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LYAPUNOV–TYPE INEQUALITY FOR n–DIMENSIONAL QUASILINEAR SYSTEMS

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Abstract. A new version of the well-known Lyapunov-type inequality for n-dimensional quasilinear systems is obtained. The results of this paper generalize some previous results on this topic.

1. Introduction

The well-known Lyapunov inequality for second-order linear differential equation

$$x'' + q(t)x = 0 \tag{1}$$

states that if $q(t) \geqslant 0$ is continuous and equation (1) has a nonzero solution $x(t)$ satisfying the boundary condition:

$$x(a) = x(b) = 0, \quad x(t) \not\equiv 0, \quad t \in (a,b)$$

then

$$\int_a^b q(t)dt > \frac{4}{b-a}.$$

This result has found many applications in the study of various properties of solutions of differential equations such as oscillation theory, disconjugacy and eigenvalue problems. Also, there have been many proofs and generalizations of the Lyapunov inequality. For example, we refer to the papers [1–8] and the references therein. However,until now, there have been only a few results obtained for differential systems. Recently, De Nápoli and Pinasco [4] have obtained the following results:

THEOREM A. *Consider the following (p,q)-quasilinear system:*

$$\begin{aligned} (\phi_p(u'))' + h_1(t)|u|^{\alpha-2}u|v|^{\beta} &= 0, \\ (\phi_q(v'))' + h_2(t)|u|^{\alpha}|v|^{\beta-2}v &= 0, \end{aligned} \tag{2}$$

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where $1 < p,\ q < +\infty$, $\phi_p(u) = |u|^{p-2}u$ *and* $h_1,\ h_2$ *are real nonnegative continuous functions and* $\alpha,\ \beta$ *are non-negative numbers satisfying*

$$\frac{\alpha}{p} + \frac{\beta}{q} = 1. \tag{3}$$

If $h_i(t) > 0$ *for* $t \in [a,b]$ *and* $i = 1,2$ *and equation (2) has a nontrivial solution* (u,v) *satisfying the boundary condition* $u(a) = u(b) = v(a) = v(b) = 0$*, then the following inequality holds:*

$$2^{\alpha+\beta} \leqslant (b-a)^{\alpha+\beta-1}\left(\int_a^b h_1(t)dt\right)^{\frac{\alpha}{p}}\left(\int_a^b h_2(t)dt\right)^{\frac{\beta}{q}}. \tag{4}$$

More recently, Çakmak and Tiryaki [1] have considered the following 2-dimensional system:

$$\begin{aligned}(r_1(t)\phi_p(u'))' + f_1(t)|u|^{\alpha-2}u|v|^{\beta} = 0,\\ (r_2(t)\phi_q(v'))' + f_2(t)|u|^{\theta}|v|^{\gamma-2}v = 0,\end{aligned} \tag{5}$$

where
(i) r_1, r_2, f_1, f_2 are real valued continuous functions such that $r_1(t) > 0$ and $r_2(t) > 0$ for all $t \in \mathbb{R}$.
(ii) the exponents satisfy $1 < p,q < +\infty$ and the positive parameters α,β,θ and γ yield $\frac{\alpha}{p} + \frac{\beta}{q} = 1$ and $\frac{\theta}{p} + \frac{\gamma}{q} = 1$.

They have obtained the following result:

THEOREM B. *If system (5) has a real nonzero solution* $(u(t),v(t))$ *such that* $u(a) = u(b) = v(a) = v(b) = 0$ *where* $a,b \in \mathbb{R}$ *with* $a < b$ *being consecutive zeros, then the following inequality holds:*

$$2^{\theta+\beta} \leqslant \left(\int_a^b (r_1(t))^{\frac{1}{1-p}}\,dt\right)^{\frac{\theta(p-1)}{p}}\left(\int_a^b (r_2(t))^{\frac{1}{1-q}}\,dt\right)^{\frac{\beta(q-1)}{q}}\left(\int_a^b f_1^+(t)dt\right)^{\frac{\theta}{p}}\left(\int_a^b f_2^+(t)dt\right)^{\frac{\beta}{q}}, \tag{6}$$

where $f_i^+(t) = \max\{f_i(t),0\}$ *for* $i = 1,2$.

Now, in this paper, we will give a generalization of the above results to n-dimensional quasilinear differential systems.

2. Main result and its proof

Let us consider the following n-dimensional quasilinear system:

$$\begin{aligned}&(r_1(t)\phi_{p_1}(x_1'))' + f_1(t)\phi_{q_{1,1}}(x_1)\psi_{q_{1,2}}(x_2)\psi_{q_{1,3}}(x_3)\psi_{q_{1,4}}(x_4)\cdots\psi_{q_{1,n}}(x_n) = 0,\\ &(r_2(t)\phi_{p_2}(x_2'))' + f_2(t)\psi_{q_{2,1}}(x_1)\phi_{q_{2,2}}(x_2)\psi_{q_{2,3}}(x_3)\psi_{q_{2,4}}(x_4)\cdots\psi_{q_{2,n}}(x_n) = 0,\\ &\big(r_3(t)\phi_{p_3}(x_3')\big)' + f_3(t)\psi_{q_{3,1}}(x_1)\psi_{q_{3,2}}(x_2)\phi_{q_{3,3}}(x_3)\psi_{q_{3,4}}(x_4)\cdots\psi_{q_{3,n}}(x_n) = 0,\\ &\qquad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ &(r_n(t)\phi_{p_n}(x_n'))' + f_n(t)\psi_{q_{n,1}}(x_1)\psi_{q_{n,2}}(x_2)\cdots\psi_{q_{n,n-1}}(x_{n-1})\phi_{q_{n,n}}(x_n) = 0,\end{aligned} \tag{7}$$

where $1 < p_k < +\infty$ for $k = 1,2,\cdots,n$; $q_{i,j}$ are nonnegative constants for $i,j = 1,2,\cdots,n$; $\phi_p(u) = |u|^{p-2}u$; $\psi_q(u) = |u|^q$ for $q \geqslant 0$; $r_k \in C^1([a,b],(0,+\infty))$ for $k = 1,2,\cdots,n$ and $f_k(t) \in C[a,b]$ for $k = 1,2,\cdots,n$.

The main result of this paper is the following theorem.

THEOREM 1. *Let $a < b$ and assume that there exist nontrivial solutions $(e_1, e_2, \cdots, e_n)$ of the following linear homogeneous equation:*

$$
\begin{aligned}
&(p_1 - q_{1,1})e_1 - q_{2,1}e_2 - q_{3,1}e_3 - \cdots - q_{n,1}e_n = 0;\\
&-q_{1,2}e_1 + (p_2 - q_{2,2})e_2 - q_{3,2}e_3 - \cdots - q_{n,2}e_n = 0;\\
&-q_{1,3}e_1 - q_{2,3}e_2 + (p_3 - q_{3,3})e_3 - \cdots - q_{n,3}e_n = 0;\\
&\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&-q_{1,n}e_1 - q_{2,n}e_2 - \cdots - q_{n-1,n}e_{n-1} + (p_n - q_{n,n})e_n = 0,
\end{aligned}
\tag{8}
$$

where $e_k \geqslant 0$ for $k = 1,2,\cdots,n$ and $\sum_{k=1}^n e_k^2 > 0$. Suppose that there exists a nonzero solution $(x_1(t), x_2(t), \cdots, x_n(t))$ of (7) satisfying $x_k(a) = x_k(b) = 0$ and $x_k(t) \not\equiv 0$ for $k = 1,2,\cdots,n$. Then we have

$$
\prod_{k=1}^{n}\left(\int_a^b f_k^+(t)dt\right)^{e_k} > 2^{Q_n}\prod_{k=1}^{n}\left(\int_a^b (r_k(t))^{\frac{1}{1-p_k}}\,dt\right)^{(1-p_k)e_k}, \tag{9}
$$

where $f_k^+(t) = \max\{f_k(t), 0\}$ for $k = 1,2,\cdots,n$ and $Q_n = \sum_{j=1}^n p_j e_j$.

COROLLARY 1. *Assume that*

$$
\sum_{j=1}^{n} q_{j,k} = p_k, \qquad k = 1,2,\cdots,n. \tag{10}
$$

If there exists a nonzero solution $(x_1(t), x_2(t), \cdots, x_n(t))$ of (7) which satisfies $x_k(a) = x_k(b) = 0$ and $x_k(t) \not\equiv 0$ for $k = 1,2,\cdots,n$, then we have

$$
\prod_{k=1}^{n}\int_a^b f_k^+(t)dt > 2^{P_n}\prod_{k=1}^{n}\left(\int_a^b (r_k(t))^{\frac{1}{1-p_k}}\,dt\right)^{1-p_k}, \tag{11}
$$

where $f_k^+(t) = \max\{f_k(t), 0\}$ for $k = 1,2,\cdots,n$ and $P_n = \sum_{j=1}^n p_j$.

3. Proof of the main result

Proof of Theorem 1. Consider the k-th equation of (7) and assume that $|x_k(c_k)| = \max_{a\leqslant t\leqslant b}|x_k(t)|$ for some $c_k \in (a,b)$ and for $k = 1,2,\cdots,n$. Then from $x_k(t) \not\equiv 0$, we see that $|x_k(c_k)| > 0$ and $x_k'(c_k) = 0$. Now from $x_k(a) = 0$ and using the Hölder's inequality, we obtain

$$\begin{aligned}|x_k(c_k)| &= \left|\int_a^{c_k} x_k'(t)dt\right| \\ &\leqslant \int_a^{c_k} |x_k'(t)|dt = \int_a^{c_k} (r_k(t))^{-\frac{1}{p_k}} (r_k(t))^{\frac{1}{p_k}} |x_k'(t)|dt \\ &\leqslant \left(\int_a^{c_k} (r_k(t))^{-\frac{p_k'}{p_k}} dt\right)^{\frac{1}{p_k'}} \left(\int_a^{c_k} r_k(t)|x_k'(t)|^{p_k}dt\right)^{\frac{1}{p_k}} \\ &= \left(\int_a^{c_k} (r_k(t))^{\frac{1}{1-p_k}} dt\right)^{\frac{1}{p_k'}} \left(\int_a^{c_k} r_k(t)|x_k'(t)|^{p_k}dt\right)^{\frac{1}{p_k}},\end{aligned} \tag{12}$$

where $p_k' = \frac{p_k}{p_k-1}$. From (12), we obtain

$$|x_k(c_k)|^{p_k} \leqslant \left(\int_a^{c_k} (r_k(t))^{\frac{1}{1-p_k}} dt\right)^{p_k-1} \int_a^{c_k} r_k(t)|x_k'(t)|^{p_k}dt. \tag{13}$$

Multiplying the k-th equation of (7) by $x_k(t)$ and integrating over $[a, c_k]$ and using integration by parts, we obtain

$$\begin{aligned}&-\int_a^{c_k} \left(r_k(t)\phi_{p_k}(x_k'(t))\right)' x_k(t)dt \\ &= -r_k(t)\phi_{p_k}(x_k'(t))x_k(t)\big|_a^{c_k} + \int_a^{c_k} r_k(t)|x_k(t)|^{p_k}dt \\ &= \int_a^{c_k} r_k(t)|x_k'(t)|^{p_k}dt \\ &= \int_a^{c_k} f_k(t)\psi_{q_{k,1}}(x_1(t))\psi_{q_{k,2}}(x_2(t))\cdots\phi_{q_{k,k}}(x_k(t))x_k(t)\cdots\psi_{q_{k,n}}(x_n(t))dt \\ &\leqslant \int_a^{c_k} f_k^+(t)|x_1(t)|^{q_{k,1}}|x_2(t)|^{q_{k,2}}\cdots|x_k(t)|^{q_{k,k}}\cdots|x_n(t)|^{q_{k,n}}dt \\ &\leqslant |x_1(c_1)|^{q_{k,1}}|x_2(c_2)|^{q_{k,2}}\cdots|x_k(c_k)|^{q_{k,k}}\cdots|x_n(c_n)|^{q_{k,n}} \int_a^{c_k} f_k^+(t)dt.\end{aligned}$$

Substituting the above inequality into (13), we obtain

$$\begin{aligned}1 &\leqslant |x_1(c_1)|^{q_{k,1}}|x_2(c_2)|^{q_{k,2}}\cdots|x_k(c_k)|^{q_{k,k}-p_k}|x_{k+1}(c_{k+1})|^{q_{k,k+1}}\cdots|x_n(c_n)|^{q_{k,n}} \\ &\quad\times\left(\int_a^{c_k} (r_k(t))^{\frac{1}{1-p_k}} dt\right)^{p_k-1} \cdot \int_a^{c_k} f_k^+(t)dt.\end{aligned}$$

Hence, from the above inequality, we obtain

$$\int_a^{c_k} f_k^+(t)dt \geqslant |x_k(c_k)|^{p_k-q_{k,k}} \prod_{j\neq k} |x_j(c_j)|^{-q_{k,j}} \left(\int_a^{c_k} (r_k(t))^{\frac{1}{1-p_k}} dt\right)^{1-p_k}. \tag{14}$$

Similarly, by using $x_k(b) = 0$, we can show that

$$\int_{c_k}^b f_k^+(t)dt \geqslant |x_k(c_k)|^{p_k-q_{k,k}} \prod_{j\neq k} |x_j(c_j)|^{-q_{k,j}} \left(\int_{c_k}^b (r_k(t))^{\frac{1}{1-p_k}} dt\right)^{1-p_k}. \tag{15}$$

Since the function $h(x) = x^{1-p}$ is convex for $x > 0$ and $p > 1$, the Jensen's inequality

$$h\left(\frac{x+y}{2}\right) < \frac{1}{2}\left[h(x)+h(y)\right],$$

implies that

$$\left(\int_a^{c_k} (r_k(t))^{\frac{1}{1-p_k}}\,dt\right)^{1-p_k} + \left(\int_{c_k}^b (r_k(t))^{\frac{1}{1-p_k}}\,dt\right)^{1-p_k} > 2^{p_k}\left(\int_a^b (r_k(t))^{\frac{1}{1-p_k}}\,dt\right)^{1-p_k}. \tag{16}$$

Now, (14), (15) and (16) imply that

$$\int_a^b f_k^+(t)dt > |x_k(c_k)|^{p_k-q_{k,k}} \prod_{j\neq k} |x_j(c_j)|^{-q_{k,j}} 2^{p_k}\left(\int_a^b (r_k(t))^{\frac{1}{1-p_k}}\,dt\right)^{1-p_k}. \tag{17}$$

Raising the both sides of the inequality (17) to the power e_k for each $k=1,2,\cdots,n$ respectively, and multiplying the resulting inequalities side by side, we obtain

$$\prod_{k=1}^n \left(\int_a^b f_k^+(t)dt\right)^{e_k} > \left[\prod_{k=1}^n |x_k(c_k)|^{\theta_k}\right] 2^{Q_n} \prod_{k=1}^n \left[\int_a^b (r_k(t))^{\frac{1}{1-p_k}}\,dt\right]^{(1-p_k)e_k}, \tag{18}$$

where $\theta_k = (p_k - q_{k,k})e_k - \sum_{j\neq k} q_{j,k}e_j$ for $k=1,2,\cdots,n$. By assumption, equation (8) has nonzero solutions $(e_1,e_2,\cdots,e_n)$ such that $\theta_k = 0$ for $k=1,2,\cdots,n$, where $e_k \geqslant 0$ for $k=1,2,\cdots,n$ and at least one $e_j > 0$ for $j\in\{1,2,\cdots,n\}$. Choosing one of the solutions $(e_1,e_2,\cdots,e_n)$, we obtain from (18) the inequality (9). This completes the proof of Theorem 1. □

Proof of Corollary 1. From the proof of Theorem 1, we see that condition (10) implies that $e_1 = e_2 = \cdots = e_n = 1$ is a nonzero solution of (8). Now Corollary 1 is a direct consequence of Theorem 1. □

EXAMPLE 1. Consider system (2). Clearly (2) is a special case of (7) where $n=2$, $r_1(t)=r_2(t)=1$, $p_1=p$, $p_2=q$, $q_{1,1}=\alpha$, $q_{1,2}=\beta$, $q_{2,1}=\alpha$, $q_{2,2}=\beta$ and $f_k^+(t)=h_k(t)$ for $k=1,2$. Note that the condition (8) of Theorem 1 is satisfied if $e_1=\alpha/p, e_2=\beta/q$ and $\alpha/p+\beta/q=1$. Under these conditions, we can see that the inequality (9) of Theorem 1 reduces the inequality (4) of Theorem A. Also, if there exists a nonzero solution $(u(t),v(t))$ of (2) satisfying condition $u(a)=u(b)=v(a)=v(b)=0$ and $p=2\alpha$, $q=2\beta$, then by Corollary 1, we have the following inequality:

$$\frac{2^{p+q}}{(b-a)^{p+q-2}} < \int_a^b f_1^+(t)dt \int_a^b f_2^+(t)dt,$$

which agrees with the inequality (4) with $p=2\alpha$ and $q=2\beta$.

EXAMPLE 2. Consider system (5). Again, (5) is a special case of (7) where $n=2$, $p_1=p$, $p_2=q$, $q_{1,1}=\alpha$, $q_{1,2}=\beta$, $q_{2,1}=\theta$ and $q_{2,2}=\gamma$. Note that the condition (8) of Theorem 1 is satisfied if $e_1=\theta/p$, $e_2=\beta/q$ and $\alpha/p+\beta/q=1$ and $\theta/p+\gamma/q=1$. Hence, also in this case the inequality (9) of Theorem 1 reduces to the inequality (6) of Theorem B. Moreover, if we assume that (i) and $(ii)'$: $p=\alpha+\theta$, $q=\beta+\gamma$ and the other conditions of Theorem B hold, then we obtain from Corollary 1 the following inequality:

$$2^{p+q}\left(\int_a^b (r_1(t))^{\frac{1}{1-p}}\,dt\right)^{1-p}\left(\int_a^b (r_2(t))^{\frac{1}{1-q}}\,dt\right)^{1-q} < \int_a^b f_1^+(t)dt \int_a^b f_2^+(t)dt.$$

EXAMPLE 3. Consider system (5) again. Assume that (i) and $(ii)''$: $(p-\alpha)(q-\gamma)=\beta\theta$. Then it is easy to see that the condition (8) of Theorem 1 is satisfied if we choose $e_1=\theta>0$, $e_2=p-\alpha>0$. Hence by Theorem 1 we can obtain the following inequality:

$$2^{p\theta+q(p-\alpha)}\left(\int_a^b (r_1(t))^{\frac{1}{1-p}}\,dt\right)^{(1-p)\theta}\left(\int_a^b (r_2(t))^{\frac{1}{1-q}}\,dt\right)^{(1-q)(p-\alpha)}$$
$$<\left(\int_a^b f_1^+(t)dt\right)^{\theta}\left(\int_a^b f_2^+(t)dt\right)^{p-\alpha}.$$

REMARK 1. It is evident that Theorem 1 is a natural generalization of Theorem A and Theorem B. Corollary 1 and examples 1–3 show that Theorem 1 yields new inequalities which are not covered by Theorem A and Theorem B even for the case $n=2$.

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