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STABILITY OF THE n–DIMENSIONAL MIXED–TYPE ADDITIVE AND QUADRATIC FUNCTIONAL EQUATION IN NON–ARCHIMEDEAN NORMED SPACES, II

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Abstract. In this paper, we prove the stability of the functional equation

$$\sum_{1\leqslant i,j\leqslant n,i\neq j}\big(f(x_i+x_j)+f(x_i-x_j)\big)=(n-1)\sum_{i=1}^{n}\big(3f(x_i)+f(-x_i)\big)$$

in non-Archimedean normed spaces.

1. Introduction

A classical question in the theory of functional equations is "when is it true that a function, which approximately satisfies a functional equation, must be somehow close to an exact solution of the equation?" Such a problem, called *a stability problem* of the functional equation, was formulated by S. M. Ulam in 1940 (see [21]). In the following year, D. H. Hyers [6] gave a partial solution of Ulam's problem for the case of approximate additive functions. Subsequently, his result was generalized by T. Aoki [1] for additive functions, and by Th. M. Rassias [19] for linear functions. Indeed, they considered the stability problem for unbounded Cauchy differences. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians (see [2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 20]).

A non-Archimedean field is a field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|:\mathbb{K}\to[0,\infty)$ such that

(F_1) $|r|=0$ if and only if $r=0$;

(F_2) $|rs|=|r||s|$;

(F_3) $|r+s|\leqslant\max\{|r|,|s|\}$ for all $r,s\in\mathbb{K}$.

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Clearly, it holds that $|1| = |-1| = 1$ and $|n| \leqslant 1$ for all $n \in \mathbb{N}$.

Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean and non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

(N_1) $\|x\| = 0$ if and only if $x = 0$;

(N_2) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$;

(N_3) $\|x+y\| \leqslant \max\left\{\|x\|, \|y\|\right\}$ for all $x, y \in X$.

Then $\left(X, \|\cdot\|\right)$ is called a non-Archimedean space. Due to the fact that

$$\|x_n - x_m\| \leqslant \max_{m \leqslant i < n} \|x_{i+1} - x_i\| \quad (n > m),$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. A complete non-Archimedean space is a non-Archimedean space in which every Cauchy sequence is convergent.

Recently, M. S. Moslehian and Th. M. Rassias [18] proved the Hyers-Ulam stability of the Cauchy functional equation

$$f(x+y) = f(x) + f(y) \tag{1}$$

and the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{2}$$

in non-Archimedean normed spaces.

We now consider the *n-dimensional mixed-type additive and quadratic functional equation*

$$\sum_{1 \leqslant i,j \leqslant n, i \neq j} \left(f(x_i + x_j) + f(x_i - x_j)\right) = (n-1) \sum_{i=1}^{n} \left(3f(x_i) + f(-x_i)\right), \tag{3}$$

whose solutions are called *additive-quadratic functions* and where n is an integer larger than 1.

In this paper, using an idea from [16], we investigate a general stability problem for the n-dimensional mixed-type additive and quadratic functional equation (3) in non-Archimedean normed spaces.

2. Solutions of Eq. (3)

Throughout this section, let X and Y be a non-Archimedean space and a complete non-Archimedean space, respectively.

For a given function $f : X \to Y$, we use the abbreviations

$$f_e(x) := \frac{f(x) + f(-x)}{2},$$

$$\begin{aligned}
f_o(x) :=& \frac{f(x)-f(-x)}{2},\\
Af(x,y) :=& f(x+y)-f(x)-f(y),\\
Qf(x,y) :=& f(x+y)+f(x-y)-2f(x)-2f(y),\\
Df(x_1,x_2,\ldots,x_n) :=& \sum_{1\leqslant i,j\leqslant n, i\neq j} \big(f(x_i+x_j)+f(x_i-x_j)\big)\\
&-(n-1)\sum_{i=1}^{n}\big(3f(x_i)+f(-x_i)\big)
\end{aligned}$$

for all $x,y,x_1,x_2,\ldots,x_n \in X$.

THEOREM 1. *Assume that $n \geqslant 2$ is an integer. Let X and Y be a non-Archimedean space and a complete non-Archimedean space, respectively. A function $f: X \to Y$ is a solution of Eq. (3) if and only if f_e is quadratic and f_o is additive.*

Proof. If a function $f: X \to Y$ is a solution of Eq. (3), then we have $f_e(0) = 0$ and $Df_e(x_1,x_2,\ldots,x_n) = Df_o(x_1,x_2,\ldots,x_n) = 0$ for all $x_1,x_2,\ldots,x_n \in X$. In this case, a routine calculation yields

$$Qf_e(x,y) = \frac{1}{2}Df_e(x,y,0,\ldots,0)+(n-2)(n+1)f_e(0) = 0$$

and

$$Af_o(x,y) = \frac{1}{2}Df_o(x,y,0,\ldots,0) = 0$$

for all $x,y \in X$, i.e., f_e is quadratic and f_o is additive.

Conversely, assume that f_e is quadratic and f_o is additive. Then it is not difficult to see that

$$\begin{aligned}
&Df_e(x_1,x_2,\ldots,x_n)\\
&= \sum_{1\leqslant i,j\leqslant n, i\neq j}\big(f_e(x_i+x_j)+f_e(x_i-x_j)\big)-4(n-1)\sum_{i=1}^{n} f_e(x_i)\\
&= \sum_{1\leqslant i,j\leqslant n, i\neq j}\big(f_e(x_i+x_j)+f_e(x_i-x_j)\big)-2\sum_{1\leqslant i,j\leqslant n, i\neq j} f_e(x_i)-2\sum_{1\leqslant i,j\leqslant n, i\neq j} f_e(x_j)\\
&= \sum_{1\leqslant i,j\leqslant n, i\neq j} Qf_e(x_i,x_j)\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
Df_o(x_1,x_2,\ldots,x_n) &= \sum_{1\leqslant i,j\leqslant n, i\neq j}\big(f_o(x_i+x_j)+f_o(x_i-x_j)\big)-2(n-1)\sum_{i=1}^{n} f_o(x_i)\\
&= \sum_{1\leqslant i,j\leqslant n, i\neq j}\big(f_o(x_i+x_j)+f_o(x_i-x_j)\big)-2\sum_{1\leqslant i,j\leqslant n, i\neq j} f_o(x_i)
\end{aligned}$$

$$= \sum_{1 \leqslant i,j \leqslant n, i \neq j} \big(Af_o(x_i,x_j) + Af_o(x_j - x_i, x_i)\big)$$
$$= 0,$$

since

$$\sum_{1 \leqslant i,j \leqslant n, i \neq j} f_e(x_i) = \sum_{1 \leqslant i,j \leqslant n, i \neq j} f_e(x_j) = (n-1)\sum_{i=1}^{n} f_e(x_i)$$

and

$$\sum_{1 \leqslant i,j \leqslant n, i \neq j} f_o(x_i) = (n-1)\sum_{i=1}^{n} f_o(x_i).$$

Hence, we get

$$Df(x_1,x_2,\ldots,x_n) = Df_e(x_1,x_2,\ldots,x_n) + Df_o(x_1,x_2,\ldots,x_n) = 0$$

for all $x_1,x_2,\ldots,x_n \in X$, i.e., f is a solution of Eq. (3). □

3. Generalized Hyers-Ulam stability of Eq. (3)

In this section, we will investigate the generalized Hyers-Ulam stability problems of the n-dimensional mixed-type additive and quadratic functional equation (3), where n is an integer larger than 1.

THEOREM 2. *Let X and Y be a non-Archimedean space and a complete non-Archimedean space, respectively. Assume that $\varphi : X^n \to [0,\infty)$ is a function such that*

$$\lim_{m\to\infty} \frac{\varphi(2^m x_1, 2^m x_2, \ldots, 2^m x_n)}{|4|^m} = 0 \tag{4}$$

for all $x_1,x_2,\ldots,x_n \in X$ and the limit

$$\tilde{\varphi}(x) := \lim_{m\to\infty} \max_{0 \leqslant i < m} \left\{ \frac{\varphi(2^i x, \ldots, 2^i x)}{|2|^{2i+3}}, \frac{\varphi(-2^i x, \ldots, -2^i x)}{|2|^{2i+3}} \right\} \tag{5}$$

exists for each $x \in X$. If a function $f : X \to Y$ satisfies the inequality

$$\|Df(x_1,x_2,\ldots,x_n)\| \leqslant \varphi(x_1,x_2,\ldots,x_n) \tag{6}$$

for all $x_1,x_2,\ldots,x_n \in X$ and $|2| < 1$, then there exists a unique additive-quadratic function $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leqslant \frac{\tilde{\varphi}(x)}{|n(n-1)|} \tag{7}$$

for all $x \in X$.

Proof. If we replace x_i in (4) with 0 for each $i \in \{1,2,\ldots,n\}$, then we get

$$\lim_{m\to\infty} \frac{\varphi(0,0,\ldots,0)}{|4|^m} = 0.$$

Thus, since $\lim_{m\to\infty} |4|^m = 0$, we conclude that $\varphi(0,0,\ldots,0) = 0$. Moreover, it follows from (6) that

$$\|2n(1-n)f(0)\| = \|Df(0,0,\ldots,0)\| \leqslant \varphi(0,0,\ldots,0) = 0,$$

i.e., it holds that $f(0) = 0$.

Let $J_m f : X \to Y$ be a function defined by

$$J_m f(x) = \frac{f(2^m x) + f(-2^m x)}{2\cdot 4^m} + \frac{f(2^m x) - f(-2^m x)}{2^{m+1}} \tag{8}$$

for all $x \in X$ and $m \in \{0,1,2,\ldots\}$. A long calculation, together with (N_3) and (6), yields

$$\begin{aligned}
&\|J_i f(x) - J_{i+1} f(x)\| \\
&= \left\| -\frac{Df(2^i x,\ldots,2^i x)}{2^{2i+3}n(n-1)} - \frac{Df(-2^i x,\ldots,-2^i x)}{2^{2i+3}n(n-1)} \right. \\
&\qquad \left. - \frac{Df(2^i x,\ldots,2^i x)}{2^{i+2}n(n-1)} + \frac{Df(-2^i x,\ldots,-2^i x)}{2^{i+2}n(n-1)} \right\| \\
&\leqslant \max\left\{ \frac{\|Df(2^i x,\ldots,2^i x)\|}{|2|^{2i+3}|n(n-1)|}, \frac{\|Df(-2^i x,\ldots,-2^i x)\|}{|2|^{2i+3}|n(n-1)|}, \right. \\
&\qquad \left. \frac{\|Df(2^i x,\ldots,2^i x)\|}{|2|^{i+2}|n(n-1)|}, \frac{\|Df(-2^i x,\ldots,-2^i x)\|}{|2|^{i+2}|n(n-1)|} \right\} \\
&\leqslant \max\left\{ \frac{\varphi(2^i x,\ldots,2^i x)}{|2|^{2i+3}|n(n-1)|}, \frac{\varphi(-2^i x,\ldots,-2^i x)}{|2|^{2i+3}|n(n-1)|} \right\}
\end{aligned} \tag{9}$$

for all $x \in X$ and any integer $i \geqslant 0$. It follows from (4) and (9) that the sequence $\{J_m f(x)\}$ is Cauchy for each $x \in X$. Since Y is complete, the sequence $\{J_m f(x)\}$ is convergent. Hence, we can define a function $T : X \to Y$ by

$$T(x) := \lim_{m\to\infty} J_m f(x)$$

for any $x \in X$.

It follows from (9) that

$$\begin{aligned}
\|J_m f(x) - f(x)\| &= \left\| \sum_{i=0}^{m-1} \big(J_{i+1} f(x) - J_i f(x)\big) \right\| \\
&\leqslant \max_{0\leqslant i<m} \left\{ \frac{\varphi(2^i x,\ldots,2^i x)}{|2|^{2i+3}|n(n-1)|}, \frac{\varphi(-2^i x,\ldots,-2^i x)}{|2|^{2i+3}|n(n-1)|} \right\}
\end{aligned} \tag{10}$$

for all $m \in \mathbb{N}$ and all $x \in X$. Letting $m \to \infty$ in (10) and using (5), we obtain the inequality (7).

If we replace x_i in (6) with $2^m x_i$ for every $i \in \{1,2,\ldots,n\}$, then a routine claculation yields

$$\begin{aligned}
&\|D(J_m f)(x_1,x_2,\ldots,x_n)\| \\
&= \left\| \frac{Df(2^m x_1,\ldots,2^m x_n) - Df(-2^m x_1,\ldots,-2^m x_n)}{2^{m+1}} \right. \\
&\qquad \left. + \frac{Df(2^m x_1,\ldots,2^m x_n) + Df(-2^m x_1,\ldots,-2^m x_n)}{2^{2m+1}} \right\| \\
&\leqslant \max\left\{ \frac{\varphi(2^m x_1,\ldots,2^m x_n)}{|2|^{m+1}}, \frac{\varphi(-2^m x_1,\ldots,-2^m x_n)}{|2|^{m+1}}, \right. \\
&\qquad \left. \frac{\varphi(2^m x_1,\ldots,2^m x_n)}{|2|^{2m+1}}, \frac{\varphi(-2^m x_1,\ldots,-2^m x_n)}{|2|^{2m+1}} \right\}
\end{aligned}$$

for any $x_1,x_2,\ldots,x_n \in X$. Letting $m \to \infty$ in the last inequality and using (4), we get $DT(x_1,x_2,\ldots,x_n) = 0$, i.e., T is a solution of Eq. (3).

If T' is another additive-quadratic function satisfying (7), then by considering the first equality in (9) we have

$$\begin{aligned}
T'(x) &= \sum_{i=0}^{k-1} \big(J_i T'(x) - J_{i+1} T'(x)\big) + J_k T'(x) \\
&= \sum_{i=0}^{k-1} \left(-\frac{DT'(2^i x,\ldots,2^i x)}{2^{2i+3} n(n-1)} - \frac{DT'(-2^i x,\ldots,-2^i x)}{2^{2i+3} n(n-1)} \right. \\
&\qquad \left. - \frac{DT'(2^i x,\ldots,2^i x)}{2^{i+2} n(n-1)} + \frac{DT'(-2^i x,\ldots,-2^i x)}{2^{i+2} n(n-1)} \right) + J_k T'(x) \\
&= J_k T'(x)
\end{aligned}$$

for any $k \in \mathbb{N}$, since $DT'(2^i x,\ldots,2^i x) = DT'(-2^i x,\ldots,-2^i x) = 0$. It also holds that $T(x) = J_k T(x)$. Thus, it follows from (4), (5), (7), and (8) that

$$\begin{aligned}
\|T(x) - T'(x)\| &= \lim_{k\to\infty} \|J_k T(x) - J_k T'(x)\| \\
&\leqslant \lim_{k\to\infty} \max\big\{ \|J_k T(x) - J_k f(x)\|, \|J_k f(x) - J_k T'(x)\| \big\} \\
&\leqslant \lim_{k\to\infty} |2|^{-2k-1} \max\big\{ \|T(2^k x) - f(2^k x)\|, \|T(-2^k x) - f(-2^k x)\|, \\
&\qquad\qquad \|f(2^k x) - T'(2^k x)\|, \|f(-2^k x) - T'(-2^k x)\| \big\} \\
&\leqslant \lim_{k\to\infty} \lim_{m\to\infty} \max_{k \leqslant i < m+k} \left\{ \frac{\varphi(2^i x,\ldots,2^i x)}{|2|^{2i+4} |n(n-1)|}, \frac{\varphi(-2^i x,\ldots,-2^i x)}{|2|^{2i+4} |n(n-1)|} \right\} \\
&= 0
\end{aligned}$$

for all $x \in X$. Therefore, we conclude that $T = T'$. $\square$

COROLLARY 1. *Let X and Y be a non-Archimedean space and a complete non-Archimedean space, respectively. If $|2|<1$ and a function $f:X\to Y$ satisfies the inequality*

$$\|Df(x_1,x_2,\ldots,x_n)\| \leqslant \theta\sum_{i=1}^{n}\|x_i\|^r$$

for all $x_1,x_2,\ldots,x_n\in X$ and for some $r>2$, then there exists a unique additive-quadratic function $T:X\to Y$ such that

$$\|f(x)-T(x)\| \leqslant \frac{n\theta}{|2|^3|n||n-1|}\|x\|^r \tag{11}$$

for all $x\in X$.

Proof. Let us define

$$\varphi(x_1,x_2,\ldots,x_n)=\theta\sum_{i=1}^{n}\|x_i\|^r$$

for all $x_1,x_2,\ldots,x_n\in X$. Since $|2|<1$ and $r-2>0$, we have

$$\lim_{m\to\infty}|2|^{-2m}\varphi(2^m x_1,2^m x_2,\ldots,2^m x_n)=\lim_{m\to\infty}|2|^{m(r-2)}\varphi(x_1,x_2,\ldots,x_n)=0$$

for all $x_1,x_2,\ldots,x_n\in X$. Therefore, the condition (4) is satisfied. Furthermore, it is easy to see that

$$\tilde{\varphi}(x)=|2|^{-3}n\theta\|x\|^r.$$

By Theorem 2, there exists a unique additive-quadratic function $T:X\to Y$ such that the inequality (11) holds. □

THEOREM 3. *Let X and Y be a non-Archimedean space and a complete non-Archimedean space, respectively. Let $\varphi:X^n\to[0,\infty)$ be a function such that*

$$\lim_{m\to\infty}|2|^m\varphi\left(\frac{x_1}{2^m},\frac{x_2}{2^m},\ldots,\frac{x_n}{2^m}\right)=0 \tag{12}$$

for all $x_1,x_2,\ldots,x_n\in X$. Assume that the limit

$$\tilde{\varphi}(x):=\lim_{m\to\infty}\max_{0\leqslant i<m}\left\{|2|^{i-1}\varphi\left(\frac{x}{2^{i+1}},\ldots,\frac{x}{2^{i+1}}\right),|2|^{i-1}\varphi\left(\frac{-x}{2^{i+1}},\ldots,\frac{-x}{2^{i+1}}\right)\right\} \tag{13}$$

exists for each $x\in X$. If a function $f:X\to Y$ satisfies the inequality

$$\|Df(x_1,x_2,\ldots,x_n)\| \leqslant \varphi(x_1,x_2,\ldots,x_n) \tag{14}$$

for all $x_1,x_2,\ldots,x_n\in X$, $f(0)=0$, and if $|2|<1$, then there exists a unique additive-quadratic function $T:X\to Y$ such that

$$\|f(x)-T(x)\| \leqslant \frac{\tilde{\varphi}(x)}{|n(n-1)|} \tag{15}$$

for any $x\in X$.

Proof. Let $J_m f : X \to Y$ be a function defined by

$$J_m f(x) = 2^{2m-1}\left(f\left(\frac{x}{2^m}\right) + f\left(\frac{-x}{2^m}\right)\right) + 2^{m-1}\left(f\left(\frac{x}{2^m}\right) - f\left(\frac{-x}{2^m}\right)\right)$$

for all $x \in X$ and $m \in \{0,1,2,\ldots\}$. Analogously to (9), we get

$$\begin{aligned}
&\|J_i f(x) - J_{i+1} f(x)\| \\
&= \left|\frac{2^{i-1}}{n(n-1)}\right| \left\|(2^i+1)Df\left(\frac{x}{2^{i+1}},\ldots,\frac{x}{2^{i+1}}\right) + (2^i-1)Df\left(\frac{-x}{2^{i+1}},\ldots,\frac{-x}{2^{i+1}}\right)\right\| \\
&\leqslant \left|\frac{2^{i-1}}{n(n-1)}\right| \max\left\{\left\|Df\left(\frac{x}{2^{i+1}},\ldots,\frac{x}{2^{i+1}}\right)\right\|, \left\|Df\left(\frac{-x}{2^{i+1}},\ldots,\frac{-x}{2^{i+1}}\right)\right\|\right\} \qquad (16) \\
&\leqslant \left|\frac{2^{i-1}}{n(n-1)}\right| \max\left\{\varphi\left(\frac{x}{2^{i+1}},\ldots,\frac{x}{2^{i+1}}\right), \varphi\left(\frac{-x}{2^{i+1}},\ldots,\frac{-x}{2^{i+1}}\right)\right\}
\end{aligned}$$

for all $x \in X$ and any integer $i \geqslant 0$. It follows from (12) and (16) that the sequence $\{J_m f(x)\}$ is Cauchy for any $x \in X$. Since Y is complete, the sequence $\{J_m f(x)\}$ is convergent. Hence, we can define a function $T : X \to Y$ by

$$T(x) := \lim_{m\to\infty} J_m f(x)$$

for every $x \in X$.

It follows from (16) that

$$\begin{aligned}
\|J_m f(x) - f(x)\| &= \left\|\sum_{i=0}^{m-1}\left(J_{i+1}f(x) - J_i f(x)\right)\right\| \\
&\leqslant \left|\frac{1}{n(n-1)}\right| \max_{0\leqslant i<m}\left\{|2|^{i-1}\varphi\left(\frac{x}{2^{i+1}},\ldots,\frac{x}{2^{i+1}}\right),\right. \qquad (17) \\
&\qquad \left. |2|^{i-1}\varphi\left(\frac{-x}{2^{i+1}},\ldots,\frac{-x}{2^{i+1}}\right)\right\}
\end{aligned}$$

for all $m \in \mathbb{N}$ and all $x \in X$. Letting $m \to \infty$ in (17) and using (13), we obtain (15).

Replacing x_i in (14) with $2^{-m}x_i$ for $i \in \{1,2,\ldots,n\}$, it follows from a tedious calculation, (N_3), and (14) that

$$\begin{aligned}
&\|D(J_m f)(x_1,x_2,\ldots,x_n)\| \\
&= \left\|2^{m-1}Df\left(\frac{x_1}{2^m},\ldots,\frac{x_n}{2^m}\right) - 2^{m-1}Df\left(\frac{-x_1}{2^m},\ldots,\frac{-x_n}{2^m}\right)\right. \\
&\qquad \left. + 2^{2m-1}Df\left(\frac{x_1}{2^m},\ldots,\frac{x_n}{2^m}\right) + 2^{2m-1}Df\left(\frac{-x_1}{2^m},\ldots,\frac{-x_n}{2^m}\right)\right\| \\
&\leqslant \max\left\{|2|^{m-1}\varphi\left(\frac{x_1}{2^m},\ldots,\frac{x_n}{2^m}\right), |2|^{m-1}\varphi\left(\frac{-x_1}{2^m},\ldots,\frac{-x_n}{2^m}\right)\right\}
\end{aligned}$$

for all $m \in \mathbb{N}$ and $x_1,x_2,\ldots,x_n \in X$. Letting $m \to \infty$ in the last inequality and using (12), we get $DT(x_1,x_2,\ldots,x_n) = 0$ for all $x_1,x_2,\ldots,x_n \in X$, i.e., T is a solution of Eq. (3).

If T' is another additive-quadratic function satisfying (15), then by a similar argument as in the proof of Theorem 2, (16) yields

$$\begin{aligned}
T'(x) &= \sum_{i=0}^{k-1}\big(J_iT'(x)-J_{i+1}T'(x)\big)+J_kT'(x)\\
&= \frac{2^{i-1}}{n(n-1)}\sum_{i=0}^{k-1}\Bigg((2^i+1)DT'\Big(\frac{x}{2^{i+1}},\ldots,\frac{x}{2^{i+1}}\Big)\\
&\qquad\qquad +(2^i-1)DT'\Big(\frac{-x}{2^{i+1}},\ldots,\frac{-x}{2^{i+1}}\Big)\Bigg)+J_kT'(x)\\
&= J_kT'(x)
\end{aligned}$$

for any $k\in\mathbb{N}$, since $DT'\big(\frac{x}{2^{i+1}},\ldots,\frac{x}{2^{i+1}}\big)=DT'\big(\frac{-x}{2^{i+1}},\ldots,\frac{-x}{2^{i+1}}\big)=0$. It also holds that $T(x)=J_kT(x)$ for all $k\in\mathbb{N}$. Thus, we have

$$\begin{aligned}
&\|T(x)-T'(x)\|\\
&\quad=\lim_{k\to\infty}\|J_kT(x)-J_kT'(x)\|\\
&\quad\leqslant\lim_{k\to\infty}\max\big\{\|J_kT(x)-J_kf(x)\|,\|J_kf(x)-J_kT'(x)\|\big\}\\
&\quad\leqslant\lim_{k\to\infty}|2|^{k-1}\max\Bigg\{\Big\|T\Big(\frac{x}{2^k}\Big)-f\Big(\frac{x}{2^k}\Big)\Big\|,\Big\|T\Big(\frac{-x}{2^k}\Big)-f\Big(\frac{-x}{2^k}\Big)\Big\|,\\
&\qquad\qquad\Big\|f\Big(\frac{x}{2^k}\Big)-T'\Big(\frac{x}{2^k}\Big)\Big\|,\Big\|f\Big(\frac{-x}{2^k}\Big)-T'\Big(\frac{-x}{2^k}\Big)\Big\|\Bigg\}\\
&\quad\leqslant\lim_{k\to\infty}\lim_{m\to\infty}\max_{k\leqslant i<m+k}\Bigg\{\frac{|2|^{i-2}}{|n(n-1)|}\varphi\Big(\frac{x}{2^{i+1}},\ldots,\frac{x}{2^{i+1}}\Big),\\
&\qquad\qquad\frac{|2|^{i-2}}{|n(n-1)|}\varphi\Big(\frac{-x}{2^{i+1}},\ldots,\frac{-x}{2^{i+1}}\Big)\Bigg\}\\
&\quad=0
\end{aligned}$$

for all $x\in X$. Therefore, we conclude that $T=T'$, which ends the proof. □

COROLLARY 2. *Let X and Y be a non-Archimedean space and a complete non-Archimedean space, respectively. If $|2|<1$ and a function $f:X\to Y$ satisfies the inequality*

$$\|Df(x_1,x_2,\ldots,x_n)\|\leqslant\theta\sum_{i=1}^{n}\|x_i\|^r$$

for all $x_1,x_2,\ldots,x_n\in X$ and for some $0\leqslant r<1$, then there exists a unique additive-quadratic function $T:X\to Y$ such that

$$\|f(x)-T(x)\|\leqslant\frac{n\theta}{|2|^{1+r}|n(n-1)|}\|x\|^r \tag{18}$$

for all $x\in X$.

Proof. Let us define

$$\varphi(x_1,x_2,\dots,x_n) := \theta\sum_{i=1}^{n}\|x_i\|^r.$$

Since $|2|<1$ and $1-r>0$, we have

$$\lim_{m\to\infty}|2|^m\varphi\Big(\frac{x_1}{2^m},\frac{x_2}{2^m},\dots,\frac{x_n}{2^m}\Big) = \lim_{m\to\infty}|2|^{m(1-r)}\theta\sum_{i=1}^{n}\|x_i\|^r = 0$$

for all $x_1,x_2,\dots,x_n\in X$. Moreover, it is easy to see that $\tilde{\varphi}(x) = |2|^{-1-r}n\theta\|x\|^r$. By Theorem 3, there exists a unique additive-quadratic function $T : X\to Y$ such that the inequality (18) holds. □

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