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AN ADDITIVE FUNCTIONAL INEQUALITY IN MATRIX NORMED SPACES

CHOONKIL PARK, DONG YUN SHIN AND JUNG RYE LEE

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Abstract. Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of an additive functional inequality in matrix normed spaces.

1. Introduction and preliminaries

The abstract characterization given for linear spaces of bounded Hilbert space operators in terms of *matricially normed spaces* [42] implies that quotients, mapping spaces and various tensor products of operator spaces may again be regarded as operator spaces. Owing in part to this result, the theory of operator spaces is having an increasingly significant effect on operator algebra theory (see [10]).

The proof given in [42] appealed to the theory of ordered operator spaces [6]. Effros and Ruan [11] showed that one can give a purely metric proof of this important theorem by using a technique of Pisier [32] and Haagerup [18] (as modified in [9]).

The stability problem of functional equations originated from a question of Ulam [44] concerning the stability of group homomorphisms.

The functional equation

$$f(x+y) = f(x) + f(y)$$

is called the *Cauchy additive functional equation*. In particular, every solution of the Cauchy additive functional equation is said to be an *additive mapping*. Hyers [19] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [35] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Rassias theorem was obtained by Găvruta [15] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach.

In 1990, Rassias [36] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geqslant 1$. In 1991, Gajda [14] following the same approach as in Rassias [35], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [14], as well

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as by Rassias and Šemrl [40] that one cannot prove a Th. M. Rassias' type theorem when $p=1$ (cf. the books of Czerwik [7], Hyers, Isac and Rassias [20]). The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 12, 21, 23, 24, 28, 34], [37]–[39]).

In [16], Gilányi showed that if f satisfies the functional inequality

$$\|2f(x)+2f(y)-f(xy^{-1})\| \leqslant \|f(xy)\| \tag{1}$$

then f satisfies the Jordan-von Neumann functional equation

$$2f(x)+2f(y)=f(xy)+f(xy^{-1}).$$

See also [41]. Gilányi [17] and Fechner [13] proved the Hyers-Ulam stability of the functional inequality (1).

Park, Cho and Han [31] proved the Hyers-Ulam stability of the following functional inequalities

$$\begin{aligned} \|f(x)+f(y)+f(z)\| &\leqslant \left\|2f\left(\frac{x+y+z}{2}\right)\right\|, \\ \|f(x)+f(y)+f(z)\| &\leqslant \|f(x+y+z)\|, \\ \|f(x)+f(y)+2f(z)\| &\leqslant \left\|2f\left(\frac{x+y}{2}+z\right)\right\|. \end{aligned} \tag{2}$$

We will use the following notations:

$M_n(X)$ is the set of all $n\times n$-matrices in X;

$e_j \in M_{1,n}(\mathbb{C})$ is that j-th component is 1 and the other components are zero;

$E_{ij} \in M_n(\mathbb{C})$ is that (i,j)-component is 1 and the other components are zero;

$E_{ij}\otimes x \in M_n(X)$ is that (i,j)-component is x and the other components are zero;

For $x\in M_n(X), y\in M_k(X)$,

$$x\oplus y = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

Note that $(X,\{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X),\|\cdot\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \leqslant \|A\|\|B\|\|x\|_n$ holds for $A\in M_{k,n}(\mathbb{C})$, $x=(x_{ij})\in M_n(X)$ and $B\in M_{n,k}(\mathbb{C})$, and that $(X,\{\|\cdot\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X,\{\|\cdot\|_n\})$ is a matrix normed space.

A matrix normed space $(X,\{\|\cdot\|_n\})$ is called an *L^∞-matrix normed space* if $\|x\oplus y\|_{n+k} = \max\{\|x\|_n,\|y\|_k\}$ holds for all $x\in M_n(X)$ and all $y\in M_k(X)$.

Let E,F be vector spaces. For a given mapping $h: E\to F$ and a given positive integer n, define $h_n: M_n(E)\to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$ for all $x,y \in X$;
(3) $d(x,z) \leqslant d(x,y) + d(y,z)$ for all $x,y,z \in X$.

We recall a fundamental result in fixed point theory.

THEOREM 1. [3, 8] *Let (X,d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $\alpha < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty, \ \forall n \geqslant n_0$;
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y,y^*) \leqslant \frac{1}{1-\alpha} d(y,Jy)$ for all $y \in Y$.*

In 1996, Isac and Rassias [22] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [4, 5, 25, 27, 29, 30, 33]).

Throughout this paper, let $(X, \{\|\cdot\|_n\})$ be a matrix normed space, $(Y, \{\|\cdot\|_n\})$ a matrix Banach space and let n be a fixed positive integer.

In Section 2, we prove the Hyers-Ulam stability of the additive functional inequality (2) in matrix normed spaces by using the direct method.

In Section 3, we prove the Hyers-Ulam stability of the additive functional inequality (2) in matrix normed spaces by using the fixed method.

2. Hyers-Ulam stability of the additive functional inequality (2) in matrix normed spaces: direct method

In this section, we prove the Hyers-Ulam stability of the additive functional inequality (2) in matrix normed spaces by using the direct method.

LEMMA 1. *Let $(X, \{\|\cdot\|_n\})$ be a matrix normed space.*

(1) $\|E_{kl} \otimes x\|_n = \|x\|$ *for* $x \in X$.
(2) $\|x_{kl}\| \leqslant \|[x_{ij}]\|_n \leqslant \sum_{i,j=1}^{n} \|x_{ij}\|$ *for* $[x_{ij}] \in M_n(X)$.
(3) $\lim_{n\to\infty} x_n = x$ *if and only if* $\lim_{n\to\infty} x_{nij} = x_{ij}$ *for* $x_n = [x_{nij}], x = [x_{ij}] \in M_k(X)$.

Proof. (1) Since $E_{kl} \otimes x = e_k^* x e_l$ and $\|e_k^*\| = \|e_l\| = 1$, $\|E_{kl} \otimes x\|_n \leqslant \|x\|$. Since $e_k(E_{kl} \otimes x)e_l^* = x$, $\|x\| \leqslant \|E_{kl} \otimes x\|_n$. So $\|E_{kl} \otimes x\|_n = \|x\|$.

(2) Since $e_k x e_l^* = x_{kl}$ and $\|e_k\| = \|e_l^*\| = 1$, $\|x_{kl}\| \leqslant \|[x_{ij}]\|_n$.
Since $[x_{ij}] = \sum_{i,j=1}^n E_{ij} \otimes x_{ij}$,

$$\|[x_{ij}]\|_n = \left\| \sum_{i,j=1}^n E_{ij} \otimes x_{ij} \right\|_n \leqslant \sum_{i,j=1}^n \|E_{ij} \otimes x_{ij}\|_n = \sum_{i,j=1}^n \|x_{ij}\|.$$

(3) By (2), we have

$$\|x_{nkl} - x_{kl}\| \leqslant \|[x_{nij} - x_{ij}]\|_n = \|[x_{nij}] - [x_{ij}]\|_n \leqslant \sum_{i,j=1}^n \|x_{nij} - x_{ij}\|.$$

So we get the result. □

We need the following result.

LEMMA 2. ([31, Proposition 2.2]) *Let $f : X \to Y$ be a mapping such that*

$$\|f(a) + f(b) + f(c)\| \leqslant \|f(a+b+c)\|$$

for all $a, b, c \in X$. Then $f : X \to Y$ is additive.

THEOREM 2. *Let $f : X \to Y$ be a mapping and let $\phi : X^3 \to [0, \infty)$ be a function such that*

$$\Phi(a,b,c) := \frac{1}{2} \sum_{l=0}^{\infty} \frac{1}{2^l} \phi(2^l a, 2^l b, 2^l c) < +\infty, \tag{3}$$

$$\left\| f_n([x_{ij}]) + f_n([y_{ij}]) + f_n([z_{ij}]) \right\|_n \leqslant \left\| f_n([x_{ij}] + [y_{ij}] + [z_{ij}]) \right\|_n + \sum_{i,j=1}^n \phi(x_{ij}, y_{ij}, z_{ij}) \tag{4}$$

for all $a, b, c \in X$ and all $x = [x_{ij}]$, $y = [y_{ij}]$, $z = [z_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\left\| f_n([x_{ij}]) - A_n([x_{ij}]) \right\|_n \leqslant \sum_{i,j=1}^n \Phi(x_{ij}, x_{ij}, -2x_{ij}) \tag{5}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. When $n = 1$, (4) is equivalent to

$$\|f(a) + f(b) + f(c)\| \leqslant \|f(a+b+c)\| + \phi(a,b,c)$$

for all $a, b, c \in X$. By the same reasoning as in the proof of [31, Theorem 3.2], one can show that there is a unique additive mapping $A : X \to Y$ such that

$$\|f(a) - A(a)\| \leqslant \Phi(a, a, -2a)$$

for all $a \in X$. The mapping $A : X \to Y$ is given by

$$A(a) = \lim_{l\to\infty} \frac{1}{2^l} f(2^l a)$$

for all $a \in X$. By Lemma 1,

$$\left\| f_n([x_{ij}]) - A_n([x_{ij}]) \right\|_n \leqslant \sum_{i,j=1}^{n} \|f(x_{ij}) - A(x_{ij})\| \leqslant \sum_{i,j=1}^{n} \Phi(x_{ij}, x_{ij}, -2x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $A : X \to Y$ is a unique additive mapping satisfying (5), as desired. $\square$

COROLLARY 1. *Let r, θ be positive real numbers with $r < 1$. Let $f : X \to Y$ be a mapping such that*

$$\begin{aligned} \left\| f_n([x_{ij}]) + f_n([y_{ij}]) + f_n([z_{ij}]) \right\|_n \leqslant & \left\| f_n\left([x_{ij}] + [y_{ij}] + [z_{ij}]\right) \right\|_n \\ & + \sum_{i,j=1}^{n} \theta\left(\|x_{ij}\|^r + \|y_{ij}\|^r + \|z_{ij}\|^r\right) \end{aligned} \tag{6}$$

for all $x = [x_{ij}], y = [y_{ij}], z = [z_{ij}] \in M_n(X)$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\left\| f_n([x_{ij}]) - A_n([x_{ij}]) \right\|_n \leqslant \sum_{i,j=1}^{n} \frac{2+2^r}{2-2^r} \theta \|x_{ij}\|^r$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. Letting $\phi(a,b,c) = \theta(\|a\|^r + \|b\|^r + \|c\|^r)$ in Theorem 2, we obtain the result. $\square$

THEOREM 3. *Let $f : X \to Y$ be a mapping and let $\phi : X^3 \to [0,\infty)$ be a function satisfying* (4) *and*

$$\Phi(a,b,c) := \frac{1}{2} \sum_{l=1}^{\infty} 2^l \phi\left(\frac{a}{2^l}, \frac{b}{2^l}, \frac{c}{2^l}\right) < +\infty, \tag{7}$$

for all $a, b, c \in X$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\left\| f_n([x_{ij}]) - A_n([x_{ij}]) \right\|_n \leqslant \sum_{i,j=1}^{n} \Phi(x_{ij}, x_{ij}, -2x_{ij})$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. The proof is similar to the proof of Theorem 2. $\square$

COROLLARY 2. *Let r,θ be positive real numbers with $r>1$. Let $f:X\to Y$ be a mapping satisfying* (6)*. Then there exists a unique additive mapping $A:X\to Y$ such that*

$$\left\|f_n([x_{ij}])-A_n([x_{ij}])\right\|_n \leqslant \sum_{i,j=1}^{n}\frac{2^r+2}{2^r-2}\theta\|x_{ij}\|^r$$

for all $x=[x_{ij}]\in M_n(X)$.

Proof. Letting $\phi(a,b,c)=\theta(\|a\|^r+\|b\|^r+\|c\|^r)$ in Theorem 3, we obtain the result. □

We need the following result.

LEMMA 3. ([43]) *If E is an L^∞-matrix normed space, then $\|[x_{ij}]\|_n \leqslant \|[\|x_{ij}\|]\|_n$ for all $[x_{ij}]\in M_n(E)$.*

THEOREM 4. *Let Y be an L^∞-normed Banach space. Let $f:X\to Y$ be a mapping and let $\phi:X^3\to[0,\infty)$ be a function satisfying* (3) *and*

$$\left\|f_n([x_{ij}])+f_n([y_{ij}])+f_n([z_{ij}])\right\|_n \leqslant \left\|f_n\left([x_{ij}]+[y_{ij}]+[z_{ij}]\right)\right\|_n + \|[\phi(x_{ij},y_{ij},z_{ij})]\|_n \tag{8}$$

for all $x=[x_{ij}]$, $y=[y_{ij}]$, $z=[z_{ij}]\in M_n(X)$. Then there exists a unique additive mapping $A:X\to Y$ such that

$$\left\|[f(x_{ij})-A(x_{ij})]\right\|_n \leqslant \|[\Phi(x_{ij},x_{ij},-2x_{ij})]\|_n \tag{9}$$

for all $x=[x_{ij}]\in M_n(X)$. Here Φ is given in Theorem 2.

Proof. By the same reasoning as in the proof of Theorem 2, there exists a unique additive mapping $A:X\to Y$ such that

$$\|f(a)-A(a)\| \leqslant \Phi(a,a,-2a)$$

for all $a\in X$. The mapping $A:X\to Y$ is given by

$$A(a)=\lim_{l\to\infty}\frac{1}{2^l}f(2^l a)$$

for all $a\in X$.

It is easy to show that if $0\leqslant a_{ij}\leqslant b_{ij}$ for all i,j, then

$$\|[a_{ij}]\|_n \leqslant \|[b_{ij}]\|_n. \tag{10}$$

By Lemma 3 and (10),

$$\left\|[f(x_{ij})-A(x_{ij})]\right\|_n \leqslant \|[\|f(x_{ij})-A(x_{ij})\|]\|_n \leqslant \|[\Phi(x_{ij},x_{ij},-2x_{ij})]\|_n$$

for all $x=[x_{ij}]\in M_n(X)$. So we obtain the inequality (9). □

COROLLARY 3. *Let Y be an L^∞-normed Banach space. Let r,θ be positive real numbers with $r<1$. Let $f:X\to Y$ be a mapping such that*

$$\left\|f_n([x_{ij}])+f_n([y_{ij}])+f_n([z_{ij}])\right\|_n \leqslant \left\|f_n\left([x_{ij}]+[y_{ij}]+[z_{ij}]\right)\right\|_n + \|[\theta(\|x_{ij}\|^r+\|y_{ij}\|^r+\|z_{ij}\|^r)]\|_n \tag{11}$$

for all $x=[x_{ij}]$, $y=[y_{ij}]$, $z=[z_{ij}]\in M_n(X)$. Then there exists a unique additive mapping $A:X\to Y$ such that

$$\left\|f_n([x_{ij}])-A_n([x_{ij}])\right\|_n \leqslant \left\|\left[\frac{2-2^r}{2-2^r}\theta\|x_{ij}\|^r\right]\right\|_n$$

for all $x=[x_{ij}]\in M_n(X)$.

Proof. Letting $\phi(a,b,c)=\theta(\|a\|^r+\|b\|^r+\|c\|^r)$ in Theorem 4, we obtain the result. □

THEOREM 5. *Let Y be an L^∞-normed Banach space. Let $f:X\to Y$ be a mapping and let $\phi:X^3\to[0,\infty)$ be a function satisfying* (7) *and* (8). *Then there exists a unique additive mapping $A:X\to Y$ such that*

$$\left\|[f(x_{ij})-A(x_{ij})]\right\|_n \leqslant \|[\Phi(x_{ij},x_{ij},-2x_{ij})]\|_n$$

for all $x=[x_{ij}]\in M_n(X)$. Here Φ is given in Theorem 3.

Proof. The proof is similar to the proof of Theorem 4. □

COROLLARY 4. *Let Y be an L^∞-normed Banach space. Let r,θ be positive real numbers with $r>1$. Let $f:X\to Y$ be a mapping satisfying* (11). *Then there exists a unique additive mapping $A:X\to Y$ such that*

$$\left\|f_n([x_{ij}])-A_n([x_{ij}])\right\|_n \leqslant \left\|\left[\frac{2^r+2}{2^r-2}\theta\|x_{ij}\|^r\right]\right\|_n$$

for all $x=[x_{ij}]\in M_n(X)$.

Proof. Letting $\phi(a,b,c)=\theta(\|a\|^r+\|b\|^r+\|c\|^r)$ in Theorem 5, we obtain the result. □

3. Hyers-Ulam stability of the additive functional inequality (2) in matrix normed spaces: fixed point approach

In this section, we prove the Hyers-Ulam stability of the additive functional inequality (2) in matrix normed spaces by the fixed point method.

THEOREM 6. *Let $\phi : X^3 \to [0,\infty)$ be a function such that there exists an $\alpha < 1$ with*

$$\phi(a,b,c) \leqslant 2\alpha\phi\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right) \tag{12}$$

for all $a,b,c \in X$. Let $f : X \to Y$ be a mapping satisfying (4). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\left\|f_n([x_{ij}]) - A_n([x_{ij}])\right\|_n \leqslant \sum_{i,j=1}^{n} \frac{1}{2-2\alpha}\phi(x_{ij},x_{ij},-2x_{ij}) \tag{13}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. When $n = 1$, (4) is equivalent to

$$\|f(a)+f(b)+f(c)\| \leqslant \|f(a+b+c)\| + \phi(a,b,c) \tag{14}$$

for all $a,b,c \in X$. It follows from (14) that

$$\|2f(a) - f(2a)\| \leqslant \phi(a,a,-2a) \tag{15}$$

for all $a \in X$. So

$$\left\|f(a) - \frac{1}{2}f(2a)\right\| \leqslant \frac{1}{2}\phi(a,a,-2a) \tag{16}$$

for all $a \in X$.

Consider the set

$$S := \{h : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : \|g(a) - h(a)\| \leqslant \mu\phi(a,a,-2a),\ \forall a \in X\},$$

where, as usual, $\inf\{\} = +\infty$. It is easy to show that (S,d) is complete (see [26]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(a) := \frac{1}{2}g(2a)$$

for all $a \in X$.

Let $g,h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(a) - h(a)\| \leqslant \phi\,(a,a,-2a)$$

for all $a \in X$. Hence

$$\|Jg(a) - Jh(a)\| = \left\| \frac{1}{2} g\,(2a) - \frac{1}{2} h\,(2a) \right\| \leqslant \alpha\phi\,(a,a,-2a)$$

for all $a \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg,Jh) \leqslant \alpha\varepsilon$. This means that

$$d(Jg,Jh) \leqslant \alpha d(g,h)$$

for all $g,h \in S$.

It follows from (16) that $d(f,Jf) \leqslant \frac{1}{2}$.

By Theorem 1, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, i.e.,

$$A\,(2a) = 2A(x) \tag{17}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set

$$M = \{g \in S : d(h,g) < \infty\}.$$

This implies that A is a unique mapping satisfying (17) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(a) - A(a)\| \leqslant \mu\phi\,(a,a,-2a)$$

for all $a \in X$;

(2) $d(J^l f, A) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} \frac{1}{2^l} f\left(2^l a\right) = A(a)$$

for all $a \in X$;

(3) $d(f,A) \leqslant \frac{1}{1-\alpha} d(f,Jf)$, which implies the inequality

$$d(f,A) \leqslant \frac{1}{2-2\alpha}.$$

So

$$\|f(a) - A(a)\| \leqslant \frac{1}{2-2\alpha}\phi(a,a,-2a) \tag{18}$$

for all $a \in X$.

It follows from (12) and (14) that

$$\lim_{l\to\infty}\frac{1}{2^l}\|f(2^la)+f(2^lb)+f(2^lc)\| \leqslant \lim_{l\to\infty}\left(\frac{1}{2^l}\|f(2^l(a+b+c))\|+\frac{1}{2^l}\phi(2^la,2^lb,2^lc)\right) \tag{19}$$

for all $a,b,c\in X$.

It follows from (19) that

$$\|A(a)+A(b)+A(c)\|\leqslant\|A(a+b+c)\|$$

for all $a,b,c\in X$. By Lemma 2, $A:X\to Y$ is additive.

By Lemma 1 and (18),

$$\left\|f_n([x_{ij}])-A_n([x_{ij}])\right\|_n\leqslant\sum_{i,j=1}^n\|f(x_{ij})-A(x_{ij})\|\leqslant\sum_{i,j=1}^n\frac{1}{2-2\alpha}\phi(x_{ij},x_{ij},-2x_{ij})$$

for all $x=[x_{ij}]\in M_n(X)$. Thus $A:X\to Y$ is a unique additive mapping satisfying (13), as desired. □

COROLLARY 5. *Let r,θ be positive real numbers with $r<1$. Let $f:X\to Y$ be a mapping satisfying* (6)*. Then there exists a unique additive mapping $A:X\to Y$ such that*

$$\left\|f_n([x_{ij}])-A_n([x_{ij}])\right\|_n\leqslant\sum_{i,j=1}^n\frac{2+2^r}{2-2^r}\theta\|x_{ij}\|^r$$

for all $x=[x_{ij}]\in M_n(X)$.

Proof. The proof follows from Theorem 6 by taking $\phi(a,b,c)=\theta(\|a\|^r+\|b\|^r+\|c\|^r)$ for all $a,b,c\in X$. Then we can choose $\alpha=2^{r-1}$ and we get the desired result. □

THEOREM 7. *Let $f:X\to Y$ be a mapping satisfying* (4) *for which there exists a function $\phi:X^3\to[0,\infty)$ such that there exists an $\alpha<1$ with*

$$\phi(a,b,c)\leqslant\frac{\alpha}{2}\phi\,(2a,2b,2c) \tag{20}$$

for all $a,b,c\in X$. Then there exists a unique additive mapping $A:X\to Y$ such that

$$\left\|f_n([x_{ij}])-A_n([x_{ij}])\right\|_n\leqslant\sum_{i,j=1}^n\frac{\alpha}{2-2\alpha}\phi(x_{ij},x_{ij},-2x_{ij})$$

for all $x=[x_{ij}]\in M_n(X)$.

Proof. Let (S,d) be the generalized metric space defined in the proof of Theorem 6.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(a) := 2g\left(\frac{a}{2}\right)$$

for all $a \in X$.

It follows from (15) that

$$\left\| f(a) - 2f\left(\frac{a}{2}\right)\right\| \leqslant \phi\left(\frac{a}{2},\frac{a}{2},-a\right) \leqslant \frac{\alpha}{2}\phi\left(a,a,-2a\right)$$

for all $a \in X$. Thus $d(f,Jf) \leqslant \frac{\alpha}{2}$. So

$$d(f,A) \leqslant \frac{\alpha}{2-2\alpha}.$$

The rest of the proof is similar to the proof of Theorem 6. □

COROLLARY 6. *Let r,θ be positive real numbers with $r > 1$. Let $f : X \to Y$ be a mapping satisfying* (6). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\left\| f_n([x_{ij}]) - A_n([x_{ij}])\right\|_n \leqslant \sum_{i,j=1}^{n} \frac{2^r+2}{2^r-2}\theta \|x_{ij}\|^r$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 7 by taking $\phi(a,b,c) = \theta(\|a\|^r + \|b\|^r + \|c\|^r)$ for all $a,b,c \in X$. Then we can choose $\alpha = 2^{1-r}$ and we get the desired result. □

From now on, assume that Y is an L^∞-normed Banach space.

THEOREM 8. *Let $f : X \to Y$ be a mapping and let $\phi : X^3 \to [0,\infty)$ be a function satisfying* (12) *and* (8). *Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\left\|[f(x_{ij}) - A(x_{ij})]\right\|_n \leqslant \left\|\left[\frac{1}{2-2\alpha}\phi(x_{ij},x_{ij},-2x_{ij})\right]\right\|_n \tag{21}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof. By the same reasoning as in the proof of Theorem 6, there exists a unique additive mapping $A : X \to Y$ such that

$$\|f(a) - A(a)\| \leqslant \frac{1}{2-2\alpha}\phi(a,a,-2a)$$

for all $a \in X$.

By Lemma 3 and (10),

$$\left\|[f(x_{ij})-A(x_{ij})]\right\|_n \leqslant \|[\|f(x_{ij})-A(x_{ij})\|]\|_n \leqslant \left\|\left[\frac{1}{2-2\alpha}\phi(x_{ij},x_{ij})\right]\right\|_n$$

for all $x=[x_{ij}] \in M_n(X)$. So we obtain the inequality (21). □

COROLLARY 7. *Let r,θ be positive real numbers with $r<1$. Let $f: X \to Y$ be a mapping satisfying* (11). *Then there exists a unique additive mapping $A: X \to Y$ such that*

$$\left\|f_n([x_{ij}])-A_n([x_{ij}])\right\|_n \leqslant \left\|\left[\frac{2-2^r}{2-2^r}\theta\|x_{ij}\|^r\right]\right\|_n$$

for all $x=[x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 8 by taking $\phi(a,b,c)=\theta(\|a\|^r+\|b\|^r+\|c\|^r)$ for all $a,b,c \in X$. Then we can choose $\alpha = 2^{r-1}$ and we get the desired result. □

THEOREM 9. *Let $f: X \to Y$ be a mapping and let $\phi: X^3 \to [0,\infty)$ be a function satisfying* (8) *and* (20). *Then there exists a unique additive mapping $A: X \to Y$ such that*

$$\left\|[f(x_{ij})-A(x_{ij})]\right\|_n \leqslant \left\|\left[\frac{\alpha}{2-2\alpha}\phi(x_{ij},x_{ij},-2x_{ij})\right]\right\|_n$$

for all $x=[x_{ij}] \in M_n(X)$.

Proof. The proof is similar to the proof of Theorem 8. □

COROLLARY 8. *Let r,θ be positive real numbers with $r>1$. Let $f: X \to Y$ be a mapping satisfying* (11). *Then there exists a unique additive mapping $A: X \to Y$ such that*

$$\left\|f_n([x_{ij}])-A_n([x_{ij}])\right\|_n \leqslant \left\|\left[\frac{2^2+2}{2^r-2}\theta\|x_{ij}\|^r\right]\right\|_n$$

for all $x=[x_{ij}] \in M_n(X)$.

Proof. The proof follows from Theorem 9 by taking $\phi(a,b,c)=\theta(\|a\|^r+\|b\|^r+\|c\|^r)$ for all $a,b,c \in X$. Then we can choose $\alpha = 2^{1-r}$ and we get the desired result. □

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Choonkil Park
Research Institute for Natural Sciences
Hanyang University
Seoul 133-791, Korea
e-mail: baak@hanyang.ac.kr

Dong Yun Shin
Department of Mathematics
University of Seoul
Seoul 130-743, Korea
e-mail: dyshin@uos.ac.kr

Jung Rye Lee
Department of Mathematics
University of Seoul
Seoul 130-743, Korea
e-mail: jrlee@daejin.ac.kr