Mathematical
Inequalities
& Applications
Volume 16, Number 4 (2013), 999–1007 doi:10.7153/mia-16-78

ON log MAJORIZATIONS FOR POSITIVE SEMIDEFINITE MATRICES

C.-S. LIN AND Y. J. CHO

(*Communicated by Josip Pečarić*)

Abstract. In this paper, we show that every log majorization for positive semidefinite matrices can be expressed in countable sets of log majorizations and all of them are equivalent to one another.

1. Introduction and preliminaries

This paper is a continuation of the author's previous one in Lin [10], where it was shown that the well-known Furuta inequality can be expressed in countable sets of operator inequalities in two forms, one is $(YXY)^{\beta}$ and the other the β-power-mean. So are the ground Furuta inequality and its generalization and the chaotic order for two operators. Generally speaking, it was shown in Lin [10] that each Furuta-type operator inequality has many expressions (countable sets, indeed) and they are all equivalent to one another. For more information on operator inequalities, refer to the papers [2], [3], [5], [6], [9] and [11]–[13].

In this paper, we show that the log majorizations for positive semidefinite matrices have the same properties, i.e., every log majorization can be expressed by countable sets of log majorizations, and they are all equivalent to one another.

Recall that the β*-power-mean* introduced by Kubo-Ando [9] was given by

$$A \natural_{\beta} B = A^{1/2}(A^{-1/2}BA^{-1/2})^{\beta}A^{1/2} \tag{1.1}$$

for two operators $A,B > 0$ and any real number β.

The β-power-mean is a useful tool in expressing alternatively the Furuta-type operator inequalities and the chaotic order of two operators as we have seen in the literature in the past twenty plus some years. In case $\alpha \in [0,1]$, we write

$$A \sharp_{\alpha} B = A^{1/2}(A^{-1/2}BA^{-1/2})^{\alpha}A^{1/2}$$

and call it the α*-power-mean*.

The following lemma is the operator expansion of operator form $(YXY^*)^{\beta}$.

Mathematics subject classification (2010): 47A63.
Keywords and phrases: Log majorization for positive semidefinte matrices, β-power-mean, α-power-mean, Furuta's inequality, Furuta-type operator inequality.

Paper MIA-16-78

LEMMA 1. ([4], Lemma 1) *Let X be positive and invertible and Y be an invertible operator. For any real number β, we have*

$$(YXY^*)^\beta = YX^{1/2}(X^{1/2}Y^*YX^{1/2})^{\beta-1}X^{1/2}Y^*. \tag{1.2}$$

Due to (1.1) and (1.2) above, we produced the following two results in Lin [10]. In what follows, let $i = 0,2,4,6,\cdots$, i.e., i is an even number and $j = 1,3,5,\cdots$, i.e., j is an odd number.

Lemmas 2 and 3 below are the β-power-mean expressed in terms of countable sets of the operators form $(YXY)^\beta$.

LEMMA 2. ([10], Lemma 1) *For any real number β and operators $A,B > 0$,*
(1) $A\natural_\beta B = (AB^{-1})^{i/2}A^{1/2}(A^{-1/2}BA^{-1/2})^{\beta+i}A^{1/2}(B^{-1}A)^{i/2}$.
(2) $A\natural_\beta B = (AB^{-1})^{\frac{j+1}{2}}B^{1/2}(B^{1/2}A^{-1}B^{1/2})^{\beta+j}B^{1/2}(B^{-1}A)^{\frac{j+1}{2}}$.

LEMMA 3. ([10], Lemma 2) *For any real number β and two operators $A,B > 0$,*
(1) $A\natural_\beta B = (BA^{-1})^{i/2}A^{1/2}(A^{-1/2}BA^{-1/2})^{\beta-i}A^{1/2}(A^{-1}B)^{i/2}$.
(2) $A\natural_\beta B = (BA^{-1})^{\frac{j-1}{2}}B^{1/2}(B^{1/2}A^{-1}B^{1/2})^{\beta-j}B^{1/2}(A^{-1}B)^{\frac{j-1}{2}}$.

In what follows, the capital letters mean the positive semidefinite matrices unless otherwise stated. For any operators $A,B \geqslant 0$, let $\{\lambda_i(A)\}_{i=1}^n$ and $\{\lambda_i(B)\}_{i=1}^n$ be the families of eigenvalues of A and B, respectively, satisfying the following:

$$\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A), \quad \lambda_1(B) \geqslant \lambda_2(B)\cdots\lambda_n(B),$$

i.e., they are arranged in decreasing order.

Following Ando and Hiai [1], we recall the standard notation that $A \succ_{(\log)} B$ is the log *majorization* for two operators $A,B \geqslant 0$ if $\Pi_{i=1}^k\lambda_i(A) \geqslant \Pi_{i=1}^k\lambda_i(B)$ for each $k = 1,2,\cdots,n-1$ and $\Pi_{i=1}^n\lambda_i(A) = \Pi_{i=1}^n\lambda_i(B)$, i.e., det A= det B.

The following fundamental result for the log majorization is due to Ando and Hiai [1]:

THEOREM 1. ([1]) *For any operators $A,B \geqslant 0$ and $0 \leqslant \alpha \leqslant 1$, we have*

$$(A\sharp_\alpha B)^r \succ_{(\log)} A^r\sharp_\alpha B^r$$

for each $r \geqslant 1$.

Some generalizations of Theorem 1 can be found by Furuta in [7]. In fact, they are just special cases of the next most extension of Theorem 1.

THEOREM 2. ([7], Theorem 2.1) *For any operators $A > 0$, $B \geqslant 0$, p_1, p_2, $\cdots$, $p_{2(n-1)}$, $p_{2n} \geqslant 1$ $(n \geqslant 2)$, $t \in [0,1]$ and $r \geqslant t$, the relation*

$$\begin{aligned}(A\sharp_{1/p_1}B)^h \succ_{(\log)} A^{1-t+r}\sharp_\beta \{A^{1-t}\natural_{p_{2n}}\{A\natural_{p_{2n-1}}\{A^{1-t}\natural_{p_{2n-2}}\{A\natural_{p_{2n-3}} \\ \{A^{1-t}\natural_{p_{2n-4}}\{A\natural_{p_{2n-5}}\{A^{1-t}\cdots\{A\natural_{p_3}(A^{1-t}\natural_{p_2}B)\}\}\cdots\}\}\end{aligned} \tag{1.3}$$

holds, where $h = \frac{p_1p_2\cdots p_{2n}(1-t+r)}{(\cdots((((p_1-t)p_2+t)p_3-t)p_4+t)p_5-\cdots-t)p_{2n}+r}$ *and* $\beta = \frac{h}{P_1p_2\cdots p_{2n}}$.

More precisely, in Theorem 2, there exist $n-1$ terms of A and n terms of A^{1-t} alternatively arranged on the left side of the term B. Also, there exists one term of the brackte $()$ and are $2n-2$ terms of the brackets $\{\}$ involved. In the denominator of h, $-t$ appears n times and t appears $n-1$ times, and there are $2n-1$ times of the brackets $()$ involved.

2. Main results

Again, as stated before, let $i = 0,2,4,6,\cdots$, i.e., i is an even number, and $j = 1,3,5,\cdots$, i.e., j is an odd number.

THEOREM 3. *For any operators $A > 0$, $B \geqslant 0$, $p_1, p_2, \cdots, p_{2(n-1)}, p_{2n} \geqslant 1$ for any natural number $n, t \in [0,1]$ and $r \geqslant t$, the following statements hold with*

$$h = \frac{p_1 p_2 \cdots p_{2n}(1-t+r)}{(\cdots((((p_1-t)p_2+t)p_3-t)p_4+t)p_5-\cdots-t)p_{2n}+r}, \quad \beta = \frac{h}{p_1 p_2 \cdots p_{2n}}.$$

Moreover, they are equivalent to one another, where Z is the second term of the right side of the log *majorization* (1.3) *in Theorem 2, i.e.,*

$$Z = \{A^{1-t} \natural_{p_{2n}} \{A \natural_{p_{2n-1}} \{A^{1-t} \natural_{p_{2n-2}} \{A \natural_{p_{2n-3}} \{A^{1-t} \natural_{p_{2n-4}} \{A \natural_{p_{2n-5}} \{A^{1-t} \cdots \{A \natural_{p_3} (A^{1-t} \natural_{p_2} B)\}\} \cdots\}\}.$$

(1) *By Theorem 2,*

$$(A \sharp_{1/p_1} B)^h \succ_{(\log)} A^{1-t+r} \sharp_\beta \{A^{1-t} \natural_{p_{2n}} \{A \natural_{p_{2n-1}} \{A^{1-t} \natural_{p_{2n-2}} \{A \natural_{p_{2n-3}} \{A^{1-t} \natural_{p_{2n-4}} \{A \natural_{p_{2n-5}} \{A^{1-t} \cdots \{A \natural_{p_3} (A^{1-t} \natural_{p_2} B)\}\} \cdots\}\},$$

i.e., shortly,

$$(A \sharp_{1/p_1} B)^h \succ_{(\log)} A^{1-t+r} \sharp_\beta Z.$$

(2) $(A \sharp_{1/p_1} B)^h \succ_{(\log)} (A^{1-t+r}Z^{-1})^{i/2}(A^{1-t+r} \natural_{\beta+i} Z)(Z^{-1}A^{1-t+r})^{i/2}.$

(3) $(A \sharp_{1/p_1} B)^h \succ_{(\log)} (A^{1-t+r}Z^{-1})^{\frac{j+1}{2}} Z(Z^{-1} \natural_{\beta+j} A^{-(1-t+r)})Z(Z^{-1}A^{1-t+r})^{\frac{j+1}{2}}.$

(4) $(A \sharp_{1/p_1} B)^h \succ_{(\log)} (ZA^{-(1-t+r)})^{i/2}(A^{1-t+r} \natural_{\beta-i} Z)(A^{-(1-t+r)}Z)^{i/2}.$

(5) $(A \sharp_{1/p_1} B)^h \succ_{(\log)} (ZA^{-(1-t+r)})^{\frac{j-1}{2}} Z(Z^{-1} \natural_{\beta-j} A^{-(1-t+r)})Z(A^{-(1-t+r)}Z)^{\frac{j-1}{2}}.$

Proof. Notice that, in each of the above five expressions, the left side of the log majorization is the same and we proof only for the right side.

(1) $\Longrightarrow$ (2) By (1) in Lemma 2 and (1.1), we have

$$\begin{aligned} A^{1-t+r} \sharp_\beta Z &= (A^{1-t+r}Z^{-1})^{i/2} A^{\frac{1-t+r}{2}} (A^{-\frac{1-t+r}{2}} Z A^{-\frac{1-t+r}{2}})^{\beta+i} A^{\frac{1-t+r}{2}} (Z^{-1}A^{1-t+r})^{i/2} \\ &= (A^{1-t+r}Z^{-1})^{i/2}(A^{1-t+r} \natural_{\beta+i} Z)(Z^{-1}A^{1-t+r})^{i/2}. \end{aligned}$$

(1) $\Longrightarrow$ (3) Due to (2) in Lemma 2 and (1.1), we have

$$\begin{aligned} A^{1-t+r}\sharp_\beta Z &= (A^{1-t+r}Z^{-1})^{\frac{j+1}{2}}Z^{1/2}(Z^{1/2}A^{-(1-t+r)}Z^{1/2})^{\beta+j}Z^{1/2}(Z^{-1}A^{1-t+r})^{\frac{j+1}{2}} \\ &= (A^{1-t+r}Z^{-1})^{\frac{j+1}{2}}Z(Z^{-1}\natural_{\beta+j}A^{-(1-t+r)})Z(Z^{-1}A^{1-t+r})^{\frac{j+1}{2}}. \end{aligned}$$

(1) $\Longrightarrow$ (4) By (1) in Lemma 3 and (1.1), we have

$$\begin{aligned} A^{1-t+r}\sharp_\beta Z &= (ZA^{-(1-t+r)})^{i/2}A^{\frac{1-t+r}{2}}(A^{-\frac{1-t+r}{2}}ZA^{-\frac{1-t+r}{2}})^{\beta-i}A^{\frac{1-t+r}{2}}(A^{-(1-t+r)}Z)^{i/2} \\ &= (ZA^{-(1-t+r)})^{i/2}(A^{1-t+r}\sharp_{\beta-1}Z)(A^{-(1-t+r)}Z)^{i/2}. \end{aligned}$$

(1) $\Longrightarrow$ (5) It follows from (2) in Lemma 3 and (1.1) that

$$\begin{aligned} A^{1-t+r}\sharp_\beta Z &= (ZA^{-(1-t+r)})^{\frac{j-1}{2}}Z^{1/2}(Z^{1/2}A^{-(1-t+r)}Z^{1/2})^{\beta-j}Z^{1/2}(A^{-(1-t+r)}Z)^{\frac{j-1}{2}} \\ &= (ZA^{-(1-t+r)})^{\frac{j-1}{2}}Z(Z^{-1}\sharp_{\beta-j}A^{-(1-t+r)}Z(A^{-(1-t+r)}Z)^{\frac{j-1}{2}}. \end{aligned}$$

(2) or (4) $\Longrightarrow$ (1) Let $i=0$ in (2) or (4).

(3) $\Longrightarrow$ (1) Let $j=1$ in (3). Then the right side of (3) is

$$A^{1-t+r}(Z^{-1}\natural_{\beta+1}A^{-(1-t+r)})A^{1-t+r},$$

which is the same as $A^{1-t+r}\sharp_\beta Z$ if we let $j=1$ in the proof of (1) $\Longrightarrow$ (3).

(5) $\Longrightarrow$ (1) Let $j=1$ in (5). Then the right side of (5) is

$$Z(Z^{-1}\natural_{\beta-1}A^{-(1-t+r)})Z,$$

which is the same as $A^{1-t+r}\sharp_\beta Z$ if we let $j=1$ in the proof of (1) $\Longrightarrow$ (5). This completes the proof.

By using the similar arguments, we could consider the left side (and keep the right side) of the log majorization in Theorem 2. In fact, the next result is equivalent to Theorem 3. $\square$

THEOREM 4. *Let the assumptions be exactly the same as stated in Theorem 3 Then the following statements hold and they are equivalent to one another.*

(1) *By Theorem 2,*

$$\begin{aligned} (A\sharp_{1/p_1}B)^h \succ_{(\log)} A^{1-t+r}\sharp_\beta\{A^{1-t}\sharp_{p_{2n}}\{A\sharp_{p_{2n-1}}\{A^{1-t}\sharp_{p_{2n-2}}\{A\sharp_{p_{2n-3}} \\ \{A^{1-t}\sharp_{p_{2n-4}}\{A\sharp_{p_{2n-5}}\{A^{1-t}\cdots\{A\sharp_{p_3}(A^{1-t}\sharp_{p_2}B)\}\}\cdots\}\}, \end{aligned}$$

i.e., in short,

$$(A\sharp_{1/p_1}B)^h \succ_{(\log)} A^{1-t+r}\sharp_\beta Z.$$

(2) $[(AB^{-1})^{i/2}(A\natural_{\frac{1}{p_1}+i}B)(B^{-1}A)^{i/2}]^h \succ_{(\log)} A^{1-t+r}\sharp_\beta Z.$

(3) $[(AB^{-1})^{\frac{j+1}{2}}B(B^{-1}\natural_{\frac{1}{p_1}+j}A^{-1})B(B^{-1}A)^{\frac{j+1}{2}}]^h \succ_{(\log)} A^{1-t+r}\sharp_\beta Z.$

(4) $[(BA^{-1})^{i/2}(A\natural_{\frac{1}{p_1}-i}B)(A^{-1}B)^{i/2}]^h \succ_{(\log)} A^{1-t+r}\sharp_\beta Z.$

(5) $[(BA^{-1})^{\frac{j-1}{2}}B(B^{-1}\natural_{\frac{1}{p_1}-j}A^{-1})B(A^{-1}B)^{\frac{j-1}{2}}]^h \succ_{(\log)} A^{1-t+r}\sharp_\beta Z.$

Proof. Now, we prove the left side of the log majorization only as the right sides are all the same.

(1) $\Longrightarrow$ (2) From (1) in Lemma 2 and (1.1), it follows that

$$\begin{aligned} A\sharp_{1/p_1} &= (AB^{-1})^{i/2}A^{1/2}(A^{-1/2}BA^{-1/2})^{\frac{1}{p_1}+i}A^{1/2}(B^{-1}A)^{i/2} \\ &= (AB^{-1})^{i/2}(A\natural_{\frac{1}{p_1}+i}B)(B^{-1}A)^{i/2}. \end{aligned}$$

(1) $\Longrightarrow$ (3) From (2) in Lemma 2 and (1.1), it follows that

$$\begin{aligned} A\sharp_{1/p_1}B &= (AB^{-1})^{\frac{j+1}{2}}B^{1/2}(B^{1/2}A^{-1}B^{1/2})^{\frac{1}{p_1}+j}B^{1/2}(B^{-1}A)^{\frac{j+1}{2}} \\ &= (AB^{-1})^{\frac{j+1}{2}}B(B^{-1}\natural_{\frac{1}{p_1}+j}A^{-1})B(B^{-1}A)^{\frac{j+1}{2}}. \end{aligned}$$

(1) $\Longrightarrow$ (4) From (1) in Lemma 3 and (1.1), it follows that

$$\begin{aligned} A\sharp_{1/p_1}B &= (BA^{-1})^{i/2}A^{1/2}(A^{-1/2}BA^{-1/2})^{\frac{1}{p_1}-i}A^{1/2}(A^{-1}B)^{i/2} \\ &= (BA^{-1})^{i/2}(A\natural_{\frac{1}{p_1}-i}B)(A^{-1}B)^{i/2}. \end{aligned}$$

(1) $\Longrightarrow$ (5) From (2) in Lemma 3 and (1.1), it follows that

$$\begin{aligned} A\sharp_{1/p_1}B &= (BA^{-1})^{\frac{j-1}{2}}B^{1/2}(B^{1/2}A^{-1}B^{1/2})^{\frac{1}{p_1}-j}B^{1/2}(A^{-1}B)^{\frac{j-1}{2}} \\ &= (BA^{-1})^{\frac{j-1}{2}}B(B^{-1}\natural_{\frac{1}{p_1}-j}A^{-1})B(A^{-1}B)^{\frac{j-1}{2}}. \end{aligned}$$

(2) or (4) $\Longrightarrow$ (1) Let $i=0$ in (2) or (4). Then we get the result.

(3) $\Longrightarrow$ (1) Let $j=1$ in (3). Then the left side of (3) is $[A(B^{-1}\natural_{\frac{1}{p_1}+1}A^{-1})A]^h$, which is the same as $(A\sharp_{1/p_1}B)^h$ if we let $j=1$ in the proof of (1) $\Longrightarrow$ (3).

(5) $\Longrightarrow$ (1) Let $j=1$ in (5). Then the left side of (5) is $[B(B^{-1}\sharp_{\frac{1}{p_1}-1}A^{-1})B]^h$, which is the same as $(A\sharp_{1/p_1}B)^h$ if we let $j=1$ in the proof of (1) $\Longrightarrow$ (5). This completes the proof. $\square$

3. Some consequences of the main results

Each result in this section are special cases of Theorems 3 and 4 in the section 2. Now, we combine both results into just one as Theorems 3 and 4 are equivalent to each other.

COROLLARY 1. *Let $A>0,\ B\geqslant 0,\ p_1,\ p_2,\ p_3,\ p_4\geqslant 1,\ t\in[0,1]$ and $r\geqslant t$. Then the following statements hold and they are equivalent to one another, where*

$$Z=\{A^{1-t}\sharp_{p_4}\{A\sharp_{p_3}(A^{1-t}\sharp_{p_2}B)\}\},$$

$$h=\frac{p_1p_2p_3p_4(1-t+r)}{(((p_1-t)p_2+t)p_3-t)p_4+r},\quad \beta=\frac{h}{p_1p_2p_3p_4}.$$

(1) *By Furuta* [8], *we have*

$$(A \sharp_{1/p_1} B)^h \succ_{(\log)} A^{1-t+r} \sharp_\beta \{A^{1-t} \natural_{p_4} \{A \natural_{p_3} (A^{1-t} \natural_{p_2} B)\}\},$$

i.e., in short,

$$(A \sharp_{1/p_1} B)^h \succ_{(\log)} A^{1-t+r} \sharp_\beta Z.$$

(2) $(A \sharp_{1/p_1} B)^h \succ_{(\log)} (A^{1-t+r}Z^{-1})^{i/2}(A^{1-t+r} \natural_{\beta+i} Z)(Z^{-1}A^{1-t+r})^{i/2}$.

(3) $(A \sharp_{1/p_1} B)^h \succ_{(\log)} (A^{1-t+r}Z^{-1})^{\frac{j+1}{2}} Z(Z^{-1} \natural_{\beta+j} A^{-(1-t+r)})Z(Z^{-1}A^{1-t+r})^{\frac{j+1}{2}}$.

(4) $(A \sharp_{1/p_1} B)^h \succ_{(\log)} (ZA^{-(1-t+r)})^{i/2}(A^{1-t+r} \natural_{\beta-i} Z)(A^{-(1-t+r)}Z)^{i/2}$.

(5) $(A \sharp_{1/p_1} B)^h \succ_{(\log)} (ZA^{-(1-t+r)})^{\frac{j-1}{2}} Z(Z^{-1} \natural_{\beta-j} A^{-(1-t+r)})Z(A^{-(1-t+r)}Z)^{\frac{j-1}{2}}$.

(6) $[(AB^{-1})^{i/2}(A \natural_{\frac{1}{p_1}+i} B)(B^{-1}A)^{i/2}]^h \succ_{(\log)} A^{1-t+r} \sharp_\beta Z$.

(7) $[(AB^{-1})^{\frac{j+1}{2}} B(B^{-1} \natural_{\frac{1}{p_1}+j} A^{-1})B(B^{-1}A)^{\frac{j+1}{2}}]^h \succ_{(\log)} A^{1-t+r} \sharp_\beta Z$.

(8) $[(BA^{-1})^{i/2}(A \natural_{\frac{1}{p_1}-i} B)(A^{-1}B)^{i/2}]^h \succ_{(\log)} A^{1-t+r} \sharp_\beta Z$.

(9) $[(BA^{-1})^{\frac{j-1}{2}} B(B^{-1} \natural_{\frac{1}{p_1}-j} A^{-1})B(A^{-1}B)^{\frac{j-1}{2}}]^h \succ_{(\log)} A^{1-t+r} \sharp_\beta Z$.

Proof. Let $n = 2$ in Theorems 3 and 4. Then we have the results. □

COROLLARY 2. *Let* $A > 0$, $B \geqslant 0$, $0 \leqslant \alpha \leqslant 1$, $t \in [0,1]$, $s \geqslant 1$ *and* $r \geqslant t$. *Then the following statements hold and they are equivalent to one another, where* $h = \frac{(1-t+r)s}{(1-\alpha t)s+\alpha r}$ *and* $\beta = \frac{h}{s}\alpha$.

(1) $(A \sharp_\alpha B)^h \succ_{(\log)} A^{1-t+r} \sharp_\beta (A^{1-t} \natural_s B)$ (*by Furuta* [8]).

(2) $(A \sharp_\alpha B)^h \succ_{(\log)} [A^{1-t+r}(A^{1-t} \natural_s B)^{-1}]^{i/2}[A^{1-t+r} \natural_{\beta+i} (A^{1-t} \natural_s B)]$
$\times [(A^{1-t} \natural_s B)^{-1}A^{1-t+r}]^{i/2}$.

(3) $(A \sharp_\alpha B)^h \succ_{(\log)} [A^{1-t+r}(A^{1-t} \natural_s B)^{-1}]^{\frac{j+1}{2}}(A^{1-t} \natural_s B)[(A^{1-t} \natural_s B)^{-1} \natural_{\beta+j}$
$\times A^{-(1-t+r)}](A^{1-t} \natural_s B)[(A^{1-t} \natural_s B)^{-1}A^{1-t+r}]^{\frac{J+1}{2}}$.

(4) $(A \sharp_\alpha B)^h \succ_{(\log)} [(A^{1-t} \natural_s B)A^{-(1-t+r)}]^{i/2}[A^{1-t+r} \natural_{\beta-i} (A^{1-t} \sharp_s B)]$
$\times [A^{-(1-t+r)}(A^{1-t} \natural_s B)]^{i/2}$.

(5) $(A \sharp_\alpha B)^h \succ_{(\log)} [(A^{1-t} \natural_s B)A^{-(1-t+r)}]^{\frac{j-1}{2}}(A^{1-t} \natural_s B)[(A^{1-t} \natural_s B)^{-1} \natural_{\beta-j}$
$\times A^{-(1-t+r)}](A^{1-t} \natural_s B)[A^{-(1-t+r)}(A^{1-t} \natural_s B)]^{\frac{j-1}{2}}$.

(6) $[(AB^{-1})^{i/2}(A \natural_{\alpha+i} B)(B^{-1}A)^{i/2}]^h \succ_{(\log)} A^{1-t+r} \sharp_\beta (A^{1-t} \natural_s B)$.

(7) $[(AB^{-1})^{\frac{j+1}{2}} B(B^{-1} \natural_{\alpha+j} A^{-1})B(B^{-1}A)^{\frac{j+1}{2}}]^h \succ_{(\log)} A^{1-t+r} \sharp_\beta (A^{1-t} \natural_s B)$.

(8) $[(BA^{-1})^{i/2}(A \natural_{\alpha-i} B)(A^{-1}B)^{i/2}]^h \succ_{(\log)} A^{1-t+r} \sharp_\beta (A^{1-t} \natural_s B)$.

(9) $[(BA^{-1})^{\frac{j-1}{2}} B(B^{-1} \natural_{\alpha-j} A^{-1})B(A^{-1}B)^{\frac{j-1}{2}}]^h \succ_{(\log)} A^{1-t+r} \sharp_\beta (A^{1-t} \natural_s B)$.

Proof. Let $\frac{1}{p_1} = \alpha \in [0,1]$, $p_2 = p_3 = 1$ and $p_4 = s$ in Corollary 1. Then we have the following:

$$Z = \{A^{1-t} \sharp_{p_4} \{A \sharp_{p_3} (A^{1-t} \sharp_{p_2} B)\}\} = A^{1-t} \natural_s B. \quad \square$$

COROLLARY 3. *Let* $A,B \geqslant 0,\ 0 \leqslant \alpha \leqslant 1,\ s \geqslant 1$ *and* $r \geqslant 1$ *with*

$$h = [\alpha s^{-1} + (1-\alpha)r^{-1}]^{-1}.$$

Then the following statements hold and they are equivalent to one another.

(1) $(A \sharp_\alpha B)^h \succ_{(\log)} A^r \sharp_{\frac{h}{s}\alpha} B^s$ (*by Furuta* [4]).

(2) $(A \sharp_\alpha B)^h \succ_{(\log)} (A^r B^{-s})^{i/2}(A^r \natural_{\frac{h}{s}\alpha+i} B^s)(B^{-s}A^r)^{i/2}$.

(3) $(A \sharp_\alpha B)^h \succ_{(\log)} (A^r B^{-s})^{\frac{j+1}{2}} B^s (B^{-s} \natural_{\frac{h}{s}\alpha+j} A^{-r}) B^s (B^{-s}A^r)^{\frac{j+1}{2}}$.

(4) $(A \sharp_\alpha B)^h \succ_{(\log)} (B^s A^{-r})^{i/2}(A^r \natural_{\frac{h}{s}\alpha-i} B^s)(A^{-r}B^s)^{i/2}$.

(5) $(A \sharp_\alpha B)^h \succ_{(\log)} (B^s A^{-r})^{\frac{j-1}{2}} B^s (B^{-s} \natural_{\frac{h}{s}\alpha-j} A^{-r}) B^s (A^{-r}B^s)^{\frac{j-1}{2}}$.

(6) $[(AB^{-1})^{i/2}(A \natural_{\alpha+i} B)(B^{-1}A)^{i/2}]^h \succ_{(\log)} A^r \sharp_{\frac{h}{s}\alpha} B^s$.

(7) $[(AB^{-1})^{\frac{j+1}{2}} B(B^{-1} \natural_{\alpha+j} A^{-1})B(B^{-1}A)^{\frac{j+1}{2}}]^h \succ_{(\log)} A^r \sharp_{\frac{h}{s}\alpha} B^s$.

(8) $[(BA^{-1})^{i/2}(A \natural_{\alpha-i} B)(A^{-1}B)^{i/2}]^h \succ_{(\log)} A^r \sharp_{\frac{h}{s}\alpha} B^s$.

(9) $[(BA^{-1})^{\frac{j-1}{2}} B(B^{-1} \natural_{\alpha-j} A^{-1})B(A^{-1}B)^{\frac{j-1}{2}}]^h \succ_{(\log)} A^r \sharp_{\frac{h}{s}\alpha} B^s$.

Proof. Let $t = 1$ in Corollary 2. Then we have $A^{1-t} \natural_s B = B^s$. □

COROLLARY 4. *Let* $A,B \geqslant 0,\ 0 \leqslant \alpha \leqslant 1,$ *and* $r \geqslant 1$. *Then the following statements hold and they are equivalent to one another.*

(1) $(A \sharp_\alpha B)^r \succ_{(\log)} A^r \sharp_\alpha B^r$ (*by Ando and Hiai* [1]).

(2) $(A \sharp_\alpha B)^r \succ_{(\log)} (A^r B^{-r})^{i/2}(A^r \natural_{\alpha+i} B^r)(B^{-r}A^r)^{i/2}$.

(3) $(A \sharp_\alpha B)^r \succ_{(\log)} (A^r B^{-r})^{\frac{j+1}{2}} B^r (B^{-r} \natural_{\alpha+j} A^{-r}) B^r (B^{-r}A^r)^{\frac{j+1}{2}}$.

(4) $(A \sharp_\alpha B)^r \succ_{(\log)} (B^r A^{-r})^{i/2}(A^r \natural_{\alpha-i} B^r)(A^{-r}B^r)^{i/2}$.

(5) $(A \sharp_\alpha B)^r \succ_{(\log)} (B^r A^{-r})^{\frac{j-1}{2}} B^r (B^{-r} \natural_{\alpha-j} A^{-r}) B^r (A^{-r}B^r)^{\frac{j-1}{2}}$.

(6) $[(AB^{-1})^{i/2}(A \natural_{\alpha+i} B)(B^{-1}A)^{i/2}]^r \succ_{(\log)} A^r \sharp_\alpha B^r$.

(7) $[(AB^{-1})^{\frac{j+1}{2}} B(B^{-1} \natural_{\alpha+j} A^{-1})B(B^{-1}A)^{\frac{j+1}{2}}]^r \succ_{(\log)} A^r \sharp_\alpha B^r$.

(8) $[(BA^{-1})^{i/2}(A \natural_{\alpha-i} B)(A^{-1}B)^{i/2}]^r \succ_{(\log)} A^r \sharp_\alpha B^r$.

(9) $[(BA^{-1})^{\frac{j-1}{2}} B(B^{-1} \natural_{\alpha-j} A^{-1})B(A^{-1}B)^{\frac{j-1}{2}}]^r \succ_{(\log)} A^r \sharp_\alpha B^r$.

Proof. Let $s = r$ in Corollary 3, so that $h = r$. □

4. Remarks

In conclusion, we would like to mention that there are interesting relations among the Furuta-type operator inequalities and the log majorizations. Indeed, Furuta proved the following five equivalent relations.

(I) Theorem 2 is equivalent to Theorem 5 as follows:

THEOREM 5. *Let* $A \geqslant B \geqslant 0,\ A > 0,\ t \in [0,1]$ *and* $p_1, p_2, \cdots, p_{2n} \geqslant 1$ *for a natural number* n. *Then the following inequality holds for* $r \geqslant t$:

$$A^{1-t+r} \geqslant \{A^{r/2}[A^{-t/2}[A^{t/2} \cdots A^{t/2}[A^{-t/2}[A^{t/2}(A^{-t/2}B^{p_1}A^{-t/2})^{p_2}A^{t/2}]^{p_3} \\ \times A^{-t/2}]^{p_4}A^{t/2}]^{p_5} \cdots A^{-t/2}]^{p_{2n-2}}A^{t/2}]^{p_{2n-1}}]A^{-t/2}]^{p_{2n}}A^{r/2}\}^{q},$$

where $q = \frac{1-t+r}{(\cdots((((p_1-t)p_2+t)p_3-t)p_4+t)p_5-\cdots-t)p_{2n}+r}$.

More precisely, in Theorem 5, there are n terms of $A^{-t/2}$ and $n-1$ terms of $A^{t/2}$ alternatively arranged on the left side of the term B^{p_1} and the same arrangement on the right side of the term B^{p_1}. For the denominator of q, there are n terms of $-t$ and $n-1$ terms of t alternatively arranged.

(II) (1) in Corollary 1 is equivalent to Theorem 6 as follows:

THEOREM 6. *Let* $A \geqslant B \geqslant 0,\ A > 0,\ t \in [0,1],\ r \geqslant t$ *and* $p_1, p_2, p_3, p_4 \geqslant 1$. *Then we have the following:*

$$A^{1-t+r} \\ \geqslant \{A^{r/2}[A^{-t/2}[A^{t/2}(A^{-t/2}B^{p_1}A^{-t/2})^{p_2}A^{t/2}]^{p_3}A^{-t/2}]^{p_4}A^{r/2}\}^{\frac{1-t+r}{(((p_1-t)p_2+t)p_3-t)p_4+r}}.$$

(III) (1) in Corollary 2 is equivalent to Theorem 7 as follows:

THEOREM 7. *Let* $A \geqslant B \geqslant 0,\ A > 0,\ r \geqslant t \in [0,1],\ p \geqslant 1$ *and* $s \geqslant 1$. *Then we have the following:*

$$A^{1-t+r} \geqslant \{A^{r/2}(A^{-t/2}B^{p}A^{-t/2})^{s}A^{r/2}]^{p_3}A^{-t/2}]^{p_4}A^{r/2}\}^{\frac{1-t+r}{(p-t)s+r}}.$$

(IV) (1) in Corollary 3 is equivalent to Theorem 8 as follows:

THEOREM 8. *Let* $A \geqslant B \geqslant 0,\ A > 0,\ r \geqslant 1,\ s \geqslant 1$ *and* $p \geqslant 1$. *Then we have the following:*

$$A^{r} \geqslant \{A^{r/2}(A^{-t/2}B^{p}A^{-t/2})^{s}A^{r/2}\}^{\frac{1-t+r}{(p-t)s+r}}.$$

(V) (1) in Corollary 4 is equivalent to Theorem 9 as follows:

THEOREM 9. *Let* $A \geqslant B \geqslant 0,\ A > 0,\ r \geqslant 1$ *and* $p \geqslant 1$. *Then we have the following:*

$$A^{r} \geqslant \{A^{r/2}(A^{-1/2}B^{p}A^{-1/2})^{r}A^{r/2}\}^{1/p}.$$

Acknowledgement.

This work was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant Number: 2012-0008170).

REFERENCES

[1] T. ANDO AND F. HIAI, *Log majorization and complementary Golden-Thomson type inequalities*, Linear Algebra Appl. **197 (198)** (1994), 113–131.
[2] R. BHATIA, *Positive Definite Matrices*, Princeton Press, 2007.
[3] T. FURUTA, *$A \geqslant B \geqslant O$ assures $(B^r A^p B^r)^{1/q} \geqslant B^{(p+2r)/q}$ for $r \geqslant 0,\ p \geqslant 0,\ q \geqslant 1$ with $(1+2r)q \geqslant p+2r$*, Proc. Amer. Math. Soc. **101** (1987), 85–88.
[4] T. FURUTA, *An extension of the Furuta inequality and Ando-Hiai* log *majorization*, Linear Algebra Appl. **219** (1995), 139–155.
[5] T. FURUTA, *Invitation to Linear Operators*, ISBN: 0-415-26799-4, Taylor & Francis, 2001.
[6] T. FURUTA, *Further extension of an order preserving operator inequality*, J. Math. Inequal. **2** (2008), 465–472.
[7] T. FURUTA, *Log majorization via an order preserving operator inequality*, Linear Algebra Appl. **431** (2009), 132–138.
[8] T. FURUTA, *An extension of order preserving operator inequality*, Math. Inequal. Appl. **13** (2010), 49–56.
[9] F. KUBO AND T. ANDO, *Means of positive linear operators*, Math. Ann. **246** (1980), 205–224.
[10] C.-S. LIN, *On operator inequalities in terms of geometric mean*, Internat. J. Pure Appl. Math. **58** (2010), 299–308.
[11] C.-S. LIN, *Generalization of β-power-mean for operators*, Internat. J. Pure Appl. Math. **62** (2010), 275–285.
[12] C.-S. LIN, *On countable sets of order preserving operator inequalities in Hilbert spaces*, Internat. J. Pure Appl. Math. **68** (2011), 189–199.
[13] C.-S. LIN AND Y. J. CHO, *Characterizations of operator inequality $A \geqslant B \geqslant C$*, Math. Inequal. Appl. **14** (2011), 575–580.

(Received September 11, 2012)

C.-S. Lin
Department of Mathematics
Bishop's University
2600 College Street Sherbrooke
QC Canada J1M 1Z7
e-mail: plin@ubishops.ca

Y. J. Cho
Department of Mathematics Education
and the RINS Gyeongsang National University
Chinju 660-701, Korea
e-mail: yjcho@gsnu.ac.kr