} \int_{\alpha_j(t_0)}^{\alpha_j(t)} p_j(s) h_j \left(\max_{\xi\in[s-h,s]} u(\xi) \right) ds, \\
&\leqslant \hat{r}_1(t) + \sum_{i=1}^{m} \int_{\alpha_i(t_0)}^{\alpha_i(t)} p_i(s) h_i(z(s)) ds + \sum_{j=m+1}^{m+n} \int_{\alpha_j(t_0)}^{\alpha_j(t)} p_j(s) h_j(z(s)) ds. \qquad (2.16)
\end{aligned}
$$

Then, from (2.16) we obtain (2.10) by Theorem 2.1 of [10], where we choose $f_i(t,s) = p_i(s)$, $a(t) = \hat{r}_1(t)$ and $\omega_i(t) = h_i(t)$, $i = 1, 2, \ldots, m+n$. This completes the proof. $\square$

Proof of Theorem 2.1. First of all, we monotonize some given functions f_i, ω_i, g, a in system (1.7) of integral inequalities. Let

$$\tilde{g}(t) := \max_{\tau\in[0,t]} \{g(\tau)\}, \ t \geqslant 0, \ \tilde{a}(t) := \max_{\tau\in[t_0,t]} \{a(\tau)\}, \ t \geqslant t_0 \qquad (2.17)$$

$$\hat{\omega}_j(t) := \max_{\tau\in[0,t]} \{\omega_j(\tau)\}, \ t \geqslant 0, \ j = m+1, \ldots, m+n. \qquad (2.18)$$

Consider a sequence of functions $\omega_i(t)$, which can be calculated recursively by

$$
\begin{cases}
\tilde{\omega}_1(t) := \max_{\tau\in[0,t]} \{\omega_1(\tau)\}, \ t \geqslant 0, \\
\tilde{\omega}_{i+1}(t) := \max_{\tau\in[0,t]} \left\{ \frac{\omega_{i+1}(\tau)}{\tilde{\omega}_i(\tau)} \right\} \tilde{\omega}_i(t), \ t \geqslant 0, \quad i = 1, 2, \ldots, m-1, \\
\tilde{\omega}_{m+1}(t) := \max_{\tau\in[0,t]} \left\{ \frac{\hat{\omega}_{m+1}(\tilde{g}(\tau))}{\tilde{\omega}_m(\tau)} \right\} \tilde{\omega}_m(t), \quad t \geqslant 0, \\
\tilde{\omega}_{j+1}(t) := \max_{\tau\in[0,t]} \left\{ \frac{\hat{\omega}_{j+1}(\tilde{g}(\tau))}{\tilde{\omega}_j(\tau)} \right\} \tilde{\omega}_j(t), \ t \geqslant 0, \quad j = m+1, \ldots, m+n-1.
\end{cases} \qquad (2.19)
$$

Obviously, $\tilde{\omega}_i$'s are nondecreasing. From (2.2) we observe that the function W_i is strictly increasing. Thus its inverse W_i^{-1} is well defined, continuous and increasing in its corresponding domain. The sequence $\{\tilde{\omega}_i(t)\}$ defined in (2.19) consists of nondecreasing nonnegative functions on $\mathbb{R}_+$ and satisfies

$$
\begin{aligned}
\omega_i(t) &\leqslant \tilde{\omega}_i(t), \quad i = 1, 2, \ldots, m, \\
\omega_i(t) &\leqslant \hat{\omega}_i(t), \quad i = m+1, 2, \ldots, m+n, \\
\hat{\omega}_i(\tilde{g}(t)) &\leqslant \tilde{\omega}_i(t), \quad i = m+1, \ldots, m+n. \qquad (2.20)
\end{aligned}
$$

Moreover,

$$\tilde{\omega}_i \propto \tilde{\omega}_{i+1}, \quad i = 1, 2, \ldots, m+n \qquad (2.21)$$

because the ratio $\tilde{\omega}_{i+1}/\tilde{\omega}_i$, $i = 1, 2, \ldots, m+n$, are all nondecreasing. Furthermore, let

$$\tilde{f}_i(t,s) := \max_{\iota\in[t_0,t]} f_i(\iota, s), \qquad (2.22)$$

which is nondecreasing in t for each fixed s and satisfies $\tilde{f}_i(t,s) \geqslant f_i(t,s) \geqslant 0$ for all $i = 1,2,\ldots,m+n$. We note that $\tilde{a}(t) \geqslant a(t)$ and $\tilde{f}_i(t,s) \geqslant f_i(t,s)$ and they are continuous and nondecreasing in t. From the monotonicity of $\tilde{g}(t)$ we obtain the inequality

$$\max_{\xi\in[s-h,s]} g(u(\xi)) \leqslant \max_{\xi\in[s-h,s]} \tilde{g}(u(\xi)) \leqslant \tilde{g}(\max_{\xi\in[s-h,s]} u(\xi)), \quad \forall s \in [J(t_0),t_1). \tag{2.23}$$

From (1.7) and the definition of $\tilde{f}_i(t,s)$ we get

$$\begin{aligned} \varphi(u(t)) &\leqslant \tilde{a}(t) + \sum_{i=1}^{m} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \tilde{f}_i(t,s)\omega_i(u(s))ds \\ &\quad + \sum_{j=m+1}^{m+n} \int_{\alpha_j(t_0)}^{\alpha_j(t)} \tilde{f}_j(t,s)\omega_j\left(\max_{\xi\in[s-h,s]} g(u(\xi))\right)ds, \quad t\in[t_0,t_1), \\ u(t) &\leqslant \psi(t), \quad t \in [J(t_0)-h,t_0]. \end{aligned} \tag{2.24}$$

Consider the auxiliary system of inequalities with (2.24)

$$\begin{aligned} \varphi(u(t)) &\leqslant \tilde{a}(\sigma) + \sum_{i=1}^{m} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \tilde{f}_i(\sigma,s)\omega_i(u(s))ds \\ &\quad + \sum_{j=m+1}^{m+n} \int_{\alpha_j(t_0)}^{\alpha_j(t)} \tilde{f}_j(\sigma,s)\omega_j\left(\max_{\xi\in[s-h,s]} g(u(\xi))\right)ds \end{aligned} \tag{2.25}$$

for all $t \in [t_0,\sigma]$, where σ is chosen arbitrarily such at $t_0 \leqslant \sigma \leqslant T$. Having $u(t) \leqslant \psi(t)$, $t \in [J(t_0)-h,t_0]$ and (2.25), we claim

$$u(t) \leqslant \varphi^{-1}\left\{W_{m+n}^{-1}\left(W_{m+n}(\tilde{r}_{m+n}(\sigma,t)) + \int_{\alpha_{m+n}(t_0)}^{\alpha_{m+n}(t)} \tilde{f}_{m+n}(\sigma,s)ds\right)\right\} \tag{2.26}$$

for all $t_0 \leqslant t \leqslant \sigma \leqslant T_3$, where

$$\begin{aligned} \tilde{r}_1(\sigma,t) &:= \tilde{a}(\sigma), \\ \tilde{r}_{i+1}(\sigma,t) &:= W_i^{-1}\left(W_i(\tilde{r}_i(\sigma,t)) + \int_{\alpha_i(t_0)}^{\alpha_i(t)} \tilde{f}_i(\sigma,s)ds\right), \quad i=1,2,\ldots,m+n-1, \end{aligned} \tag{2.27}$$

and $T_3 < t_1$ is the largest number such that

$$W_i(\tilde{r}_i(\sigma,T_3)) + \int_{\alpha_i(t_0)}^{\alpha_i(T_3)} f_i(\sigma,s)ds \leqslant \int_{u_i}^{\infty} \frac{dz}{\tilde{\omega}_i(\varphi^{-1}(z))}, \quad i=1,2,3,\ldots,m+n. \tag{2.28}$$

Notice that $T \leqslant T_3$. In fact, W_i is strictly increasing by (2.2), so its inverse W_i^{-1} is continuous and increasing in its corresponding domain by (2.2). It follows from (2.22) and (2.27) that $\tilde{f}_i(\sigma,s)$ and $\tilde{r}_i(\sigma,t)$ are nondecreasing in σ. Thus, T_3 satisfying (2.28) gets smaller as σ is chosen larger. In particular, T_3 satisfies the same (2.4) as T when $\sigma = T$. From (2.23), (2.25) and the definitions of $\tilde{g}(t)$, $\tilde{\omega}_i(t)$ and $\hat{\omega}_i(t)$, we obtain

$$\begin{aligned} \varphi(u(t)) &\leqslant \tilde{a}(\sigma) + \sum_{i=1}^{m} \int_{\alpha_i(t_0)}^{\alpha_i(t)} \tilde{f}_i(\sigma,s)\omega_i(u(s))ds \\ &\quad + \sum_{j=m+1}^{m+n} \int_{\alpha_j(t_0)}^{\alpha_j(t)} \tilde{f}_j(\sigma,s)\hat{\omega}_j\left(\tilde{g}(\max_{\xi\in[s-h,s]} u(\xi))\right)ds \end{aligned}$$

$$\leqslant \tilde{a}(\sigma)+\sum_{i=1}^{m}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\tilde{f}_i(\sigma,s)\tilde{\omega}_i(u(s))ds+\sum_{j=m+1}^{m+n}\int_{\alpha_j(t_0)}^{\alpha_j(t)}\tilde{f}_j(\sigma,s)$$
$$\times\tilde{\omega}_j\left(\max_{\xi\in[s-h,s]}u(\xi)\right)ds,\quad t\in[t_0,\sigma],$$
$$u(t)\leqslant\psi(t),\qquad t\in[J(t_0)-h,t_0]. \tag{2.29}$$

Notice that $\max_{s\in[J(t_0)-h,t_0]}\psi(s)\leqslant\varphi^{-1}(\tilde{a}(\sigma))$ because $\max_{s\in[J(t_0)-h,t_0]}\psi(s)\leqslant\varphi^{-1}(a(t_0))$ and $a(t_0)=\tilde{a}(t_0)\leqslant\tilde{a}(\sigma)$. Define a function $z(t):[J(t_0)-h,\sigma]\to\mathbb{R}_+$ such that

$$z(t)=\begin{cases}\tilde{a}(\sigma)+\sum\limits_{i=1}^{m}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\tilde{f}_i(\sigma,s)\tilde{\omega}_i(u(s))ds\\ +\sum\limits_{j=m+1}^{m+n}\int_{\alpha_j(t_0)}^{\alpha_j(t)}\tilde{f}_j(\sigma,s)\tilde{\omega}_j\left(\max\limits_{\xi\in[s-h,s]}u(\xi)\right)ds, & t\in[t_0,\sigma],\\ \tilde{a}(\sigma), & t\in[J(t_0)-h,t_0].\end{cases}$$

Clearly, $z(t)$ is nondecreasing. By (2.29) and the definition of $z(t)$ we have

$$u(t)\leqslant\varphi^{-1}(z(t)),\quad t\in[J(t_0)-h,\sigma]. \tag{2.30}$$

Since $z(t)$ is nondecreasing and $\varphi(t)$ is strictly increasing, from (2.30) we obtain

$$\max_{\xi\in[s-h,s]}u(\xi)\leqslant\max_{\xi\in[s-h,s]}\varphi^{-1}(z(\xi))\leqslant\max_{\xi\in[s-h,s]}\varphi^{-1}(z(s))$$
$$=\varphi^{-1}(\max_{\xi\in[s-h,s]}z(\xi)),\quad s\in[J(t_0),\sigma]. \tag{2.31}$$

It follows from (2.30), (2.31) and the definition of $z(t)$ that

$$\begin{aligned}z(t)&\leqslant\tilde{a}(\sigma)+\sum_{i=1}^{m}\int_{\alpha_i(t_0)}^{\alpha_i(t)}\tilde{f}_i(\sigma,s)\tilde{\omega}_i(\varphi^{-1}(u(s)))ds\\&\quad+\sum_{j=m+1}^{m+n}\int_{\alpha_j(t_0)}^{\alpha_j(t)}\tilde{f}_j(\sigma,s)\tilde{\omega}_j\left(\varphi^{-1}(\max_{\xi\in[s-h,s]}u(\xi))\right)ds, t\in[t_0,\sigma],\\z(t)&\leqslant\tilde{a}(\sigma),\qquad t\in[J(t_0)-h,t_0].\end{aligned}\tag{2.32}$$

In order to demonstrate the basic condition of monotonicity, let $b(t):=\varphi^{-1}(t)$, which is clearly a continuous and nondecreasing function on $\mathbb{R}_+$. Thus, for each i, $\tilde{\omega}_i(b(t))$ is continuous and nondecreasing on $\mathbb{R}_+$ and $\tilde{\omega}_i(b(t))>0$ for $t>0$. Moreover, since $\tilde{\omega}_i(t)\propto\tilde{\omega}_{i+1}(t)$, we see that the ratio $\tilde{\omega}_{i+1}(b(t))/\tilde{\omega}_i(b(t))$ is also a continuous and nondecreasing function on $\mathbb{R}_+$ and postive on the $(0,\infty)$, implying that $\tilde{\omega}_i(b(t))\propto\tilde{\omega}_{i+1}(b(t))$ for $i=1,2,\ldots,m+n-1$. By Lemma 1 and (2.32), we have

$$u(t)\leqslant W_{m+n}^{-1}\left(W_{m+n}(\tilde{r}_{m+n}(\sigma,t))+\int_{\alpha_{m+n}(t_0)}^{\alpha_{m+n}(t)}\tilde{f}_{m+n}(\sigma,s)ds\right) \tag{2.33}$$

for $t_0\leqslant t\leqslant\sigma\leqslant T_3$. It follows from (2.30) and (2.33) that

$$u(t)\leqslant\varphi^{-1}\left\{W_{m+n}^{-1}\left(W_{m+n}(\tilde{r}_{m+n}(\sigma,t))+\int_{\alpha_{m+n}(t_0)}^{\alpha_{m+n}(t)}\tilde{f}_{m+n}(\sigma,s)ds\right)\right\} \tag{2.34}$$

for $t_0 \leqslant t \leqslant \sigma \leqslant T_3$. This proves (2.26).

Finally, from (1.7) we have

$$\varphi(u(\sigma)) \leqslant \tilde{a}(\sigma)+\sum_{i=1}^{m}\int_{\alpha_i(t_0)}^{\alpha_i(\sigma)}\tilde{f}_i(\sigma,s)\tilde{\omega}_i(u(s))ds+\sum_{j=m+1}^{m+n}\int_{\alpha_j(t_0)}^{\alpha_j(\sigma)}\tilde{f}_j(\sigma,s)$$
$$\times\tilde{\omega}_j\left(\max_{\xi\in[s-h,s]}u(\xi)\right)ds,$$
$$u(t) \leqslant \psi(t), \quad t\in[J(t_0)-h,t_0],$$

namely, the auxiliary system of integral inequalities (2.25) hold for $t=\sigma$. By (2.26), we get for $\sigma\in[t_0,T]$

$$u(\sigma) \leqslant \varphi^{-1}\left\{W_{m+n}^{-1}\left(W_{m+n}(\tilde{r}_{m+n}(\sigma,\sigma))+\int_{\alpha_{m+n}(t_0)}^{\alpha_{m+n}(\sigma)}\tilde{f}_{m+n}(\sigma,s)ds\right)\right\}$$
$$\leqslant \varphi^{-1}\left\{W_{m+n}^{-1}\left(W_{m+n}(r_{m+n}(\sigma))+\int_{\alpha_{m+n}(t_0)}^{\alpha_{m+n}(\sigma)}\tilde{f}_{m+n}(\sigma,s)ds\right)\right\},$$

where we apply the facts that $\tilde{r}(\sigma,\sigma)=r(\sigma)$ and $T_3=T$, which can be easily verified and found in the sentences after (2.4) respectively. This proves (2.1) because σ is arbitrarily chosen. This completes the proof. □

Remark that T is defined by (2.4). In particular, (2.1) is true for all $t\in[t_0,t_1)$ when all $\tilde{\omega}_i$ $(i=1,2,\ldots,m+n)$ and φ satisfy $\int_{u_i}^{\infty}\frac{ds}{\tilde{\omega}_i(\varphi^{-1}(s))}=\infty$. In particular, we have the following:

COROLLARY 2.2. *Suppose that $(H_1$-$H_5)$ hold, and $u\in C(J(t_0)-h,t_1),\mathbb{R}_+)$ satisfies*

$$\begin{aligned}\varphi(u(t)) &\leqslant c+\sum_{i=1}^{m}\int_{\alpha_i(t_0)}^{\alpha_i(t)}f_i(t,s)\omega_i(u(s))ds\\ &\quad+\sum_{j=m+1}^{m+n}\int_{\alpha_j(t_0)}^{\alpha_j(t)}f_j(t,s)\omega_j\left(\max_{\xi\in[s-h,s]}g(u(\xi))\right)ds, \quad t\in[t_0,t_1),\\ u(t) &\leqslant \psi(t), \qquad t\in[J(t_0)-h,t_0],\end{aligned} \tag{2.35}$$

where $c\geqslant 0$ is a constant. Then

$$u(t)\leqslant\varphi^{-1}\left\{W_{m+n}^{-1}\left(W_{m+n}(\bar{r}_{m+n}(t))+\int_{\alpha_{m+n}(t_0)}^{\alpha_{m+n}(t)}\tilde{f}_{m+n}(t,s)ds\right)\right\} \tag{2.36}$$

for all $t\in[t_0,t_3)$, where W_i^{-1} is the inverse of W_i, W_i is defined in (2.2), $\bar{r}_i(t)$ is defined by $\bar{r}_1(t):=\varphi(M)$ and

$$\bar{r}_{i+1}(t):=W_i^{-1}\left(W_i(\bar{r}_i(t))+\int_{\alpha_i(t_0)}^{\alpha_i(t)}\tilde{f}_i(t,s)ds\right), \quad i=1,2,\ldots,m+n-1, \tag{2.37}$$

$M := \max(\max_{s\in[J(t_0)-h,t_0]}\psi(s), \varphi^{-1}(c))$, $t_3 < t_1$ *is the largest number such that*

$$W_i(r_i(t_3)) + \int_{\alpha_i(t_0)}^{\alpha_i(t_3)} f_i(t_3,s)ds \leqslant \int_{u_i}^{\infty} \frac{dz}{\tilde{\omega}_i(\varphi^{-1}(z))}, \quad i = 1,2,3,\ldots,m+n, \tag{2.38}$$

and $\tilde{\omega}_i$ *and* $\tilde{f}_i$ *are defined by (2.19) and (2.22) respectively.*

Proof. From (2.35) and the definition of M we get

$$\begin{aligned}
\varphi(u(t)) &\leqslant \varphi(M) + \sum_{i=1}^{m} \int_{\alpha_i(t_0)}^{\alpha_i(t)} f_i(t,s)\omega_i(u(s))ds \\
&\quad + \sum_{j=m+1}^{m+n} \int_{\alpha_j(t_0)}^{\alpha_j(t)} f_j(t,s)\omega_j\left(\max_{\xi\in[s-h,s]} g(u(\xi))\right)ds, \quad t\in[t_0,t_1), \\
u(t) &\leqslant \psi(t), \quad t\in[J(t_0)-h,t_0].
\end{aligned} \tag{2.39}$$

Then, from (2.39) we obtain (2.36) by our Theorem 2.1, where we choose $a(t)\equiv M$. This completes the proof. □

3. Applications

In this section, we apply our result to prove boundedness of solutions for a differential equation with the maxima and an integral equation with maxima separately.

3.1. Differential equation with the maxima

Consider a system of differential equations with maxima

$$\begin{cases} x'(t) = F(t,x(t), \max_{s\in[\beta(t),\alpha(t)]} g(x(s))), & t\geqslant t_0, \\ x(t) = \psi_1(t), & t\in[\alpha(t_0)-h,t_0], \end{cases} \tag{3.1}$$

where $\psi_1 \in C([\alpha(t_0)-h,t_0],\mathbb{R})$, $F\in C(\mathbb{R}_+\times\mathbb{R}^2,\mathbb{R})$, $\alpha,\beta\in C^1([t_0,\infty),\mathbb{R}_+)$, $\alpha(t)$ is a nondecreasing function, $\beta(t)\leqslant t$, $\alpha(t)\leqslant t$ and $0<\alpha(t)-\beta(t)\leqslant h$ for $t\geqslant t_0$, and both $t_0\geqslant 0$ and $h>0$ are constants.

Equation (3.1) is more general than the equation considered in Section 3 of [24] such that results of integral inequalities obtained in [24] do not work. We will give an estimate for solutions of system (3.1).

COROLLARY 3.1. *Suppose in system (3.1) that*

$$|F(t,x,y)| \leqslant h_1(t)\mu_1(|x|) + h_2(t)\mu_2(|y|), \tag{3.2}$$

where $h_i\in C([t_0,\infty),\mathbb{R}_+)$, $g\in C([0,\infty),\mathbb{R}_+)$ *and* $\mu_i\in C(\mathbb{R}_+,(0,\infty))$ *such that* $\mu_i(u) > 0$ *for* $u>0$, $i=1,2$. *For given* $u_1>0$ *and* $u_2>0$, *let*

$$Q_1(u) := \int_{u_1}^{u} ds/\max_{\tau\in[0,s]}\{\mu_1(\tau)\},\ u\geqslant u_1,$$

$$Q_2(u) := \int_{u_2}^{u} ds/\{\max_{\tau\in[0,s]}\{\tilde{\mu}_2(\tilde{g}(\tau))/\max_{\tau_1\in[0,\tau]}\{\mu_1(\tau_1)\}\}\max_{\tau\in[0,s]}\{\mu_1(\tau)\}\},\ u \geqslant u_2,$$
$$\tilde{g}(t) := \max_{s\in[0,t]}\{g(s)\},\ t\geqslant 0,\ \ \tilde{\mu}_2(t) := \max_{s\in[0,t]}\{\mu_2(s)\},\ t\geqslant 0.$$

Then every solution $x(t,t_0,\psi_1)$ of system (3.1) has the estimate

$$|x(t,t_0,\psi_1)| \leqslant Q_2^{-1}\left(Q_2(\gamma_2(t)) + \int_{t_0}^{t} h_2(s)ds\right), \quad \forall t\in[t_0,t^*], \tag{3.3}$$

where $\gamma_i(t)$ are defined by $\gamma_1(t) := \max_{s\in[\alpha(t_0)-h,t_0]}|\psi_1(s)|$ and

$$\gamma_2(t) := Q_1^{-1}\left(Q_1(\gamma_1(t)) + \int_{t_0}^{t} h_1(s)ds\right)$$

and t^* is the largest number such that

$$Q_1(\gamma_1(t^*)) + \int_{t_0}^{t^*} h_1(s)ds \leqslant \int_{u_1}^{\infty}\frac{ds}{\max_{\tau\in[0,s]}\{\mu_1(\tau)\}},$$
$$Q_2(r_2(t^*)) + \int_{t_0}^{t^*} h_2(s)ds \leqslant \int_{u_2}^{\infty}\frac{ds}{\max_{\tau\in[0,s]}\{\frac{\tilde{\mu}_2(\tilde{g}(\tau))}{\max_{\tau_1\in[0,\tau]}\mu_1(\tau_1)}\}\max_{\tau\in[0,s]}\{\mu_1(\tau)\}}. \tag{3.4}$$

Proof. Let $M = \max_{s\in[\alpha(t_0)-h,t_0]}|\psi_1(s)|$ and $x(t) := x(t,t_0,\psi_1)$, the solution of system (3.1) defined for all $t\geqslant\alpha(t_0)-h$. Function $x(t)$ satisfies the integral equation

$$x(t) = \psi_1(t_0) + \int_{t_0}^{t} F(s,x(s),\max_{\xi\in[\beta(s),\alpha(s)]} g(x(\xi)))ds, \quad t\geqslant t_0,$$
$$x(t) = \psi_1(t), \qquad t\in[\alpha(t_0)-h,t_0]. \tag{3.5}$$

By (3.2) we get from (3.5) that

$$|x(t)| \leqslant |\psi_1(t_0)| + \int_{t_0}^{t}|F(s,x(s),\max_{\xi\in[\beta(s),\alpha(s)]} x(\xi))|ds$$
$$\leqslant M + \int_{t_0}^{t} h_1(s)\mu_1(x(s))ds + \int_{t_0}^{t} h_2(s)\mu_2(|\max_{\xi\in[\beta(s),\alpha(s)]} g(x(\xi))|)ds$$
$$\leqslant M + \int_{t_0}^{t} h_1(s)\mu_1(x(s))ds + \int_{t_0}^{t} h_2(s)\tilde{\mu}_2(\max_{\xi\in[\beta(s),\alpha(s)]}\tilde{g}(|x(\xi)|))ds, \quad t\geqslant t_0,$$
$$|x(t)| \leqslant |\psi_1(t)| \leqslant M, \qquad t\in[\alpha(t_0)-h,t_0]. \tag{3.6}$$

Set $u(t) := |x(t)|$ for $t\in[\alpha(t_0)-h,\infty)$ and change the variable $\eta=\alpha(s)$ in the second integral of (3.6). Then, using the inequality $\max_{\xi\in[\beta(s),\alpha(s)]} u(\xi) \leqslant \max_{\xi\in[\alpha(s)-h,\alpha(s)]} u(\xi)$, we obtain

$$u(t) \leqslant M + \int_{t_0}^{t} h_1(s)\mu_1(u(s))ds$$
$$+ \int_{\alpha(t_0)}^{\alpha(t)} h_2(\alpha^{-1}(\eta))(\alpha^{-1}(\eta))'\tilde{\mu}_2(\max_{\xi\in[\eta-h,\eta]}\tilde{g}(u(\xi)))d\eta, \quad t\geqslant t_0,$$
$$u(t) \leqslant M, \qquad t\in[\alpha(t_0)-h,t_0]. \tag{3.7}$$

Using our Corollary 2.2 to specified $m=n=1$, $\varphi(u)=u$, $f_1(t,s)=h_1(s)$, $\alpha_1(t)=t$, $\alpha_2(t)=\alpha(t)$, $f_2(t,s)=h_2(s)(\alpha^{-1}(s))'$, $c=M$ and $\omega_i(u)=\mu_i(u)$, $i=1,2$, from (3.7) we obtain

$$|u(t)| \leqslant Q_2^{-1}\left(Q_2(\gamma_2(t))+\int_{t_0}^{t} h_2(s)ds\right) \tag{3.8}$$

for all $t\in[t_0,t^*]$, where t^* is given as in (3.4). Inequality (3.8) proves the validity of inequality (3.3). $\square$

Our Corollary 3.1 actually gives a condition for boundedness of solutions. Concretely, observing from (3.3), we see that if

$$\int_{t_0}^{t} h_1(s)ds<\infty, \qquad \int_{t_0}^{t} h_2(s)ds<\infty \quad \forall t\in[t_0,t^*],$$

then every solution $x(t,t_0,\psi)$ of (3.1) is bounded on $[t_0,t^*]$.

Next, we discuss the uniqueness of solutions for system (3.1).

COROLLARY 3.2. *Suppose that $g(s)=s$ and*

$$|F(t,x_1,y_1)-F(t,x_2,y_2)| \leqslant h_1(t)\mu_1(|x_1-x_2|)+h_2(t)\mu_2(|y_1-y_2|) \tag{3.9}$$

for all $t\in[t_0,t_1]$ and all $x_i,y_i\in\mathbb{R}$ (i=1,2), where $h_i\in C([t_0,\infty),\mathbb{R}_+)$ and $\mu_i\in C(\mathbb{R}_+,\mathbb{R}_+)$ are both nondecreasing such that $\mu_i(0)=0$, $\mu_i(u)>0$ for $u>0$, μ_2/μ_1 is also nondecreasing and $\int_0^1 ds/\mu_i(s)=+\infty, i=1,2$. Then system (3.1) has at most one solution on $[t_0,t_1]$.

Proof. $g(s)=s$. From (3.1) we get

$$\begin{cases} x'(t)=F(t,x(t),\max\limits_{s\in[\beta(t),\alpha(t)]} x(s)), & t\geqslant t_0,\\ x(t)=\psi_1(t), & t\in[\alpha(t_0)-h,t_0]. \end{cases} \tag{3.10}$$

Assume that (3.10) has two different solutions $u(t)=u(t,t_0,\psi_1)$ and $v(t)=v(t,t_0,\psi_1)$, defined for $t\geqslant\alpha(t_0)-h$. Then $u(t)$ and $v(t)$ satisfy the integral equations

$$u(t) = \psi_1(t_0)+\int_{t_0}^{t} F(s,u(s),\max_{\xi\in[\beta(s),\alpha(s)]} u(\xi))ds, \quad t\in[t_0,t_1],$$
$$v(t) = \psi_1(t_0)+\int_{t_0}^{t} F(s,v(s),\max_{\xi\in[\beta(s),\alpha(s)]} v(\xi))ds, \quad t\in[t_0,t_1],$$

and $u(t)=v(t)=\psi_1(t)$ for $t\in[\alpha(t_0)-h,t_0]$. It implies that

$$\begin{aligned} |u(t)-v(t)| &\leqslant \int_{t_0}^{t} |F(s,u(s),\max_{\xi\in[\beta(s),\alpha(s)]} u(\xi))-F(s,v(s),\max_{\xi\in[\beta(s),\alpha(s)]} v(\xi))|ds\\ &\leqslant \int_{t_0}^{t} h_1(s)\mu_1(|u(s)-v(s)|)ds\\ &\quad +\int_{t_0}^{t} h_2(s)\mu_2(|\max_{\xi\in[\beta(s),\alpha(s)]} u(\xi)-\max_{\xi\in[\beta(s),\alpha(s)]} v(\xi)|)ds \end{aligned}$$

$$\leqslant \int_{t_0}^{t} h_1(s)\mu_1(|u(s)-v(s)|)ds$$
$$+\int_{t_0}^{t} h_2(s)\mu_2(\max_{\xi\in[\beta(s),\alpha(s)]}|u(\xi)-v(\xi)|)ds \quad \forall t\in[t_0,t_1]. \tag{3.11}$$

Let $\phi(t) := |u(t)-v(t)|$ for $t \geqslant \alpha(t_0)-h$. Noting that

$$\max_{\xi\in[\beta(s),\alpha(s)]} u(\xi) \leqslant \max_{\xi\in[\alpha(s)-h,\alpha(s)]} u(\xi),$$

from (3.11) we obtain

$$\phi(t) \leqslant \int_{t_0}^{t} h_1(s)\mu_1(\phi(s))ds \tag{3.12}$$
$$+\int_{\alpha(t_0)}^{\alpha(t)} h_2(\alpha^{-1}(\eta))(\alpha^{-1}(\eta))'\mu_2(\max_{\xi\in[\eta-h,\eta]}\phi(\xi))d\eta, \quad t\geqslant t_0,$$
$$\phi(t)\leqslant 0, \qquad t\in[\alpha(t_0)-h,t_0]. \tag{3.13}$$

Using our Corollary 2.2 to specified $m=n=1$, $\varphi(u)=u$, $\varphi(u)=u$, $f_1(t,s)=h_1(s)$, $\alpha_1(t)=t$, $\alpha_2(t)=\alpha(t)$, $f_2(t,s)=h_2(s)(\alpha^{-1}(s))'$, $g(t)=t$, $c=0$, and $\omega_i(t)=h_i(t)$, $i=1,2$, from (3.12) we obtain

$$\phi(t) \leqslant \hat{Q}_2^{-1}\left(\hat{Q}_2(\bar{\gamma}_2(t)) + \int_{t_0}^{t} h_2(s)ds\right) \tag{3.14}$$

for all $t\in[t_0,t_1]$, where

$$\hat{Q}_1(u) := \int_1^u \frac{ds}{\mu_1(s)}, \quad \hat{Q}_2(u) := \int_1^u \frac{ds}{\mu_2(\tau)}, \quad u\geqslant 1, \tag{3.15}$$
$$\bar{r}_1(t) := 0, \tag{3.16}$$
$$\bar{r}_2(t) := \hat{Q}_1^{-1}\left(\hat{Q}_1(\bar{r}_1(t)) + \int_{t_0}^{t} h_1(s)ds\right). \tag{3.17}$$

By the definition of $\hat{Q}_i$ and properties of μ_i, noting that $\int_0^1 ds/\mu_i(s) = +\infty$ $(i=1,2)$, we obtain

$$\lim_{u\to 0^+}\hat{Q}_i(u) = -\infty, \quad \lim_{u\to-\infty}\hat{Q}_i^{-1}(u)=0, \quad i=1,2. \tag{3.18}$$

Since $\int_{t_0}^{t} h_1(s)ds$ is finite on a finite interval $[t_0,t_1]$, by (3.16) we obtain

$$\hat{Q}_1(\bar{r}_1(t)) + \int_{t_0}^{t} h_1(s)ds = -\infty. \tag{3.19}$$

Thus, we obtain $\bar{\gamma}_2(t)=0$ from (3.17), (3.18) and (3.19) immediately. Similarly, noting that $\int_{t_0}^{t} h_2(s)ds$ is finite on finite interval $[t_0,t_1]$, from (3.18) we obtain

$$\hat{Q}_2(\bar{r}_2(t)) + \int_{\alpha(t_0)}^{\alpha(t)} h_2(s)ds = -\infty. \tag{3.20}$$

Thus, we conclude from (3.14), (3.18) and (3.20) that $|u(t)-v(t)|\leqslant 0$, which implies that $u(t)=v(t)$ for all $t\in[t_0,t_1]$. The uniqueness is proved. □

3.2. Integral equation with maxima

Consider the system of integral equations with maxima

$$\begin{cases} x^p(t) = a(t) + \int_{t_0}^t f(t,s,x(s), \max_{\xi\in[\beta(s),\alpha(s)]} x^2(\xi))ds, & t \geqslant t_0, \\ x(t) = \psi_2(t), & t \in [\alpha(t_0)-h, t_0], \end{cases} \tag{3.21}$$

where $r \in C([t_0,\mathbb{R}),\ \psi_2 : [\alpha(t_0)-h,t_0] \to \mathbb{R},\ f : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$ and both $t_0 \geqslant 0$ and $h > 0$ are constants. Suppose that

(a_1) $|f(t,s,x,y)| \leqslant p_1(t,s)|x|^q + p_2(t,s)|y|^{\frac{q}{2}}$, where $p_i \in C([t_0,\infty)\times[t_0,\infty),\mathbb{R}_+)$, $p_i(t,s)$ is nondecreasing in t for each fixed s;

(a_2) $\alpha,\beta \in C^1([t_0,\infty),\mathbb{R}_+)$, $\alpha(t)$ is nondecreasing function, $\beta(t) \leqslant t$, $\alpha(t) \leqslant t$ and $0 < \alpha(t)-\beta(t) \leqslant h$ for $t \geqslant t_0$;

(a_3) $a(t)$ is continuous $[t_0,\infty)$ and p,q are constants such that $p \geqslant q > 0$.

Firstly, we give an estimate for solutions of (3.21).

COROLLARY 3.3. *Suppose that* (a_1), (a_2) *and* (a_3) *hold and*

$$\max_{s\in[\alpha(t_0)-h,t_0]} |\psi_2(s)| \leqslant |a(t_0)|^{\frac{1}{p}}.$$

Let $x(t,t_0,\varphi)$ *be a solution of the Cauchy problem* (3.21) *defined for* $t \geqslant \alpha(t_0)-h$. *Then*

1) In the case $p > q$,

$$|x(t,t_0,\varphi)| \leqslant \left\{ (\max_{\tau\in[0,t]}\{|a(\tau)|\})^{\frac{p-q}{p}} + \int_{t_0}^t (p_1(t,s)+p_2(t,s))ds \right\}^{\frac{1}{p-q}}; \tag{3.22}$$

2) In the case $p = q$,

$$|x(t,t_0,\varphi)| \leqslant (\max_{\tau\in[0,t]}\{|a(\tau)|\})^{\frac{1}{p}} \exp\left(\frac{1}{p}\int_{t_0}^t (p_1(t,s)+p_2(t,s))ds \right), \tag{3.23}$$

for $t \geqslant t_0$.

Proof. Let $\tilde{a}(t) := \max_{\tau\in[0,t]}\{|a(\tau)|\}$. Then $\tilde{a}(t)$ is a continuous and nondecreasing function on $[t_0,\infty)$. From (3.21) and condition (a_1) we obtain

$$\begin{aligned} |x(t)|^p &\leqslant |a(t)| + \int_{t_0}^t |f(t,s,x(s), \max_{\xi\in[\beta(s),\alpha(s)]} x(\xi))|ds \\ &\leqslant \tilde{a}(t) + \int_{t_0}^t p_1(t,s)|x(s)|^q ds + \int_{t_0}^t p_2(t,s)|\max_{\xi\in[\beta(s),\alpha(s)]} x^2(\xi)|^{q/2} ds \\ &\leqslant \tilde{a}(t) + \int_{t_0}^t p_1(t,s)|x(s)|^q ds + \int_{t_0}^t p_2(t,s)(\max_{\xi\in[\beta(s),\alpha(s)]} |x(\xi)|^2)^{q/2} ds \end{aligned} \tag{3.24}$$

for all $t \geqslant t_0$. Let $u(t) = |x(t)|$ for $t \in [J(t_0) - h, \infty)$. Then

$$\begin{aligned} u^p(t) &\leqslant \tilde{a}(t) + \int_{t_0}^{t} p_1(t,s)u^q(s)ds \\ &\qquad + \int_{t_0}^{t} p_2(t,s)(\max_{\xi \in [\beta(s),\alpha(s)]} u^2(\xi))^{q/2}ds, \quad t \geqslant t_0, \\ u(t) &= |\psi_2(t)|, \quad t \in [\alpha(t_0) - h, t_0]. \end{aligned} \tag{3.25}$$

Making the change of variables $s = \alpha^{-1}(\eta)$ in the second integral of (3.25) and using the inequality $\max_{\xi \in [\beta(s),\alpha(s)]} u(\xi) \leqslant \max_{\xi \in [\alpha(s)-h,\alpha(s)]} u(\xi)$, which follows from condition (a_2), we obtain

$$\begin{aligned} u^p(t) &\leqslant \tilde{a}(t) + \int_{t_0}^{t} p_1(t,s)u^q(s)ds \\ &\qquad + \int_{\alpha(t_0)}^{\alpha(t)} p_2(t,\alpha^{-1}(\eta))(\alpha^{-1}(\eta))'(\max_{\xi \in [\eta-h,\eta]} u^2(\xi))^{q/2}d\eta, \quad t \geqslant t_0, \\ u(t) &= |\psi_2(t)|, \quad t \in [\alpha(t_0) - h, t_0]. \end{aligned} \tag{3.26}$$

Notice that $\max_{s \in [\alpha(t_0)-h,t_0]} |\psi_2(s)| \leqslant (\tilde{a}(t_0))^{1/p}$ because $\max_{s \in [\alpha(t_0)-h,t_0]} |\psi_2(s)| \leqslant |a(t_0)|^{\frac{1}{p}}$ and $|a(t_0)|^{\frac{1}{p}} \leqslant (\tilde{a}(t_0))^{1/p}$. Using Theorem 2.1 to specified $f_1(t,s) = p_1(t,s)$, $f_2(t,s) = p_2(t,\alpha^{-1}(s))(\alpha^{-1}(s))'$, $\omega_1(u) = u^q$, $\omega_2(u) = u^{q/2}$, $g(u) = u^2$ and $\varphi(u) = u^p$, from (3.26) we obtain that

1) if $p > q$ then

$$u(t) \leqslant \left\{ (\tilde{a}(t))^{\frac{p-q}{p}} + \frac{p-q}{p} \int_{t_0}^{t} (p_1(t,s) + p_2(t,s))ds \right\}^{\frac{1}{p-q}}, \quad t \geqslant t_0;$$

2) if $p = q$ then

$$u(t) \leqslant (\tilde{a}(t))^{1/p} \exp\left(\frac{1}{p} \int_{t_0}^{t} (p_1(t,s) + p_2(t,s))ds \right), \quad t \geqslant t_0.$$

This completes the proof. □

Our Corollary 3.3 actually gives a condition of boundedness for solutions. Concretely, observing from (3.22) and (3.23), we see that if

$$\tilde{a}(t) < \infty, \int_{t_0}^{t} (p_1(t,s) + p_2(t,s))ds < \infty \quad \forall t \in [t_0, \infty),$$

then every solution $x(t, t_0, \psi)$ of (3.21) is bounded on $[t_0, \infty)$.

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REFERENCES

[1] T. H. GRONWALL, *Note on the derivatives with respect to a parameter of the solutions of a system of differential equations*, Ann. of Math. **20** (1919), 292–296.

[2] R. BELLMAN, *The stability of solutions of linear differential equations*, Duke. Math. J. **10** (1943), 643–647.

[3] D. BAINOV, P. SIMEONOV, *Integral Inequalities and Applications*, Kluwer Academic, Dordrecht, 1992.

[4] D. BAINOV, S. HRISTOVA, *Differential Equations Equations with Maxima*, CRC Press, Francis & Taylor, 2011.

[5] B. G. PACHPATTE, *Integral and Finite Difference Inequalities and Applications*, North-Holland Mathematics Studies, vol. **205**, Elsevier Science, Amsterdam, 2006.

[6] B. G. PACHPATTE, *Inequalities for Differential and Integral Equations*, Academic Press, London, 1998.

[7] W. ZHANG, S. DENG, *Projected Gronwall-Bellman's inequality for integrable functions*, Math. Comput. Modelling **34** (2001), 393–402.

[8] I. A. BIHARI, *A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations*, Acta. Math. Acad. Hang. Sci. Hungar. **7** (1956), 81–94.

[9] O. LIPOVAN, *A retarded Gronwall-like inequality and its applications*, J. Math. Anal. Appl. **252** (2000), 389–401.

[10] R. P. AGARWAL, S. DENG, W. ZHANG, *Generalization of a retarded Gronwall-like inequality and its applications*, Appl. Math. Comput. **165** (2005), 599–612.

[11] W. S. CHEUNG, Q. H. MA, *On certain new Gronwall-Ou-Iang type integral inequalities in two variables and their applications*, J. Inequal. Appl. **8** (2005), 347–361.

[12] O. LIPOVAN, *A retarded integral inequality and its applications*, J. Math. Anal. Appl. **285** (2003), 436–443.

[13] Q. H. MA, E. H. YANG, *Some new nonlinear delay integral inequalities*, J. Math. Anal. Appl. **252** (2000), 864–878.

[14] F. W. MENG, W. N. LI, *On some new integral inequalities and their applications*, Appl. Math. Comput. **148** (2004), 381–392.

[15] B. G. PACHPATTE, *On some new inequalities Related to certain inequalities in the theory of differential Equations*, J. Math. Anal. Appl. **189** (1995), 128–144.

[16] L. OU-IANG, *The boundedness of solutions of linear differential equations* $y'' + A(t)y' = 0$, Adv. Math. (China) **3** (1957), 409–418 (in Chinese).

[17] W. S. CHEUNG, *Some new nonlinear inequalities and applications to boundary value problems*, Nonlinear Anal. **64** (2006), 2112–2128.

[18] Y.-H. KIM, GRONWALL, *Bellman and Pachpatte type integral inequalities with applications*, Nolinear Anal. **71** (2009), 2641–2656.

[19] W.-S. WANG, *A generalized retarded Gronwall-like inequality in two variables and applications to BVP*, Appl. Math. Comput. **191** (2007), 144–154.

[20] A. GOLEV, S. HRISTOVA, A. RAHNEV, *An algorithm for approximate solving of differential equations with maxima*, Comput. Math. Appl. **60** (10)(2010), 2771–2778.

[21] S. G. HRISTOVA, L. F. ROBERTS, *Boundedness of the solutions of differential equations with maxima*, Int. J. Appl. Math. **4**, 2 (2000), 231–240.

[22] V. G. ANGELOV, D. D. BAINOV, *On the functional differential equations with maximums*, Appl. Anal. **16** (1983), 187–194.

[23] E. P. POPOV, *Automatic Regulation and Control*, Nauka, Moscow, 1966, in Russian.

[24] S. G. HRISTOVA, K. V. STEFANOVA, *Linear integral inequalities involving maxima of the unknown scalar functions[J]*, Funkcialaj Ekvacioj **53** (2010), 381–394.

[25] S. G. HRISTOVA, K. V. STEFANOVA, *Linear integral inequalities involving maxima of the unknown scalar functions*, J. Math. Inequal. **4**, 4 (2010), 523–535.

[26] J. HENDERSON, S. HRISTOVA, *Nonlinear integral inequalities involving maxima of unknown scalar functions*, Math. Comput. Modelling **53** (2011), 871–882.

[27] M. BOHNER, S. HRISTOVA, K. STEFANOVA, *Nonlinear Integral Inequalities Involving Maxima of the Unknown Scalar Functions*, Mathematical Inequalities and Applications **12** (2012), 811–825.

[28] M. PINTO, *Integral inequalities of Bihari-type and applications*, Funkcialaj Ekvacioj **33** (1990), 387–403.

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