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DIFFERENCE INEQUALITY FOR ATTRACTING AND QUASI–INVARIANT SETS FOR A CLASS OF IMPULSIVE STOCHASTIC DIFFERENCE EQUATIONS WITH CONTINUOUS TIME

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Abstract. So far there have been few results presented on the attracting and quasi-invariant sets for impulsive stochastic difference equations with continuous time. The main aim of this work is to close this gap. By establishing a difference inequality with continuous time, we obtain the attracting and quasi-invariant sets of systems under consideration. An example is given to illustrate the theory.

1. Introduction

Difference equations with continuous time are difference equations in which the unknown function is a function of continuous time. In practice, time is often involved as the independent variable in difference equations with continuous time. In view of this fact, we may refer to them as difference equations with continuous time. Difference equations with continuous time appear as natural descriptions of observed evolution phenomena in many branches of natural sciences [1, 2]. Deterministic and stochastic difference equations with continuous time are very popular with researchers [3, 4, 10, 5, 6, 7, 8, 9].

However, besides the stochastic effect, an impulsive effect likewise exists in a wide variety of evolutionary processes in which states are changed abruptly at certain moments of time, involving such fields as medicine and biology, economics, mechanics, electronics and telecommunications. Recently, the asymptotic behaviors of impulsive difference equations have attracted considerable attention. Many interesting results on impulsive effect have been obtained [11, 12, 13]. In [14], some stability conditions on impulsive stochastic difference equations with continuous time are given. However, under impulsive perturbation, an equilibrium point sometimes does not exist in many physical systems, especially, in nonlinear systems. Therefore, an interesting subject is to discuss the invariant sets and the attracting sets of impulsive systems. Some significant progress has been made in the techniques and methods of determining the invariant

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sets and attracting sets for delay difference equations, delay differential equations and impulsive functional differential equations [16, 15, 17]. Unfortunately, the corresponding problems for impulsive stochastic difference equations with continuous time have not been considered.

Motivated by the above discussion, we here make a first attempt to arrive at results on the invariant sets and attracting sets of impulsive stochastic difference equations with continuous time.

2. Model description

For convenience, we introduce several notations and recall some basic definitions.

$C(X,Y)$ denotes the space of continuous mappings from the topological space X to the topological space Y. Especially, let $C \triangleq C([-h,0],R)$ with a norm $\|\varphi\| = \sup_{-h\leqslant s\leqslant 0} |\varphi(s)|$, where h is a positive constant.

$$PC(J,H) = \Big\{ \psi(t) : J \to H \mid \psi(t) \text{ is continuous for all but at most countable points } s \in J \\ \text{and at these points } s \in J,\ \psi(s^+) \text{ and } \psi(s^-) \text{ exist, } \psi(s^+) = \psi(s) \Big\},$$

where $J \subset R$ is an interval, H is a complete metric space, $\psi(s^+)$ and $\psi(s^-)$ denote the right-hand and left-hand limit of the function $\psi(s)$, respectively. Especially, let $PC \triangleq PC([-h,0],R)$.

Let R^n be the space of n-dimensional real column vectors and $R_+ = [0,+\infty)$. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geqslant 0}, P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e, it is right continuous and $\mathscr{F}_0$ contains all P-null sets). If $x(t)$ is an R-valued stochastic process on $t \in [-\tau - h, \infty)$, we let $x_t = x(t+s) : -h \leqslant s \leqslant 0$, which is regarded as a PC-valued stochastic process for $t \geqslant -\tau$. $\xi(t)$ is a $\mathscr{F}_t$-measurable stationary and mutually independent stochastic process satisfying

$$E\xi(t) = 0, \qquad E\xi^2(t) = 1,$$

where E be the mathematical expectation. Denote by $PC^b_{\mathscr{F}_0}([-h,0],R)$ the family of all bounded $\mathscr{F}_0$-measurable, PC-valued random variables φ, satisfying $\|\varphi\|^2_{L^2} = \sup_{s\in[-h,0]} E|\varphi(s)|^2 < \infty$.

In this paper, we mainly consider the following impulsive stochastic difference equations with continuous time

$$\begin{cases} x(t+\tau) = F(t, x(t-h_m), \ldots, x(t-h_1), x(t)) \\ \qquad\qquad + G(t, x(t-h_m), \ldots, x(t-h_1), x(t))\, \xi(t+\tau), \quad t > -\tau, \quad t \neq t_k, \\ x(t) = H_k(x(t^-)), \quad t = t_k, \end{cases} \tag{1}$$

with initial condition

$$x_t = \varphi_t, \quad t \in [-(\tau+h), 0],$$

where m is a positive integer, $h=\max_{1\leqslant i\leqslant m} h_i$, τ is a positive constant, $F,G:[-\tau,\infty)\times R^{m+1}\to R$, $H_k:R\to R$, and for fixed $t\in[-\tau,0]$, $\varphi_t\in PC^b_{\mathscr{F}_0}([-h,0],R)$. The fixed moments of time t_k satisfy $0<t_1<t_2<\ldots,\lim\limits_{k\to\infty}t_k=\infty$.

Throughout this paper, we assume that for any $\varphi_t\in PC^b_{\mathscr{F}_0}([-h,0],R)$, $t\in[-\tau,0]$, there exists at least one solution of (1), which is denoted by $x(t,-\tau,\varphi)$ or $x_t(-\tau,\varphi)$ (simply $x(t)$ and x_t if no confusion should occur).

DEFINITION 2.1. The set $S\subset PC^b_{\mathscr{F}_0}([-h,0],R)$ is called a quasi-invariant set of (1), if there exists a constant k such that for any initial value $\varphi_t\in S$, $t\in[-\tau,0]$, the solution $kx_t(-\tau,\varphi)\in S$, $t\geqslant-\tau$. Especially, if $k=1$, S is called a invariant set.

DEFINITION 2.2. The set $S\subset PC^b_{\mathscr{F}_0}([-h,0],R)$ is called a global attracting set of (1), if for any initial value $\varphi_t\in PC^b_{\mathscr{F}_0}([-h,0],R)$, $t\in[-\tau,0]$, the solution $x_t(-\tau,\varphi)$ satisfies

$$\mathrm{dist}(x_t,S)\to 0,\quad \text{as}\quad t\to\infty,$$

where

$$\mathrm{dist}(\varphi,S)=\inf_{\psi\in S}\rho(\varphi(s),\psi(s))\quad \text{for}\quad \varphi\in PC^b_{\mathscr{F}_0}([-h,0],R),$$

where $\rho(\cdot,\cdot)$ is any distance in $PC^b_{\mathscr{F}_0}([-h,0],R)$.

DEFINITION 2.3. The zero solution of Eq. (1) is called mean square exponential stable if there are positive constants λ and $M\geqslant 1$ such that for any initial condition $\varphi_t\in PC^b_{\mathscr{F}_0}([-h,0],R)$, $t\in[-\tau,0]$,

$$Ex^2(t)\leqslant M\max_{t\in[-\tau,0]}\|\varphi_t\|^2_{L_2}e^{-\lambda t},\quad t\geqslant-\tau.$$

Here λ is called the exponential convergence rate. Of course, conditions are needed to ensure that the zero function is a solution of (1).

To establish the main results of system (1), we will employ the following assumptions.
(A_1) For any $t\geqslant-\tau$, there exist nonnegative functions $a_j(t)$, $b_j(t)$ and constants J_1, J_2 such that

$$|F(t,x(t-h_m),\ldots,x(t-h_1),x(t))|\leqslant\sum_{j=0}^{m}a_j(t)\left|x(t-h_j)\right|+J_1,$$
$$|G(t,x(t-h_m),\ldots x(t-h_1),x(t))|\leqslant\sum_{j=0}^{m}b_j(t)\left|x(t-h_j)\right|+J_2,$$

where $h_0=0$.
(A_2) $\sup\limits_{t\geqslant-\tau}2\left\{a^2(t)+b^2(t)\right\}=\mu<1$, where $a(t)=\sum\limits_{j=0}^{m}a_j(t)$ and $b(t)=\sum\limits_{j=0}^{m}b_j(t)$.
(A_3) There exist constants $d_k\geqslant 1$ such that

$$\left|H_k\left(x\left(t_k^-\right)\right)\right|\leqslant d_k\left|x(t_k^-)\right|,\quad k=1,2,\ldots.$$

(A_4) There exists constant $\alpha^* \geqslant 0$ such that

$$\frac{2\ln d_k}{t_k - t_{k-1}} \leqslant \alpha^* < \lambda^*, \quad k = 1, 2, \ldots,$$

where $t_0 = 0$ and λ^* satisfies

$$0 < \lambda^* = \frac{1}{h+\tau} \ln \frac{1}{\mu},$$

and

$$\sigma = 2\sum_{k=1}^{\infty} \ln d_k < \infty, \quad k = 1, 2, \ldots.$$

(A_5) There exist nonnegative constants $d_k \leqslant 1$ such that

$$\left|H_k\left(x\left(t_k^-\right)\right)\right| \leqslant d_k \left|x(t_k^-)\right|.$$

(A_6) For any $t \geqslant -\tau$, there exist positive functions $a_j(t)$ and $b_j(t)$ such that

$$\begin{array}{l} |F(t, x(t-h_m), \ldots, x(t-h_1), x(t))| \leqslant \sum\limits_{j=0}^{m} a_j(t) \left|x(t-h_j)\right|, \\ |G(t, x(i-h_m), \ldots, x(t-h_1), x(t))| \leqslant \sum\limits_{j=0}^{m} b_j(t) \left|x(t-h_j)\right|, \end{array},$$

where $h_0 = 0$.

(A_7) $\sup\limits_{t \geqslant -\tau} \left\{a^2(t) + b^2(t)\right\} = \mu < 1$, where $a(t) = \sum\limits_{j=0}^{m} a_j(t)$ and $b(t) = \sum\limits_{j=0}^{m} b_j(t)$.

3. Main results

In this section, we shall present the main results and complete the proof. Unlike earlier studies, we does not make use of general methods such as Lyapunov methods, It$\hat{o}$ formula methods and so forth. However, we firstly establish a difference inequality with continuous time, which plays an important role in this section, for obtaining our desired results.

LEMMA 3.1. *Suppose* $c_j(t) \in R_+$, $t \geqslant t_0 \geqslant 0$, $\sup\limits_{t \geqslant t_0} \left\{ \sum\limits_{j=0}^{m} c_j(t) \right\} = \eta < 1$ *and* $b > 0$. *Let continuous function* $u(t)$ *satisfy the following difference inequality with continuous time:*

$$u(t+\tau) \leqslant \sum_{j=0}^{m} c_j(t) u(t-h_j) + b, \quad t \geqslant t_0. \tag{2}$$

(a) *Then*

$$u(t) \leqslant de^{-\lambda t} + (1-\eta)^{-1} b, \quad t \geqslant t_0, \tag{3}$$

provided that the initial condition satisfies

$$u(t) \leqslant de^{-\lambda t} + (1-\eta)^{-1}b, \quad t \in [t_0-\tau-h, t_0], \tag{4}$$

where $d \in R_+$ *and* λ *satisfies*

$$0 < \lambda \leqslant \frac{1}{h+\tau} \ln \frac{1}{\eta}. \tag{5}$$

(b) *Then*

$$u(t) \leqslant \gamma(1-\eta)^{-1}b, \quad t \geqslant t_0, \tag{6}$$

provided the initial condition

$$u(t) \leqslant \gamma(1-\eta)^{-1}b, \quad t \in [t_0-\tau-h, t_0], \tag{7}$$

where $\gamma \geqslant 1$.

Proof. (a) Since $\eta < 1$, there exists a constant λ satisfying the inequality (5). Then,

$$e^{\lambda(h+\tau)}\eta \leqslant 1. \tag{8}$$

We first shall prove that for any positive ε:

$$u(t) < de^{-\lambda t} + (1-\eta)^{-1}b + \varepsilon, \quad t \geqslant t_0. \tag{9}$$

If (9) is not true, then there must be a positive number $t^* + \tau > t_0$ such that

$$u(t^*+\tau) = de^{-\lambda(t^*+\tau)} + (1-\eta)^{-1}b + \varepsilon \quad \text{and} \quad u(t) < de^{-\lambda t} + (1-\eta)^{-1}b + \varepsilon,$$
$$t \in [t_0-\tau-h, t^*+\tau). \tag{10}$$

By (2), (8) and (3), we have

$$\begin{aligned} u(t^*+\tau) &\leqslant \sum_{j=0}^{m} c_j(t^*)u(t^*-h_j) + b \\ &< \sum_{j=0}^{m} c_j(t^*)\left[de^{-\lambda\left(t^*-h_j\right)} + (1-\eta)^{-1}b + \varepsilon\right] + b \\ &\leqslant e^{\lambda(h+\tau)}\eta de^{-\lambda(t^*+\tau)} + \eta(1-\eta)^{-1}b + b + \eta\varepsilon \\ &= e^{\lambda(h+\tau)}\eta de^{-\lambda(t^*+\tau)} + (1-\eta)^{-1}b + \eta\varepsilon \\ &< de^{-\lambda(t^*+\tau)} + (1-\eta)^{-1}b + \varepsilon, \end{aligned}$$

which contradicts the first inequality of (3). So (9) holds. Letting $\varepsilon \to 0$ in (9), we can get (3). The proof of part (a) is complete.

(b) We first shall prove that for any positive ε:

$$u(t) \leqslant \gamma(1-\eta)^{-1}b + \varepsilon, \quad t \geqslant t_0, \tag{11}$$

If (11) is not true, then there must be a positive number $t^*+\tau>t_0$ such that

$$u(t^*+\tau)=\gamma(1-\eta)^{-1}b+\varepsilon \quad \text{and} \quad u(t)<\gamma(1-\eta)^{-1}b+\varepsilon, \quad t\in[t_0-\tau-h,t^*+\tau). \tag{12}$$

By (2) and (12), we have

$$\begin{aligned} u(t^*+\tau) &\leqslant \sum_{j=0}^{m} c_j(t^*)u(t^*-h_j)+b \\ &< \gamma\eta(1-\eta)^{-1}b+b+\eta\varepsilon \\ &\leqslant \gamma\Big(\eta(1-\eta)^{-1}b+b\Big)+\eta\varepsilon \\ &< \gamma(1-\eta)^{-1}b+\varepsilon, \end{aligned}$$

which contradicts the first inequality of (12). So (11) holds. Letting $\varepsilon\to 0$ in (11), we can get (6). The proof of part (b) is complete. □

THEOREM 3.1. *If $(A_1)-(A_4)$ hold, then*

$$S=\left\{\phi\in PC^b_{\mathscr{F}_0}([-h,0],R)\,\Big|\,\|\phi\|^2_{L^2}\leqslant e^{\sigma}(1-\mu)^{-1}J\right\}$$

is a global attracting set of (1), where $J=2\left(J_1^2+J_2^2\right)$.

Proof. From (1), Condition (A_1), $(a+b)^2\leqslant 2\left(a^2+b^2\right)$ and the Hölder inequality , we have

$$\begin{aligned} Ex^2(t+\tau) &= EF^2(t,x(t-h_m),\ldots,x(t-h_1),x(t)) \\ &\quad +EG^2(t,x(t-h_m),\ldots,x(t-h_1),x(t)) \\ &\leqslant E\left(\sum_{j=0}^{m}a_j(t)\left|x(t-h_j)\right|+J_1\right)^2+E\left(\sum_{j=0}^{m}b_j(t)\left|x(t-h_j)\right|+J_2\right)^2 \\ &\leqslant 2E\left(\sum_{j=0}^{m}a_j(t)\left|x(t-h_j)\right|\right)^2+2E\left(\sum_{j=0}^{m}b_j(t)\left|x(t-h_j)\right|\right)^2+2\left(J_1^2+J_2^2\right) \\ &\leqslant 2\sum_{j=0}^{m}a_j(t)\sum_{j=0}^{m}a_j(t)E\left|x(t-h_j)\right|^2+2\sum_{j=0}^{m}b_j(t)\sum_{j=0}^{m}b_j(t)E\left|x(t-h_j)\right|^2+J \\ &= 2\sum_{j=0}^{m}\left[a(t)a_j(t)+b(t)b_j(t)\right]Ex^2(t-h_j)+J,\ t\neq t_k,\ k=1,2,\ldots. \end{aligned} \tag{13}$$

From Condition (A_2), we obtain

$$\sup_{t\geqslant-\tau}2\sum_{j=0}^{m}\left[a(t)a_j(t)+b(t)b_j(t)\right]=\sup_{t\geqslant-\tau}2\left\{a^2(t)+b^2(t)\right\}=\mu<1. \tag{14}$$

For the initial conditions $x_t = \varphi_t$, $t \in [-\tau, 0]$, where $\varphi_t \in PC^b_{\mathscr{F}_0}([-h,0],R)$, we have a positive constant K such that

$$Ex^2(t) \leqslant Ke^{-\lambda^* t} + (1-\mu)^{-1}J, \quad t \in [-\tau - h, 0]. \tag{15}$$

Then, all the conditions of the part (a) of Lemma 2.1 are satisfied by $(13)-(15)$. So, we can obtain

$$Ex^2(t) \leqslant Ke^{-\lambda^* t} + (1-\mu)^{-1}J, \quad t \in [0, t_1).$$

Suppose for all $q = 1, 2, \ldots, k$, the inequalities

$$Ex^2(t) \leqslant d_0^2 d_1^2 \cdots d_{q-1}^2 Ke^{-\lambda^* t} + d_0^2 d_1^2 \cdots d_{q-1}^2 (1-\mu)^{-1}J, \quad t \in \left[t_{q-1}, t_q\right), \tag{16}$$

hold, where $d_0 = 1$ and $t_0 = 0$. Then from Condition (A_3) and (16), we have

$$\begin{aligned} Ex^2(t_k) &= E\left|H_k\left(x\left(t_k^-\right)\right)\right|^2 \\ &\leqslant d_k^2 Ex^2\left(t_k^-\right) \\ &\leqslant d_0^2 d_1^2 \cdots d_{k-1}^2 d_k^2 Ke^{-\lambda^* t} + d_0^2 d_1^2 \cdots d_{k-1}^2 d_k^2 (1-\mu)^{-1}J. \end{aligned}$$

This, together with (16) and $d_k \geqslant 1$, $k = 1, 2, \ldots$, leads to

$$Ex^2(t) \leqslant d_0^2 d_1^2 \cdots d_{k-1}^2 d_k^2 Ke^{-\lambda^* t} + d_0^2 d_1^2 \cdots d_{k-1}^2 d_k^2 (1-\mu)^{-1}J, \quad t \in [t_k - \tau - h, t_k]. \tag{17}$$

It follows from (13), (14), (17) and the part (a) of Lemma 2.1 that

$$Ex^2(t) \leqslant d_0^2 d_1^2 \cdots d_{k-1}^2 d_k^2 Ke^{-\lambda^* t} + d_0^2 d_1^2 \cdots d_{k-1}^2 d_k^2 (1-\mu)^{-1}J, \quad t \in [t_k, t_{k+1}).$$

By mathematical induction, we can conclude that

$$Ex^2(t) \leqslant d_0^2 d_1^2 \cdots d_{k-1}^2 Ke^{-\lambda^* t} + d_0^2 d_1^2 \cdots d_{k-1}^2 (1-\mu)^{-1}J, \quad t \in [t_{k-1}, t_k], \ k = 1, 2, \ldots. \tag{18}$$

Noticing that $d_k^2 \leqslant e^{\alpha^*(t_k - t_{k-1})}$ and $e^\sigma = \prod_{k=1}^{\infty} d_k^2 < \infty$, by Condition (A_4), we can use (18) to conclude that

$$\begin{aligned} Ex^2(t) &\leqslant e^{\alpha^*(t_1 - t_0)} \cdots e^{\alpha^*(t_{k-1} - t_{k-2})} Ke^{-\lambda^* t} + d_0^2 d_1^2 \cdots d_{k-1}^2 (1-\mu)^{-1}J \\ &\leqslant Ke^{\alpha^* t} e^{-\lambda^* t} + e^\sigma (1-\mu)^{-1}J \\ &= Ke^{-(\lambda^* - \alpha^*)t} + e^\sigma (1-\mu)^{-1}J, \quad t \in [t_{k-1}, t_k), \ k = 1, 2, \ldots. \end{aligned}$$

This implies that the conclusion holds and the proof is complete. □

THEOREM 3.2. *If* $(A_1)-(A_4)$ *hold, then*

$$S = \left\{ \phi \in PC^b_{\mathscr{F}_0}([-h,0],R) \,\middle|\, \|\phi\|^2_{L^2} \leqslant \gamma(1-\mu)^{-1}J, \ \gamma \geqslant 1 \right\}$$

is a quasi-invariant set of (1).

Proof. For the initial conditions $x_t = \varphi_t$, $t \in [-\tau, 0]$, where $\varphi_t \in PC^b_{\mathscr{F}_0}([-h,0], R)$, we have

$$Ex^2(t) \leqslant \gamma(1-\mu)^{-1}J, \quad t \in [-\tau - h, 0]. \tag{19}$$

By (19) and the part (b) of Lemma 2.1, we have

$$Ex^2(t) \leqslant \gamma(1-\mu)^{-1}J, \quad t \in [t_0, t_1).$$

Suppose for all $q = 1, 2, \ldots, k$, the inequalities

$$Ex^2(t) \leqslant d_0^2 d_1^2 \cdots d_{q-1}^2 \gamma(1-\mu)^{-1}J, \quad t \in \left[t_{q-1}, t_q\right), \tag{20}$$

hold, where $d_0 = 1$ and $t_0 = 0$. Then from Condition (A_3) and (20), we have

$$\begin{aligned} Ex^2(t_k) &= E\left|H_k\left(x\left(t_k^-\right)\right)\right|^2 \\ &\leqslant d_k^2 Ex^2\left(t_k^-\right) \\ &\leqslant d_0^2 d_1^2 \cdots d_{k-1}^2 d_k^2 \gamma(1-\mu)^{-1}J. \end{aligned}$$

This, together with (20) and $d_k \geqslant 1$, $k = 1, 2, \ldots$, leads to

$$Ex^2(i) \leqslant d_0^2 d_1^2 \cdots d_{k-1}^2 d_k^2 \gamma(1-\mu)^{-1}J, \quad t \in [t_k - \tau - h, t_k]. \tag{21}$$

It follows from (21) and the part (b) of Lemma 2.1 that

$$Ex^2(x) \leqslant d_0^2 d_1^2 \cdots d_{k-1}^2 d_k^2 \gamma(1-\mu)^{-1}J, \quad t \in [t_k, t_{k+1}),$$

By mathematical induction, we can conclude that

$$Ex^2(t) \leqslant d_0^2 d_1^2 \cdots d_{k-1}^2 \gamma(1-\mu)^{-1}J, \quad t \in [t_{k-1}, t_k], \ k = 1, 2, \ldots. \tag{22}$$

Noticing that $e^\sigma = \prod_{k=1}^{\infty} d_k^2 < \infty$, by Condition (A_4), we can use (22) to conclude that

$$\begin{aligned} Ex^2(t) &\leqslant d_0^2 d_1^2 \cdots d_{k-1}^2 \gamma(1-\mu)^{-1}J \\ &\leqslant e^\sigma \gamma(1-\mu)^{-1}J, \quad t \in [t_{k-1}, t_k), \ k = 1, 2, \ldots. \end{aligned}$$

This implies that the conclusion holds and the proof is complete. □

THEOREM 3.3. *If $(A_1) - (A_2)$ and (A_5) hold, then*

$$S = \left\{ \phi \in PC^b_{\mathscr{F}_0}([-h,0], R) \,\middle|\, \|\phi\|^2_{L^2} \leqslant (1-\mu)^{-1}J \right\}$$

is a invariant set and also a global attracting set of (1).

Proof. Since $d_k \leqslant 1$, a direct calculation shows that $\alpha^* = 0$ and $\sigma = 0$ in Theorem 3.1 and Theorem 3.2. It follows from Theorem 3.1 the set S is a global attracting set of (1). It follows from Theorem 3.2 the set S is a invariant set of (1). □

If $H_k(x(t_k)) \equiv x(t_k)$, $k = 1,2,\ldots,$ the system (1) reduces to the following system without impulses

$$\begin{aligned} x(t+\tau) = {} & F(t, x(t-h_m), \ldots, x(t-h_m), x(t)) \\ & + G(t, x(t-h), \ldots, x(t-h_m), x(t))\, \xi(t+\tau), \quad t \geqslant -\tau, \end{aligned} \tag{23}$$

with initial condition

$$x_t = \varphi_t, \quad t \in [-\tau, 0].$$

By Theorem 3.3, we can obtain the following result.

COROLLARY 3.1. *If* (A_1) *and* (A_2) *hold, then*

$$S = \left\{ \phi \in PC^b_{\mathscr{F}_0}([-h,0], R) \,\middle|\, \|\phi\|^2_{L_2} \leqslant (1-\mu)^{-1} J \right\}$$

is a invariant set and also a global attracting set of (23).

We easily observe $x(t) = 0$ is a solution of Eq. (1) from (A_3) and (A_6). In the following, we give the attractivity of the zero solution and the proof is similar to that of Theorem 3.1.

THEOREM 3.4. *If* (A_3), (A_4), (A_6) *and* (A_7) *hold, then the zero solution of Eq. (1) is mean square exponential stable and the exponential convergence rate is equal to* $\lambda^* - \alpha^*$.

4. Example

In this section, we shall discuss an example in order to illustrate the effectiveness of our results.

EXAMPLE 4.1. Consider the following impulsive stochastic difference equation with continuous time:

$$\begin{cases} x(t+1) = a\sin(x(t)) - bx(t-1) + 1 + cx(t)\,\xi(t+1),\, t \neq t_k, \quad t \geqslant -1, \\ x(t_k) = e^{d^k} x(t_k^-), \quad t = t_k, \end{cases} \tag{24}$$

where $t_k = t_{k-1} + \rho k$, $k = 1,2,\ldots,$ a,b,c,d are nonnegative constants and ρ is a positive constant. Thus

$$\begin{aligned} & h = 1, \quad \tau = 1, \qquad F(t, x(t-h_m), \ldots, x(t-h_1) x(t)) = a\sin(x(t)) - bx(t-1) + 1, \\ & G(t, x(t-h_m), \ldots, x(t-h_1), x(t)) = cx(t), \qquad H_k(x(t_k)) = e^{d^k} x(t_k^-), \end{aligned}$$

yielding

$$\begin{aligned} & |F(t, x(t-h_m), \ldots, x(t-h_1) x(t))| \leqslant a|x(t)| + b|x(t-1)| + 1 \\ & |G(t, x(t-h_m), \ldots, x(t-h_1), x(t))| \leqslant c|x(t)|, \qquad |H_k(x(t_k))| = e^{d^k} \left| x(t_k^-) \right|. \end{aligned}$$

So, the parameters of Conditions $(A_1), (A_2)$ and (A_3) are as follows:

$$a_0(t) = a, \quad a_1(t) = b, \quad b_0(t) = c, \quad b_1(t) = 0, \quad J_1 = 1, \quad J_2 = 0, \quad t \geqslant -1.$$

If $\mu = (a+b)^2 + c^2 < 1$, $\frac{2\ln d_k}{t_k - t_{k-1}} = \frac{2d^k}{\rho k} < \frac{1}{2}\ln\frac{1}{\mu}$, $k = 1,2,\ldots$, and $\sigma = 2\sum_{k=1}^{\infty}\ln d_k = 2\sum_{k=1}^{\infty} d^k < \infty$. Then the conditions $(A_1),(A_2)$, (A_3) and (A_4) are satisfied. So, by Theorem 3.1, we can get that

$$S = \left\{ \phi \in PC^b_{\mathscr{F}_0}([-1,0],R) \,\middle|\, \|\phi\|^2_{L^2} \leqslant 2e^{\sigma}(1-\mu)^{-1} \right\}$$

is a global attracting set of (24). By Theorem 3.2, we can get that S is a quasi-invariant set of (24).

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