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n–EXPONENTIAL CONVEXITY OF DIVIDED DIFFERENCES AND RELATED STOLARSKY TYPE MEANS

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Abstract. In this paper we consider functionals with divided differences. Two of them use majorization type results, where one is related with Schur convexity. The others are related to Jensen inequality and Hermite-Hadamard inequalities. We use them in studying Stolarsky type means. A method of producing n-exponentially convex functions is applied using divided differences.

1. Introduction

Let f be a real-valued function defined on the segment $[a,b]$. The divided difference of order n of the function f at distinct points $x_0,...,x_n \in [a,b]$, is defined recursively (see [3], [12]) by

$$f[x_i] = f(x_i), \quad (i = 0,\dots,n)$$

and

$$f[x_0,\dots,x_n] = \frac{f[x_1,\dots,x_n] - f[x_0,\dots,x_{n-1}]}{x_n - x_0}.$$

The value $f[x_0,\dots,x_n]$ is independent of the order of the points $x_0,\dots,x_n$.

The definition may be extended to include the case that some (or all) of the points coincide. Assuming that $f^{(j-1)}(x)$ exists, we define

$$f[\underbrace{x,\dots,x}_{j-\text{times}}] = \frac{f^{(j-1)}(x)}{(j-1)!}. \tag{1}$$

For divided difference the following holds:

$$f[x_0,\dots,x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\omega'(x_i)}, \text{ where } \omega(x) = \prod_{j=0}^{n}(x - x_j),$$

so we have that

$$f[x_0,\dots,x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\prod_{j=0,j\neq i}^{n}(x_i - x_j)}.$$

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If the the function f has continuous n-th derivative on $[a,b]$, the divided difference $f[x_0,\dots,x_n]$ can be represented in integral form by

$$f[x_0,\dots,x_n] = \int_{\Delta_n} f^{(n)}\left(\sum_{i=0}^{n} u_i x_i\right) du_0 \dots du_{n-1}$$

where

$$\Delta_n = \left\{ (u_0,\dots,u_{n-1}) : u_i \geqslant 0, \sum_{i=0}^{n-1} u_i \leqslant 1 \right\}$$

and $u_n = 1 - \sum_{i=0}^{n-1} u_i$.

In [13] authors gave one result for the difference of two divided differences:

THEOREM 1. *Let f,g be two n-times continuously differentiable functions on interval $I \subseteq \mathbb{R}$ and $(n+1)$-times differentiable on the interior I° of I with the properties that, $g^{(n+1)} > 0$ on I°, and the function $\frac{f^{(n+1)}}{g^{(n+1)}}$ is bounded on I°. Then for $x_i, y_i \in I$ $(i = 0,\dots,n)$ and $\sum_{i=0}^{n}(x_i - y_i) \neq 0$, the following estimation holds true:*

$$\inf_{x \in I^\circ} \frac{f^{(n+1)}(x)}{g^{(n+1)}(x)} \leqslant \frac{f[x_0,\dots,x_n] - f[y_0,\dots,y_n]}{g[x_0,\dots,x_n] - g[y_0,\dots,y_n]} \leqslant \sup_{x \in I^\circ} \frac{f^{(n+1)}(x)}{g^{(n+1)}(x)}. \tag{2}$$

A function $f : [a,b] \to \mathbb{R}$ is said to be *n-convex* if n-th order divided difference of f satisfies

$$f[x_0,\dots,x_n] \geqslant 0 \quad \text{for all} \quad a \leqslant x_0 < \dots < x_n \leqslant b.$$

Let $\mathbf{x} = (x_0,\dots,x_n)$ and $\mathbf{y} = (y_0,\dots,y_n)$ denote two real $(n+1)$-tuples. We say that $\mathbf{x}$ *majorizes* $\mathbf{y}$ and put $\mathbf{x} \succ \mathbf{y}$ if

$$\sum_{i=0}^{k} x_{[i]} \geqslant \sum_{i=0}^{k} y_{[i]} \quad \text{for} \quad k = 0,1,\dots,n-1$$

and

$$\sum_{i=0}^{n} x_i = \sum_{i=0}^{n} y_i,$$

where

$$x_{[0]} \geqslant x_{[1]} \geqslant \dots \geqslant x_{[n]} \quad \text{and} \quad y_{[0]} \geqslant y_{[1]} \geqslant \dots \geqslant y_{[n]}$$

are the nonincreasing ordered components of $\mathbf{x}$ and $\mathbf{y}$.

Let $\mathscr{D}$ be an open, convex and permutation symmetric (invariant under each permutation of the coordinates) subset of $\mathbb{R}^{n+1}$. A function $F : \mathscr{D} \to \mathbb{R}$ is said to be *Schur convex* if $\mathbf{x} \succ \mathbf{y}$ implies $F(\mathbf{x}) \geqslant F(\mathbf{y})$. A Schur convex function is always permutation symmetric. Every permutation symmetric and convex function on $\mathscr{D}$ is Schur convex. The following theorem gives the sufficient condition which ensures the divided difference is Schur convex (see [14]):

THEOREM 2. *Let f be an $(n+2)$-convex function on (a,b) and $\mathbf{x},\mathbf{y} \in (a,b)^{n+1}$. If $\mathbf{x} \succ \mathbf{y}$, then*

$$f[x_0,\ldots,x_n] \geqslant f[y_0,\ldots,y_n], \tag{3}$$

that is, the function $F(\mathbf{x}) = f[x_0,\ldots,x_n]$ is Schur convex.

In [5] is proved the following Jensen inequality for divided differences:

THEOREM 3. *Let f be an $(n+2)$-convex function on (a,b) and $\mathbf{x} \in (a,b)^{n+1}$. Then*

$$G(\mathbf{x}) = f[x_0,\ldots,x_n]$$

is a convex function of the vector $\mathbf{x} = (x_0,\ldots,x_n)$. Consequently,

$$f\left[\sum_{i=0}^m a_i x_0^i,\ldots,\sum_{i=0}^m a_i x_n^i\right] \leqslant \sum_{i=0}^m a_i f[x_0^i,\ldots,x_n^i] \tag{4}$$

holds for all $a_i \geqslant 0$ such that $\sum_{i=0}^m a_i = 1$.

Farwig and Zwick (see [5]) also proved:

THEOREM 4. *Let $a \leqslant x_0 < \cdots < x_n \leqslant b$ and a_i be arbitrary but fixed such that $\sum_{i=0}^n a_i = 1$. Then*

$$f[x_0,\ldots,x_n] \leqslant \frac{1}{n}\sum_{i=0}^n a_i f'[x_0,\ldots,x_{i-1},x_{i+1},\ldots,x_n] \tag{5}$$

holds for every f that is $(n+2)$-convex in $[a,b]$ iff $\sum_{i=0}^n a_i x_i = \frac{1}{n+1}\sum_{i=0}^n x_i$. Furthermore, equality in (5) holds iff f is polynomial with degree at most $n+1$.

We note that given the above two conditions on the a_i, the value on the right-hand side of (5) is independent of the choice of a_i, thus they may be taken to be

$$\frac{1}{n(n+1)}\sum_{i=0}^n f'[x_0,\ldots,x_{i-1},x_{i+1},\ldots,x_n].$$

The following corollary then follows by a straightforward induction (see [12]):

COROLLARY 1. *If f is $(n+2)$-convex in (a,b), then for all $x_0,\ldots,x_n \in (a,b)$*

$$\begin{aligned} f[x_0,\ldots,x_n] &\leqslant \frac{1}{n(n+1)}\sum_{i=0}^n f'[x_0,\ldots,x_{i-1},x_{i+1},\ldots,x_n] \\ &\leqslant \cdots \leqslant \frac{1}{\binom{n}{k}\binom{n+1}{k}} \sum_{0\leqslant j_0<\cdots<j_{n-k}\leqslant n} \frac{1}{k!} f^{(k)}[x_{j_0},\ldots,x_{j_{n-k}}] \\ &\leqslant \cdots \leqslant \sum_{i=0}^n \frac{f^{(n)}(x_i)}{(n+1)!}. \end{aligned} \tag{6}$$

Schur polynomial in $n+1$ variables $x_0,\ldots,x_n$ of degree $d=d_0+\ldots+d_n$ (d_j's form nonincreasing sequence non-negative integers, i.e. $d_0\geqslant\ldots\geqslant d_n$) is defined as

$$S_{(d_0,\ldots,d_n)}(x_0,\ldots,x_n)=\frac{\det\left[x_i^{d_{n-j}+j}\right]_{i,j=0}^{n}}{\det\left[\,x_i^j\,\right]_{i,j=0}^{n}}.$$

The numerator consists of alternating polynomials (they change the sign under any transposition of the variables) and so they are all divisible by the denominator which is Vandermonde determinant. Schur polynomial is also symmetric because the numerator and denominator are both alternating. For example, Schur polynomials in two variables x_0 and x_1 of degree $d=3$ are

$$S_{(2,1)}(x_0,x_1)=x_0^2x_1+x_0x_1^2$$

and

$$S_{(3,0)}(x_0,x_1)=x_0^3+x_0^2x_1+x_0x_1^2+x_1^3.$$

Using Schur polynomial and Vandermonde determinant (extended with logarithmic function)

$$V(\mathbf{x};p,q)=\det\begin{bmatrix}1 & x_0 & {x_0}^2 & \ldots & {x_0}^{n-1} & {x_0}^p\ln^q x_0\\ 1 & x_1 & {x_1}^2 & \ldots & {x_1}^{n-1} & {x_1}^p\ln^q x_1\\ 1 & x_2 & {x_2}^2 & \ldots & {x_2}^{n-1} & {x_2}^p\ln^q x_2\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & x_n & {x_n}^2 & \ldots & {x_n}^{n-1} & {x_n}^p\ln^q x_n\end{bmatrix}$$

we obtain:

PROPOSITION 1. For monomial function $h(x)=x^{n+k}$, where $k\geqslant 1$ is an integer, holds

$$h[x_0,\ldots,x_n]=S_{(k,\underbrace{0,\ldots,0}_{n-\text{times}})}(x_0,\ldots,x_n)=\frac{V(\mathbf{x};n+k,0)}{V(\mathbf{x};n,0)}.$$

For potential function $f(x)=x^p=x^{n+p-n}$, where p is a real number, holds

$$f[x_0,\ldots,x_n]=\frac{V(\mathbf{x};p,0)}{V(\mathbf{x};n,0)}.$$

Further, for a partition π the Schur polynomial can be expressed by a sum

$$S_\pi(x_0,\ldots,x_n)=\sum_T x^T=\sum_T x_0^{t_0}\ldots x_n^{t_n}.$$

The summation is over all semistandard Young tableaux T of shape π where the exponents $t_0,\ldots t_n$ give the weight of T in which each t_j counts the occurrences of the number j in T (see [1] and [7]). So, we have the next proposition:

PROPOSITION 2. For monomial function $h(x) = x^{n+k}$, where $k \geqslant 1$ is an integer, holds

$$h[x_0, \ldots, x_n] = S_{(k,\underbrace{0,\ldots,0}_{n-\text{times}})}(x_0, \ldots, x_n) = \sum_{i_1=0}^{n} \sum_{i_2=0}^{i_1} \cdots \sum_{i_k=0}^{i_{k-1}} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

In the next section we obtain an inequality with divided differences using the Jensen integral inequality. Also, we give mean value theorems using that inequality. In [8], [9] and [10] authors gave mean value theorems using the inequalities (2), (3), (4), (5) and (6). The goal of this paper is to use obtained Cauchy type mean value theorems in studying Stolarsky type means defined by divided differences. We introduce the notion of n-exponentially convex functions and deduce a method of producing n-exponentially convex functions use some known families of functions of the same type.

2. Integral Jensen-type inequality for divided differences

THEOREM 5. *Let* $p, g_i : \Omega \to [a,b]$, $(i = 0, \ldots, n)$ *be functions from* $L_1(\mu)$ *and let* f *be an* n*-times continuously differentiable on* $[a,b]$ *and* $(n+2)$*-convex function on* (a,b)*. Then*

$$f\left[\int_\Omega p(x)g_0(x)d\mu(x), \ldots, \int_\Omega p(x)g_n(x)d\mu(x)\right] \leqslant \int_\Omega p(x)f[g_0(x), \ldots, g_n(x)]d\mu(x) \tag{7}$$

holds for all $p(x) \geqslant 0$ *such that* $\int_\Omega p(x)d\mu(x) = 1$.

Proof. Using the integral Jensen inequality for the convex function $f^{(n)}$, we have the following conclusion

$$\begin{aligned}
& f\left[\int_\Omega p(x)g_0(x)d\mu(x), \ldots, \int_\Omega p(x)g_n(x)d\mu(x)\right] \\
&= \int_{\Delta_n} f^{(n)}\left(\sum_{i=0}^{n} u_i \int_\Omega p(x)g_i(x)d\mu(x)\right) du_0 \ldots du_{n-1} \\
&= \int_{\Delta_n} f^{(n)}\left(\int_\Omega p(x) \sum_{i=0}^{n} u_i g_i(x)d\mu(x)\right) du_0 \ldots du_{n-1} \\
&\leqslant \int_{\Delta_n} \left(\int_\Omega p(x) f^{(n)}\left(\sum_{i=0}^{n} u_i g_i(x)\right) d\mu(x)\right) du_0 \ldots du_{n-1} \\
&= \int_\Omega p(x)\left(\int_{\Delta_n} f^{(n)}\left(\sum_{i=0}^{n} u_i g_i(x)\right) du_0 \ldots du_{n-1}\right) d\mu(x) \\
&= \int_\Omega p(x)f[g_0(x), \ldots, g_n(x)]d\mu(x). \quad \square
\end{aligned}$$

THEOREM 6. *Let $f \in C^{n+2}([a,b])$ and $p, g_i : \Omega \to [a,b]$, $(i = 0,\dots,n)$ be functions from $L_1(\mu)$. If $p(x) \geqslant 0$ such that $\int_\Omega p(x) = 1$, then there exists $\xi \in [a,b]$ such that*

$$\int_\Omega p(x) f[g_0(x),\dots,g_n(x)]d\mu(x) - f\left[\int_\Omega p(x)g_0(x)d\mu(x),\dots,\int_\Omega p(x)g_n(x)d\mu(x)\right]$$
$$= \frac{f^{(n+2)}(\xi)}{(n+2)!}\sum_{i=0}^{n}\sum_{j=0}^{i}\left(\int_\Omega p(x)g_i(x)g_j(x)d\mu(x) - \int_\Omega p(x)g_i(x)d\mu(x)\int_\Omega p(x)g_j(x)d\mu(x)\right).$$

Proof. Let us denote $\alpha = \min f^{(n+2)}$ and $\beta = \max f^{(n+2)}$. We first consider the following function $\phi_1(x) = \frac{\beta x^{n+2}}{(n+2)!} - f(x)$. Then $\phi_1^{(n+2)}(x) = \beta - f^{(n+2)}(x) \geqslant 0$, $x \in [a,b]$, so ϕ_1 is an $(n+2)$-convex function. Applying Theorem 5 on an $(n+2)$-convex function ϕ_1 with $\phi(x) = \frac{x^{n+2}}{(n+2)!}$ we have

$$\beta \cdot \int_\Omega p(x)\phi[g_0(x),\dots,g_n(x)]d\mu(x) - \int_\Omega p(x)f[g_0(x),\dots,g_n(x)]d\mu(x)$$
$$\geqslant \beta \cdot \phi\left[\int_\Omega p(x)g_0(x)d\mu(x),\dots,\int_\Omega p(x)g_n(x)d\mu(x)\right]$$
$$-f\left[\int_\Omega p(x)g_0(x)d\mu(x),\dots,\int_\Omega p(x)g_n(x)d\mu(x)\right],$$

i.e.

$$\int_\Omega p(x)f[g_0(x),\dots,g_n(x)]d\mu(x) - f\left[\int_\Omega p(x)g_0(x)d\mu(x),\dots,\int_\Omega p(x)g_n(x)d\mu(x)\right]$$
$$\leqslant \beta\left(\int_\Omega p(x)\phi[g_0(x),\dots,g_n(x)]d\mu(x) - \phi\left[\int_\Omega p(x)g_0(x)d\mu(x),\dots,\int_\Omega p(x)g_n(x)d\mu(x)\right]\right).$$

Similarly, a function $\phi_2(x) = f(x) - \alpha \cdot \phi(x)$ is an $(n+2)$-convex function. Inequality from the Theorem 5 with $(n+2)$-convex function ϕ_2 becomes

$$\int_\Omega p(x)f[g_0(x),\dots,g_n(x)]d\mu(x) - \alpha \cdot \int_\Omega p(x)\phi[g_0(x),\dots,g_n(x)]d\mu(x)$$
$$\geqslant f\left[\int_\Omega p(x)g_0(x)d\mu(x),\dots,\int_\Omega p(x)g_n(x)d\mu(x)\right]$$
$$-\alpha \cdot \phi\left[\int_\Omega p(x)g_0(x)d\mu(x),\dots,\int_\Omega p(x)g_n(x)d\mu(x)\right],$$

i.e.

$$\int_\Omega p(x)f[g_0(x),\dots,g_n(x)]d\mu(x) - f\left[\int_\Omega p(x)g_0(x)d\mu(x),\dots,\int_\Omega p(x)g_n(x)d\mu(x)\right]$$
$$\geqslant \alpha\left(\int_\Omega p(x)\phi[g_0(x),\dots,g_n(x)]d\mu(x) - \phi\left[\int_\Omega p(x)g_0(x)d\mu(x),\dots,\int_\Omega p(x)g_n(x)d\mu(x)\right]\right).$$

From Proposition 2 with $k=2$ and the fact that the function $\phi(x)$ is $(n+2)$-convex we have

$$\begin{aligned}
&\int_\Omega p(x)\phi[g_0(x),\ldots,g_n(x)]d\mu(x) - \phi\left[\int_\Omega p(x)g_0(x)d\mu(x),\ldots,\int_\Omega p(x)g_n(x)d\mu(x)\right]\\
&= \frac{1}{(n+2)!}\sum_{i=0}^{n}\sum_{j=0}^{i}\left(\int_\Omega p(x)g_i(x)g_j(x)d\mu(x) - \int_\Omega p(x)g_i(x)d\mu(x)\int_\Omega p(x)g_j(x)d\mu(x)\right)\\
&> 0.
\end{aligned}$$

We can now conclude that there exists $\xi \in [a,b]$ that we are looking for in (8). □

REMARK 1. Let us note that the left-hand side in equation (8) is greater than or equal to zero if $f^{(n+2)} \geqslant 0$ which is the statement of Theorem 5.

COROLLARY 2. *Let $f,\hat{f} \in C^{n+2}([a,b])$ and $p,g_i : \Omega \to [a,b]$, $(i=0,\ldots,n)$ be functions from $L_1(\mu)$. If $p(x) \geqslant 0$ such that $\int_\Omega p(x) = 1$, then there exists $\xi \in [a,b]$ such that*

$$\begin{aligned}
&\frac{\int_\Omega p(x)f[g_0(x),\ldots,g_n(x)]d\mu(x) - f\left[\int_\Omega p(x)g_0(x)d\mu(x),\ldots,\int_\Omega p(x)g_n(x)d\mu(x)\right]}{\int_\Omega p(x)\hat{f}[g_0(x),\ldots,g_n(x)]d\mu(x) - \hat{f}\left[\int_\Omega p(x)g_0(x)d\mu(x),\ldots,\int_\Omega p(x)g_n(x)d\mu(x)\right]}\\
&= \frac{f^{(n+2)}(\xi)}{\hat{f}^{(n+2)}(\xi)} \qquad (8)
\end{aligned}$$

provided that both denominators not equal zero.

Proof. We use the following standard technique: Let us define the linear functional $L(h) = \int_\Omega p(x)h[g_0(x),\ldots,g_n(x)]d\mu(x) - h\left[\int_\Omega p(x)g_0(x)d\mu(x),\ldots,\int_\Omega p(x)g_n(x)d\mu(x)\right]$. Next, we define $\psi(t) = f(t)L(\hat{f}) - \hat{f}(t)L(f)$. According to Theorem 6, applied on ψ, there exists $\xi \in [a,b]$ so that

$$\begin{aligned}
L(\psi) = \frac{\psi^{(n+2)}(\xi)}{(n+2)!}\sum_{i=0}^{n}\sum_{j=0}^{i}&\left(\int_\Omega p(x)g_i(x)g_j(x)d\mu(x)\right.\\
&\left. - \int_\Omega p(x)g_i(x)d\mu(x)\int_\Omega p(x)g_j(x)d\mu(x)\right).
\end{aligned}$$

From $L(\psi)=0$, it follows $f^{(n+2)}(\xi)L(\hat{f}) - \hat{f}^{(n+2)}(\xi)L(f) = 0$ and (8) is proved. □

3. n-exponential convexity of divided differences

We begin this section by notions which are going to be explored here and some characterizations of these properties:

DEFINITION 1. A function $\psi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\left(\frac{x_i + x_j}{2}\right) \geqslant 0$$

hold for all choices $\xi_1, \ldots, \xi_n \in \mathbb{R}$ and all choices $x_1, \ldots, x_n \in I$.

A function $\psi : I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

REMARK 2. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex function in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leqslant n$.

By definition of positive semi-definite matrices and some basic linear algebra we have the following proposition:

PROPOSITION 3. If ψ is an n-exponentially convex in the Jensen sense, then the matrix $\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ is positive semi-definite matrix for all $k \in \mathbb{N}$, $k \leqslant n$. Particularly, $\det\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k} \geqslant 0$ for all $k \in \mathbb{N}$, $k \leqslant n$.

DEFINITION 2. A function $\psi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $\psi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

REMARK 3. It is known (and easy to show) that $\psi : I \to \mathbb{R}$ is a log-convex in the Jensen sense if and only if

$$\alpha^2 \psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2 \psi(y) \geqslant 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

A positive function is log-convex if and only if it is 2-exponentially convex.

Motivated by inequalities (2)–(6) an (7), under the same assumptions, we define following functionals:

$$\Phi_1(\mathbf{x}, \mathbf{y}, f) = f[x_0, \ldots, x_n] - f[y_0, \ldots, y_n], \tag{9}$$

$$\Phi_2(\mathbf{X},\mathbf{a},f)=\sum_{i=0}^{m}a_i f[x_0^i,\ldots,x_n^i]-f\left[\sum_{i=0}^{m}a_i x_0^i,\ldots,\sum_{i=0}^{m}a_i x_n^i\right], \tag{10}$$

$$\Phi_3(\mathbf{x},\mathbf{a},f)=\frac{1}{n}\sum_{i=0}^{n}a_i f'[x_0,\ldots,x_{i-1},x_{i+1},\ldots,x_n]-f[x_0,\ldots,x_n], \tag{11}$$

$$\Phi_4(\mathbf{x},f)=\frac{1}{\binom{n}{m}\binom{n+1}{m}}\sum_I\frac{1}{m!}f^{(m)}[x_{i_0},\ldots,x_{i_{n-m}}]-\frac{1}{\binom{n}{k}\binom{n+1}{k}}\sum_J\frac{1}{k!}f^{(k)}[x_{j_0},\ldots,x_{j_{n-k}}] \tag{12}$$

and

$$\begin{aligned}\Phi_5(\mathbf{g}(x),f)&=\int_\Omega p(x)f[g_0(x),\ldots,g_n(x)]d\mu(x)\\&\quad -f\left[\int_\Omega p(x)g_0(x)d\mu(x),\ldots,\int_\Omega p(x)g_n(x)d\mu(x)\right],\end{aligned} \tag{13}$$

where $0\leqslant k<m\leqslant n$, $I=\{i_0,\ldots,i_{n-m}\}$, $J=\{j_0,\ldots,j_{n-k}\}$,

$$\mathbf{X}=\begin{bmatrix} x_0^0 & x_1^0 & x_2^0 & \ldots & x_{n-1}^0 & x_n^0\\ x_0^1 & x_1^1 & x_2^1 & \ldots & x_{n-1}^1 & x_n^1\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ x_0^m & x_1^m & x_2^m & \ldots & x_{n-1}^m & x_n^m\end{bmatrix}$$

and $\mathbf{g}(x)=(g_0(x),\ldots,g_n(x))$.

REMARK 4. If $f^{(n)}$ is convex in (a,b), $a\leqslant x_0\leqslant\cdots\leqslant x_n\leqslant b$, from (9) we get functional related to inequality (see [5]):

$$\frac{f^{(n)}(\bar{x})}{n!}=f[\underbrace{\bar{x},\ldots,\bar{x}}_{n+1\ times}]\leqslant f[x_0,\ldots,x_n], \tag{14}$$

where $\bar{x}=\frac{1}{n+1}\sum_{i=0}^{n}x_i$.

If in (10) we put $x_0^i=x^i$, $x_j^i=x^i+h_j$ and $a_i=\frac{p_i}{P_m}$, where $i=0,\ldots,m$, $P_m=\sum_{i=0}^{m}p_i$, $j=1,\ldots,n$ and $x^i+h_j\in(a,b)$, we get functional related to inequality (see [18]):

$$\frac{1}{P_m}\sum_{i=0}^{m}p_i f[x^i,x^i+h_1,\ldots,x^i+h_n]\geqslant f[\bar{x},\bar{x}+h_1,\ldots,\bar{x}+h_n],$$

where $\bar{x}=\frac{1}{P_m}\sum_{i=0}^{m}p_i x^i$.

If we put $n=1$, $h=x_1^i-x_0^i$ and $x_i=x_0^i$, $y_i=x_1^i$ for $i=0,\ldots,m$ in above case we get functional related to inequality (see [11]):

$$\frac{1}{P_m}\sum_{i=0}^{m}p_i f(x_i)-f\left(\frac{1}{P_m}\sum_{i=0}^{m}p_i x_i\right)\leqslant\frac{1}{P_m}\sum_{i=0}^{m}p_i f(y_i)-f\left(\frac{1}{P_m}\sum_{i=0}^{m}p_i y_i\right).$$

If in (10) we put $n=1$, $x_0^i = 2a - x_i$, $x_1^i = x_i$, where $i=0,\dots,m$ and $x_i \in [0,2a]$, we get functional related to inequality (see [16] and [12]):

$$\sum_{i=0}^{m} a_i \frac{f(x_i) - f(2a - x_i)}{x_i - a} \leqslant \frac{f\left(\sum_{i=0}^{m} a_i x_i\right) - f\left(2a - \sum_{i=0}^{m} a_i x_i\right)}{\sum_{i=0}^{m} a_i x_i - a}.$$

We use an idea from [6] to give an elegant method of producing an n-exponentially convex functions and exponentially convex functions applying the above functionals on a given family with the same property (see [15]):

THEOREM 7. *Let $\Upsilon = \{f_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto f_s[z_0,\dots,z_{m+1}]$ is n-exponentially convex in the Jensen sense on J for every $(m+2)$ mutually different points $z_0,\dots,z_{m+1} \in I$. Let $\Phi_1(\mathbf{x},\mathbf{y},f)$ be linear functional defined as in (9). Then $s \mapsto \Phi_1(\mathbf{x},\mathbf{y},f_s)$ is an n-exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_1(\mathbf{x},\mathbf{y},f_s)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. For $\xi_i \in \mathbb{R}$, $i=1,\dots,n$ and $s_i \in J$, $i=1,\dots,n$, we define the function

$$g(z) = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{s_i+s_j}{2}}(z).$$

Using the assumption that the function $s \mapsto f_s[z_0,\dots,z_{m+1}]$ is n-exponentially convex in the Jensen sense, we have

$$g[z_0,\dots,z_{m+1}] = \sum_{i,j=1}^{n} \xi_i \xi_j f_{\frac{s_i+s_j}{2}}[z_0,\dots,z_{m+1}] \geqslant 0,$$

which in turn implies that g is a $(m+1)$-convex function on I and therefore using results from [8] we have $\Phi_1(\mathbf{x},\mathbf{y},g) \geqslant 0$. Hence

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Phi_1\left(\mathbf{x},\mathbf{y},f_{\frac{s_i+s_j}{2}}\right) \geqslant 0.$$

We conclude that the function $s \mapsto \Phi_1(\mathbf{x},\mathbf{y},f_s)$ is n-exponentially convex on J in the Jensen sense.

If the function $s \mapsto \Phi_1(\mathbf{x},\mathbf{y},f_s)$ is also continuous on J, then $s \mapsto \Phi_1(\mathbf{x},\mathbf{y},f_s)$ is n-exponentially convex by definition. □

THEOREM 8. *Let $\Upsilon = \{f_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto f_s[z_0,\dots,z_{m+2}]$ is n-exponentially convex in the Jensen sense on J for every $(m+3)$ mutually different points $z_0,\dots,z_{m+2} \in I$. Let $\Phi_i(\cdot,f)$, $i=1,2,3,4,5$ be linear functional defined as in (9)–(13). Then $s \mapsto \Phi_i(\cdot,f_s)$ is an n-exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(\cdot,f_s)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. Similar as the proof of Theorem 7 using results from [8], [9], [10] and Section 2. $\square$

The following corollaries are an immediate consequences of the above theorems:

COROLLARY 3. *Let $\Upsilon = \{f_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto f_s[z_0, \dots, z_{m+1}]$ is exponentially convex in the Jensen sense on J for every $(m+2)$ mutually different points $z_0, \dots, z_{m+1} \in I$. Let $\Phi_1(\mathbf{x}, \mathbf{y}, f)$ be linear functional defined as in (9). Then $s \mapsto \Phi_1(\mathbf{x}, \mathbf{y}, f_s)$ is an exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_1(\mathbf{x}, \mathbf{y}, f_s)$ is continuous on J, then it is exponentially convex on J.*

COROLLARY 4. *Let $\Upsilon = \{f_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto f_s[z_0, \dots, z_{m+2}]$ is exponentially convex in the Jensen sense on J for every $(m+3)$ mutually different points $z_0, \dots, z_{m+2} \in I$. Let $\Phi_i(\cdot, f)$ $i = 1,2,3,4,5$ be linear functional defined as in (9)–(13). Then $s \mapsto \Phi_i(\cdot, f_s)$ is an exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Phi_i(\cdot, f_s)$ is continuous on J, then it is exponentially convex on J.*

COROLLARY 5. *Let $\Upsilon = \{f_s : s \in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto f_s[z_0, \dots, z_{m+1}]$ is 2-exponentially convex in the Jensen sense on J for every $(m+2)$ mutually different points $z_0, \dots, z_{m+1} \in I$. Let $\Phi_1(\mathbf{x}, \mathbf{y}, f)$ be linear functional defined as in (9). Then the following statements hold:*

(i) If the function $s \mapsto \Phi_1(\mathbf{x}, \mathbf{y}, f_s)$ is continuous on J, then it is 2-exponentially convex function on J, and thus log-convex function and for $r, s, q \in J$ such that $r < s < q$ we have

$$(\Phi_1(\mathbf{x}, \mathbf{y}, f_s))^{q-r} \leqslant (\Phi_1(\mathbf{x}, \mathbf{y}, f_r))^{q-s} (\Phi_1(\mathbf{x}, \mathbf{y}, f_q))^{s-r}. \tag{15}$$

(ii) If the function $s \mapsto \Phi_1(\mathbf{x}, \mathbf{y}, f_s)$ is strictly positive and differentiable on J, then for every $s, q, u, v \in J$, such that $s \leqslant u$ and $q \leqslant v$, we have

$$\mu_{s,q}(\mathbf{x}, \mathbf{y}, \Phi_1, \Upsilon) \leqslant \mu_{u,v}(\mathbf{x}, \mathbf{y}, \Phi_1, \Upsilon), \tag{16}$$

where

$$\mu_{s,q}(\mathbf{x}, \mathbf{y}, \Phi_1, \Upsilon) = \begin{cases} \left(\frac{\Phi_1(\mathbf{x}, \mathbf{y}, f_s)}{\Phi_1(\mathbf{x}, \mathbf{y}, f_q)} \right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\frac{d}{ds}\Phi_1(\mathbf{x}, \mathbf{y}, f_s)}{\Phi_1(\mathbf{x}, \mathbf{y}, f_q)} \right), & s = q, \end{cases} \tag{17}$$

for $f_s, f_q \in \Upsilon$.

Proof.

(i) This is an immediate consequence of Theorem 7 and Remark 3.

(ii) Since by (i) the function $s \mapsto \Phi_1(\mathbf{x},\mathbf{y},f_s)$ is log-convex on J, that is, the function $s \mapsto \log\Phi_1(\mathbf{x},\mathbf{y},f_s)$ is convex on J. So, we get

$$\frac{\log\Phi_1(\mathbf{x},\mathbf{y},f_s)-\log\Phi_1(\mathbf{x},\mathbf{y},f_q)}{s-q} \leqslant \frac{\log\Phi_1(\mathbf{x},\mathbf{y},f_u)-\log\Phi_1(\mathbf{x},\mathbf{y},f_v)}{u-v} \tag{18}$$

for $s \leqslant u, q \leqslant v, s \neq q, u \neq v$, and there form conclude that

$$\mu_{s,q}(\mathbf{x},\mathbf{y},\Phi_1,\Upsilon) \leqslant \mu_{u,v}(\mathbf{x},\mathbf{y},\Phi_1,\Upsilon).$$

Cases $s=q$ and $u=v$ follows from (18) as limit cases. □

COROLLARY 6. *Let $\Upsilon=\{f_s : s\in J\}$, where J an interval in $\mathbb{R}$, be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s\mapsto f_s[z_0,\ldots,z_{m+2}]$ is 2-exponentially convex in the Jensen sense on J for every $(m+3)$ mutually different points $z_0,\ldots,z_{m+2}\in I$. Let $\Phi_i(\cdot,f)$, $i=1,2,3,4,5$ be linear functional defined as in (9)–(13). Then the following statements hold:*

(i) If the function $s\mapsto\Phi_i(\cdot,f_s)$ is continuous on J, then it is 2-exponentially convex function on J, and thus log-convex function and for $r,s,q\in J$ such that $r<s<q$ we have

$$(\Phi_i(\cdot,f_s))^{q-r} \leqslant (\Phi_i(\cdot,f_r))^{q-s}(\Phi_i(\cdot,f_q))^{s-r}. \tag{19}$$

(ii) If the function $s\mapsto\Phi_i(\cdot,f_s)$ is strictly positive and differentiable on J, then for every $s,q,u,v\in J$, such that $s\leqslant u$ and $q\leqslant v$, we have

$$\nu_{s,q}(\cdot,\Phi_i,\Upsilon) \leqslant \nu_{u,v}(\cdot,\Phi_i,\Upsilon), \tag{20}$$

where

$$\nu_{s,q}(\cdot,\Phi_i,\Upsilon)=\begin{cases}\left(\frac{\Phi_i(\cdot,f_s)}{\Phi_i(\cdot,f_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left(\frac{\frac{d}{ds}\Phi_i(\cdot,f_s)}{\Phi_i(\cdot,f_q)}\right), & s=q,\end{cases} \tag{21}$$

for $f_s,f_q\in\Upsilon$.

Proof. Similar as the proof of Corollary 5. □

REMARK 5. Note that the results from above theorems and corollaries still hold when two of the points $z_0,\ldots,z_{m+2}\in I$ coincide, say $z_1=z_0$, for a family of differentiable functions f_s such that the function $s\mapsto f_s[z_0,\ldots,z_{m+1}]$ (or $s\mapsto f_s[z_0,\ldots,z_{m+2}]$) is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense), and furthermore, they still hold when all $(m+2)$ (or $(m+3)$) points coincide for a family of $(m+1)$ (or $(m+2)$) differentiable functions with the same property. The proofs are obtained by (1) and suitable characterization of convexity.

4. Applications to Stolarsky type means

In this section, we present several families of functions which fulfil the conditions of Theorem 7, Corollary 3 and Corollary 5 (or Theorem 8, Corollary 4 and Corollary 6) (and Remark 5). This enable us to construct a large families of functions which are exponentially convex. For a discussion related to this problem see [4].

EXAMPLE 1. Consider a family of functions

$$\Omega_1 = \{g_s : \mathbb{R} \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$g_s(x) = \begin{cases} \frac{e^{sx}}{s^{n+1}}, & s \neq 0, \\ \frac{x^{n+1}}{(n+1)!}, & s = 0. \end{cases}$$

We have $\frac{d^{n+1}g_s}{dx^{n+1}}(x) = e^{sx} > 0$ which shows that g_s is $(n+1)$-convex on $\mathbb{R}$ for every $s \in \mathbb{R}$ and $s \mapsto \frac{d^{n+1}g_s}{dx^{n+1}}(x)$ is exponentially convex by definition. Using analogous arguing as in the proof of Theorem 7 we also have that $s \mapsto g_s[z_0,\ldots,z_{n+1}]$ is exponentially convex (and so exponentially convex in the Jensen sense). Using Corollary 3 we conclude that $s \mapsto \Phi_1(\mathbf{x},\mathbf{y},g_s)$ is exponentially convex in the Jensen sense. It is easy to verify that this mapping is continuous (although mapping $s \mapsto g_s$ is not continuous for $s = 0$), so it is exponentially convex.

For this family of functions, $\mu_{s,q}(\mathbf{x},\mathbf{y},\Phi_1,\Omega_1)$ from (17), becomes

$$\mu_{s,q}(\mathbf{x},\mathbf{y},\Phi_1,\Omega_1) = \begin{cases} \left(\frac{\Phi_1(\mathbf{x},\mathbf{y},g_s)}{\Phi_1(\mathbf{x},\mathbf{y},g_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\Phi_1(\mathbf{x},\mathbf{y},id\cdot g_s)}{\Phi_1(\mathbf{x},\mathbf{y},g_s)} - \frac{n+1}{s}\right), & s = q \neq 0, \\ \exp\left(\frac{1}{n+2}\frac{\Phi_1(\mathbf{x},\mathbf{y},id\cdot g_0)}{\Phi_1(\mathbf{x},\mathbf{y},g_0)}\right), & s = q = 0. \end{cases}$$

Now, using (16) it is monotonous function in parameters s and q. Using Proposition 2 we have:

$$\Phi_1(\mathbf{x},\mathbf{y},id\cdot g_0) = \frac{1}{(n+1)!}\sum_{i=0}^{n}\sum_{j=0}^{i}(x_ix_j - y_iy_j) \text{ and } \Phi_1(\mathbf{x},\mathbf{y},g_0) = \frac{1}{(n+1)!}\sum_{i=0}^{n}(x_i - y_i).$$

For a family of functions

$$\tilde{\Omega}_1 = \{\tilde{g}_s : \mathbb{R} \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$\tilde{g}_s(x) = \begin{cases} \frac{e^{sx}}{s^{n+2}}, & s \neq 0, \\ \frac{x^{n+2}}{(n+2)!}, & s = 0. \end{cases}$$

using Theorem 8, Corollary 4 and Corollary 6, analogous as above we get that

$$\nu_{s,q}(\cdot,\Phi_i,\tilde{\Omega}_1)=\begin{cases}\left(\frac{\Phi_i(\cdot,\tilde{g}_s)}{\Phi_i(\cdot,\tilde{g}_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left(\frac{\Phi_i(\cdot,id\cdot\tilde{g}_s)}{\Phi_i(\cdot,\tilde{g}_s)}-\frac{n+2}{s}\right), & s=q\neq 0,\\ \exp\left(\frac{1}{n+3}\frac{\Phi_i(\cdot,id\cdot\tilde{g}_0)}{\Phi_i(\cdot,\tilde{g}_0)}\right), & s=q=0.\end{cases}$$

These are monotonous functions in parameters s and q. Particular, we obtain:

$$\Phi_1(\mathbf{x},\mathbf{y},id\cdot\tilde{g}_0)=\frac{1}{(n+2)!}\sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{k=0}^{j}(x_ix_jx_k-y_iy_jy_k),$$

$$\Phi_1(\mathbf{x},\mathbf{y},\tilde{g}_0)=\frac{1}{(n+2)!}\sum_{i=0}^{n}\sum_{j=0}^{i}(x_ix_j-y_iy_j),$$

$$\Phi_2(\mathbf{X},\mathbf{a},id\cdot\tilde{g}_0)=\frac{1}{(n+2)!}\left(\sum_{i=0}^{m}a_i\frac{V(\mathbf{x}^i;n+3,0)}{V(\mathbf{x}^i;n,0)}-\frac{V(\mathbf{aX};n+3,0)}{V(\mathbf{aX};n,0)}\right),$$

$$\Phi_2(\mathbf{X},\mathbf{a},\tilde{g}_0)=\frac{1}{(n+2)!}\left(\sum_{i=0}^{m}a_i\frac{V(\mathbf{x}^i;n+2,0)}{V(\mathbf{x}^i;n,0)}-\frac{V(\mathbf{aX};n+2,0)}{V(\mathbf{aX};n,0)}\right),$$

$$\Phi_3(\mathbf{x},\mathbf{a},id\cdot\tilde{g}_0)=\frac{1}{(n+2)!}\left(\frac{n+3}{n}\sum_{i=0}^{m}a_i\frac{V_i(n+2,0)}{V_i(n-1,0)}-\frac{V(n+3,0)}{V(n,0)}\right),$$

$$\Phi_3(\mathbf{x},\mathbf{a},\tilde{g}_0)=\frac{1}{(n+2)!}\left(\frac{n+2}{n}\sum_{i=0}^{m}a_i\frac{V_i(n+1,0)}{V_i(n-1,0)}-\frac{V(n+2,0)}{V(n,0)}\right),$$

$$\begin{aligned}\Phi_4(\mathbf{x},id\cdot\tilde{g}_0)=\frac{1}{(n+2)!}\Bigg(&\frac{(n+3)!}{(n+3-m)!n_m}\sum_I\frac{V_I(n+3-m,0)}{V_I(n-m,0)}\\&-\frac{(n+3)!}{(n+3-k)!n_k}\sum_J\frac{V_J(n+3-k,0)}{V_J(n-k,0)}\Bigg),\end{aligned}$$

$$\begin{aligned}\Phi_4(\mathbf{x},\tilde{g}_0)=\frac{1}{(n+2)!}\Bigg(&\frac{(n+2)!}{(n+2-m)!n_m}\sum_I\frac{V_I(n+2-m,0)}{V_I(n-m,0)}\\&-\frac{(n+2)!}{(n+2-k)!n_k}\sum_J\frac{V_J(n+2-k,0)}{V_J(n-k,0)}\Bigg),\end{aligned}$$

$$\Phi_5(\mathbf{g}(x),id\cdot\tilde{g}_0)=\frac{1}{(n+2)!}\left(\int_\Omega p(x)\frac{V(\mathbf{g}(x);n+3,0)}{V(\mathbf{g}(x);n,0)}d\mu(x)-\frac{V(\hat{\mathbf{g}};n+3,0)}{V(\hat{\mathbf{g}};n,0)}\right),$$

$$\Phi_5(\mathbf{g}(x),\tilde{g}_0)=\frac{1}{(n+2)!}\left(\int_\Omega p(x)\frac{V(\mathbf{g}(x);n+2,0)}{V(\mathbf{g}(x);n,0)}d\mu(x)-\frac{V(\hat{\mathbf{g}};n+2,0)}{V(\hat{\mathbf{g}};n,0)}\right),$$

where

$$V_i(p,q)=\det\begin{bmatrix}1 & x_0 & x_0^2 & \dots & x_0^{n-2} & x_0^p\ln^q x_0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & x_{i-1} & x_{i-1}^2 & \dots & x_{i-1}^{n-2} & x_{i-1}^p\ln^q x_{i-1}\\ 1 & x_{i+1} & x_{i+1}^2 & \dots & x_{i+1}^{n-2} & x_{i+1}^p\ln^q x_{i+1}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & x_n & x_n^2 & \dots & x_n^{n-2} & x_n^p\ln^q x_n\end{bmatrix},$$

$$V_J(p,q)=\det\begin{bmatrix}1 & x_{j_0} & x_{j_0}^2 & \dots & x_{j_0}^{n-1-k} & x_{j_0}^p\ln^q x_{j_0}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & x_{j_{n-k}} & x_{j_{n-k}}^2 & \dots & x_{j_{n-k}}^{n-1-k} & x_{j_{n-k}}^p\ln^q x_{j_{n-k}}\end{bmatrix},$$

$n_k=k!\binom{n}{k}\binom{n+1}{k}$ and $\hat{\mathbf{g}}=\left(\int_\Omega p(x)g_0(x)d\mu(x),\dots,\int_\Omega p(x)g_n(x)d\mu(x)\right)$.

We observe here that $\left(\frac{\frac{d^{n+1}g_s}{dx^{n+1}}}{\frac{d^{n+1}g_q}{dx^{n+1}}}\right)^{\frac{1}{s-q}}(\ln x)=x$ so after the substitution $x_i\to\ln x_i$, $y_i\to\ln y_i$ and $g_i(x)\to\ln g_i(x)$ in above expressions we will get means for $\mathbf{x}$ (and $\mathbf{y}$) or for $\mathbf{g}(x)$:

$$M_{s,q}(\mathbf{x},\mathbf{y},\Phi_1,\Omega_1)=\mu_{s,q}(\ln\mathbf{x},\ln\mathbf{y},\Phi_1,\Omega_1)\ \text{ and}$$
$$N_{s,q}(\cdot,\Phi_i,\tilde{\Omega}_1)=\nu_{s,q}(\ln(\cdot),\Phi_i,\tilde{\Omega}_1),\quad i=1,2,3,4,5.$$

REMARK 6. For $\tilde{q}_s(x)$, $i=1$, $\mathbf{x}=\{x,\underbrace{0,\dots,0}_{n-\text{times}}\}$, $\mathbf{y}=\underbrace{\left\{\frac{x}{n+1},\dots,\frac{x}{n+1}\right\}}_{n+1-\text{times}}$ and $0<r<s$, then (19) implies that

$$\frac{e^{rx}-\sum\limits_{k=0}^{n-1}\frac{r^kx^k}{k!}}{r^{n+2}x^n}-\frac{e^{\frac{rx}{n+1}}}{n!r^2}\leqslant\left(\frac{n}{2(n+1)}\frac{x^2}{(n+2)!}\right)^{1-\frac{r}{s}}\left(\frac{e^{sx}-\sum\limits_{k=0}^{n-1}\frac{s^kx^k}{k!}}{s^{n+2}x^n}-\frac{e^{\frac{sx}{n+1}}}{n!s^2}\right)^{\frac{r}{s}},\quad(22)$$

which is a refinement of the left inequality given in [12, Example 2.58] and reverse inequality will holds for $s<r<0$ or $r<0<s$ or $x<0$ and $n-$odd.

Similar results can be obtained for function $g_s(x)$.

For $\tilde{q}_s(x)$, $i=4$, $\mathbf{x}=\{x,\underbrace{0,\dots,0}_{n-\text{times}}\}$, $k=0$, $m=n$, and $0<r<s$, (19) implies

$$\frac{n+e^{rx}}{(n+1)!r^2}-\frac{e^{rx}-\sum\limits_{k=0}^{n-1}\frac{r^kx^k}{k!}}{r^{n+2}x^n}\leqslant\left(\frac{n}{2}\frac{x^2}{(n+2)!}\right)^{1-\frac{r}{s}}\left(\frac{n+e^{sx}}{(n+1)!s^2}-\frac{e^{sx}-\sum\limits_{k=0}^{n-1}\frac{s^kx^k}{k!}}{s^{n+2}x^n}\right)^{\frac{r}{s}},\quad(23)$$

which is a refinement of the right inequality given in [12, Example 2.58] and reverse inequality will holds for $s<r<0$ or $r<0<s$ or $x<0$ and $n-$odd.

EXAMPLE 2. Consider a family of functions

$$\Omega_2=\{f_s:(0,\infty)\to\mathbb{R}:s\in\mathbb{R}\}$$

defined by

$$f_s(x)=\begin{cases}\frac{x^s}{s(s-1)\cdots(s-n)}, & s\notin\{0,1,\dots,n\}\\ \frac{x^j\ln x}{(-1)^{n-j}j!(n-j)!}, & s=j\in\{0,1,\dots,n\}.\end{cases}$$

Here, $\frac{d^{n+1}f_s}{dx^{n+1}}(x)=x^{s-n-1}=e^{(s-n-1)\ln x}>0$ which shows that f_s is $(n+1)$-convex for $x>0$ and $s\mapsto\frac{d^{n+1}f_s}{dx^{n+1}}(x)$ is exponentially convex by definition. Arguing as in Example 1 we get that the mapping $s\mapsto\Phi_1(\mathbf{x},\mathbf{y},f_s)$ is exponentially convex. In this case we assume that $x_j>0$ and $y_j>0$, $j=0,\dots,n$. Functions (17) now are equal to:

$$\mu_{s,q}(\mathbf{x},\mathbf{y},\Phi_1,\Omega_2)=\begin{cases}\left(\frac{\Phi_1(\mathbf{x},\mathbf{y},f_s)}{\Phi_1(\mathbf{x},\mathbf{y},f_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left((-1)^n n!\frac{\Phi_1(\mathbf{x},\mathbf{y},f_0f_s)}{\Phi_1(\mathbf{x},\mathbf{y},f_s)}+\sum_{k=0}^{n}\frac{1}{k-s}\right), & s=q\notin\{0,1,\dots,n\},\\ \exp\left((-1)^n n!\frac{\Phi_1(\mathbf{x},\mathbf{y},f_0f_s)}{2\Phi_1(\mathbf{x},\mathbf{y},f_s)}+\sum_{\substack{k=0\\k\neq s}}^{n}\frac{1}{k-s}\right), & s=q\in\{0,1,\dots,n\}.\end{cases}$$

For a family of functions

$$\tilde{\Omega}_2=\{\tilde{f}_s:(0,\infty)\to\mathbb{R}:s\in\mathbb{R}\}$$

defined by

$$\tilde{f}_s(x)=\begin{cases}\frac{x^s}{s(s-1)\cdots(s-n-1)}, & s\notin\{0,1,\dots,n+1\}\\ \frac{x^j\ln x}{(-1)^{n+1-j}j!(n+1-j)!}, & s=j\in\{0,1,\dots,n+1\}\end{cases}$$

using Theorem 8, Corollary 4 and Corollary 6, analogous as above we get that

$$\nu_{s,q}(\cdot,\Phi_i,\tilde{\Omega}_2)=\begin{cases}\left(\frac{\Phi_i(\cdot,\tilde{f}_s)}{\Phi_i(\cdot,\tilde{f}_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left((-1)^{n+1}(n+1)!\frac{\Phi_i(\cdot,\tilde{f}_0\tilde{f}_s)}{\Phi_i(\cdot,\tilde{f}_s)}+\sum_{k=0}^{n+1}\frac{1}{k-s}\right), & s=q\notin\{0,1,\dots,n+1\},\\ \exp\left((-1)^{n+1}(n+1)!\frac{\Phi_i(\cdot,\tilde{f}_0\tilde{f}_s)}{2\Phi_i(\cdot,\tilde{f}_s)}+\sum_{\substack{k=0\\k\neq s}}^{n+1}\frac{1}{k-s}\right), & s=q\in\{0,1,\dots,n+1\}.\end{cases}$$

We observe that $\left(\frac{\frac{d^{n+1}f_s}{dx^{n+1}}}{\frac{d^{n+1}f_q}{dx^{n+1}}}\right)^{\frac{1}{s-q}}(x)=x$, so $\mu_{s,q}(\mathbf{x},\mathbf{y},\Phi_1,\Omega_2)$ are $2(n+1)$-Stolarsky means which explicit forms are given in [8].

In the same way we obtain that $\nu_{s,q}(\cdot,\Phi_i,\tilde{\Omega}_2)$ are Schur means ($i=1$), Jensen means for divided differences ($i=2$) and means for divided differences of first ($i=3$) and higher order derivation ($i=4$), which explicit forms are given in [8], [9] and [10].

For $i=5$ we get integral Jensen means for divided differences defined as

$$\nu_{s,q}(\mathbf{g}(x),\Phi_5,\tilde{\Omega}_2)=\begin{cases}\left(\prod_{i=0}^{n+1}\frac{q-i}{s-i}\frac{\int_\Omega p(x)\frac{V\mathbf{g}(x)s0}{V\mathbf{g}(x)n0}d\mu(x)-\frac{V\hat{\mathbf{g}}s0}{V\hat{\mathbf{g}}n0}}{\int_\Omega p(x)\frac{V\mathbf{g}(x)q0}{V\mathbf{g}(x)n0}d\mu(x)-\frac{V\hat{\mathbf{g}}q0}{V\hat{\mathbf{g}}n0}}\right)^{\frac{1}{s-q}} & \text{for } s\neq q;\ s\notin N,\ q\notin N\\ \left(\prod_{\substack{i=0\\ i\neq k}}^{n+1}\frac{(q-k)(q-i)}{k-i}\frac{\int_\Omega p(x)\frac{V\mathbf{g}(x)k1}{V\mathbf{g}(x)n0}d\mu(x)-\frac{V\hat{\mathbf{g}}k1}{V\hat{\mathbf{g}}n0}}{\int_\Omega p(x)\frac{V\mathbf{g}(x)q0}{V\mathbf{g}(x)n0}d\mu(x)-\frac{V\hat{\mathbf{g}}q0}{V\hat{\mathbf{g}}n0}}\right)^{\frac{1}{k-q}} & \text{for } s\neq q;\ s=k\in N, q\notin N\\ \left(\prod_{\substack{i=0\\ i\neq l}}^{n+1}\frac{l-i}{(s-l)(s-i)}\frac{\int_\Omega p(x)\frac{V\mathbf{g}(x)s0}{V\mathbf{g}(x)n0}d\mu(x)-\frac{V\hat{\mathbf{g}}s0}{V\hat{\mathbf{g}}n0}}{\int_\Omega p(x)\frac{V\mathbf{g}(x)l1}{V\mathbf{g}(x)n0}d\mu(x)-\frac{V\hat{\mathbf{g}}l1}{V\hat{\mathbf{g}}n0}}\right)^{\frac{1}{s-l}} & \text{for } s\neq q;\ p\notin N,\ q=l\in N\\ \left(\prod_{\substack{i=0\\ i\neq k,l}}^{n+1}\frac{l-i}{i-k}\frac{\int_\Omega p(x)\frac{V\mathbf{g}(x)k1}{V\mathbf{g}(x)n0}d\mu(x)-\frac{V\hat{\mathbf{g}}k1}{V\hat{\mathbf{g}}n0}}{\int_\Omega p(x)\frac{V\mathbf{g}(x)l1}{V\mathbf{g}(x)n0}d\mu(x)-\frac{V\hat{\mathbf{g}}l1}{V\hat{\mathbf{g}}n0}}\right)^{\frac{1}{k-l}} & \text{for } s\neq q;\ s=k\in N,\ q=l\in N\\ \exp\left(\sum_{i=0}^{n+1}\frac{1}{i-s}+\frac{\int_\Omega p(x)\frac{V\mathbf{g}(x)s1}{V\mathbf{g}(x)n0}d\mu(x)-\frac{V\hat{\mathbf{g}}s1}{V\hat{\mathbf{g}}n0}}{\int_\Omega p(x)\frac{V\mathbf{g}(x)s0}{V\mathbf{g}(x)n0}d\mu(x)-\frac{V\hat{\mathbf{g}}s0}{V\hat{\mathbf{g}}n0}}\right) & \text{for } s=q;\ s\notin N,\ q\notin N\\ \exp\left(\sum_{\substack{i=0\\ i\neq k}}^{n+1}\frac{1}{i-k}+\frac{1}{2}\frac{\int_\Omega p(x)\frac{V\mathbf{g}(x)k2}{V\mathbf{g}(x)n0}d\mu(x)-\frac{V\hat{\mathbf{g}}k2}{V\hat{\mathbf{g}}n0}}{\int_\Omega p(x)\frac{V\mathbf{g}(x)k1}{V\mathbf{g}(x)n0}d\mu(x)-\frac{V\hat{\mathbf{g}}k1}{V\hat{\mathbf{g}}n0}}\right) & \text{for } s=q;\ s=q=k\in N\end{cases}$$

where $N=\{0,1,\ldots,n+1\}$.

A mapping $\nu_{s,q}:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function in two variables s and q. We have

$$\lim_{s\to k}\frac{\int_\Omega p(x)\frac{V\mathbf{g}(x)s0}{V\mathbf{g}(x)n0}d\mu(x)-\frac{V\hat{\mathbf{g}}s0}{V\hat{\mathbf{g}}n0}}{s-k}=\int_\Omega p(x)\frac{V\mathbf{g}(x)k1}{V\mathbf{g}(x)n0}d\mu(x)-\frac{V\hat{\mathbf{g}}k1}{V\hat{\mathbf{g}}n0}\qquad\text{for } k\in N. \tag{24}$$

The quotient under limit becomes an indeterminate form when $s\to k\in N$ because $V\mathbf{g}(x)k0=0$ for $k=0,1,\ldots,n-1$ and $\int_\Omega p(x)\frac{V\mathbf{g}(x)k0}{V\mathbf{g}(x)n0}d\mu(x)-\frac{V\hat{\mathbf{g}}k0}{V\hat{\mathbf{g}}n0}=1-1=0$ for $k=n$. When $s\to k=n+1$:

$$\int_\Omega p(x)\frac{V\mathbf{g}(x)k0}{V\mathbf{g}(x)n0}d\mu(x)-\frac{V\hat{\mathbf{g}}k0}{V\hat{\mathbf{g}}n0}=\int_\Omega p(x)\sum_{i=0}^{n}g_i(x)d\mu(x)-\sum_{i=0}^{n}\int_\Omega p(x)g_i(x)d\mu(x)=0.$$

So, we can apply L'Hospital's rule and also the formula $\frac{d}{ds}V\mathbf{x}sq = V\mathbf{x}\ s\ q{+}1$ to get result (24). If $n=0$ the expressions for means $\nu_{s,q}(\mathbf{g}(x),\Phi_5,\tilde{\Omega}_2)$ proceed from the continuous extensions of function

$$(s,q)\mapsto\left(\frac{q(q-1)}{s(s-1)}\frac{\int_\Omega p(x)(g_0(x))^s d\mu(x)-(\int_\Omega p(x)g_0(x)d\mu(x))^s}{\int_\Omega p(x)(g_0(x))^q d\mu(x)-(\int_\Omega p(x)g_0(x)d\mu(x))^q}\right)^{\frac{1}{s-q}}.$$

Particularly,

$$\min\{\mathbf{x},\mathbf{y}\}\leqslant\left(\frac{\Phi_1(\mathbf{x},\mathbf{y},f_s)}{\Phi_1(\mathbf{x},\mathbf{y},f_q)}\right)^{\frac{1}{s-q}}\leqslant\max\{\mathbf{x},\mathbf{y}\}.$$

We now impose one additional parameter $r\in\mathbb{R}$. After substitutions $s\to\frac{s}{r}$, $q\to\frac{q}{r}$ $x_i\to x_i^r$ and $y_i\to y_i^r$ $(i=0,\ldots,n,\ r\neq 0)$ we have

$$\min\{\mathbf{x^r},\mathbf{y^r}\}\leqslant\left(\frac{\Phi_1(\mathbf{x^r},\mathbf{y^r},f_{\frac{s}{r}})}{\Phi_1(\mathbf{x^r},\mathbf{y^r},f_{\frac{q}{r}})}\right)^{\frac{r}{s-q}}\leqslant\max\{\mathbf{x^r},\mathbf{y^r}\}.$$

Hence, we get (generalized) Cauchy means as

$$\mu_{s,q;r}(\mathbf{x},\mathbf{y},\Phi_1,\Omega_1)=\begin{cases}\left(\mu_{\frac{s}{r},\frac{q}{r}}(\mathbf{x^r},\mathbf{y^r},\Phi_1,\Omega_1)\right)^{\frac{1}{r}}, & r\neq 0,\\ \mu_{s,q}(\log\mathbf{x},\log\mathbf{y},\Phi_1,\Omega_1), & r=0.\end{cases}$$

In the identical manners we can get $\nu_{s,q;r}(\cdot,\Phi_i,\tilde{\Omega}_2)$ $(i=1,2,3,4,5)$.

For $i=5$ and $n=0$ the expressions for means $\nu_{s,q;r}(\mathbf{g}(x),\Phi_5,\tilde{\Omega}_2)$, proceed from the continuous extensions of function

$$(s,q,r)\mapsto\left(\frac{q(q-r)}{s(s-r)}\frac{\int_\Omega p(x)(g_0(x))^s d\mu(x)-(\int_\Omega p(x)(g_0(x))^r d\mu(x))^{\frac{s}{r}}}{\int_\Omega p(x)(g_0(x))^q d\mu(x)-(\int_\Omega p(x)(g_0(x))^r d\mu(x))^{\frac{q}{r}}}\right)^{\frac{1}{s-q}},$$

which are the weighted versions of new means of Cauchy type given explicitly in [2].

EXAMPLE 3. Consider a family of functions

$$\Omega_3=\{h_s:(0,\infty)\to\mathbb{R}:s\in(0,\infty)\}$$

defined by

$$h_s(x)=\begin{cases}\frac{s^{-x}}{(-\ln s)^{n+1}}, & s\neq 1\\ \frac{x^{n+1}}{(n+1)!}, & s=1.\end{cases}$$

Since $\frac{d^{n+1}h_s}{dx^{n+1}}(x)=s^{-x}$ is the Laplace transform of a non-negative function (see [17]) it is exponentially convex. Obviously h_s are $(n+1)$-convex functions for every $s>0$.

For this family of functions, $\mu_{s,q}(\mathbf{x},\mathbf{y},\Phi_1,\Omega_3)$, in this case for $x_j > 0$ and $y_j > 0$, $j=0,\ldots,n$, from (17) becomes

$$\mu_{s,q}(\mathbf{x},\mathbf{y},\Phi_1,\Omega_3)=\begin{cases}\left(\frac{\Phi_1(\mathbf{x},\mathbf{y},h_s)}{\Phi_1(\mathbf{x},\mathbf{y},h_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left(-\frac{\Phi_1(\mathbf{x},\mathbf{y},id\cdot h_s)}{s\Phi_1(\mathbf{x},\mathbf{y},h_s)}-\frac{n+1}{s\ln s}\right), & s=q\neq 1,\\ \exp\left(-\frac{1}{n+2}\frac{\Phi_1(\mathbf{x},\mathbf{y},id\cdot h_1)}{\Phi_1(\mathbf{x},\mathbf{y},h_1)}\right), & s=q=1.\end{cases}$$

It is monotonous function in parameters s and q by (16).

For a family of functions

$$\tilde{\Omega}_3=\{\tilde{h}_s:(0,\infty)\to\mathbb{R}:s\in(0,\infty)\}$$

defined by

$$\tilde{h}_s(x)=\begin{cases}\frac{s^{-x}}{(-\ln s)^{n+2}}, & s\neq 1,\\ \frac{x^{n+2}}{(n+2)!}, & s=1.\end{cases}$$

using Theorem 8, Corollary 4 and Corollary 6, analogous as above we get that

$$\nu_{s,q}(\cdot,\Phi_i,\tilde{\Omega}_3)=\begin{cases}\left(\frac{\Phi_i(\cdot,\tilde{h}_s)}{\Phi_i(\cdot,\tilde{h}_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left(-\frac{\Phi_i(\cdot,id\cdot\tilde{h}_s)}{s\Phi_i(\cdot,\tilde{h}_s)}-\frac{n+2}{s\ln s}\right), & s=q\neq 1,\\ \exp\left(-\frac{1}{n+3}\frac{\Phi_i(\cdot,id\cdot\tilde{h}_1)}{\Phi_i(\cdot,\tilde{h}_1)}\right), & s=q=1.\end{cases}$$

These are monotonous function in parameters s and q.

After the substitution $x_i\to-\ln x_i$, $y_i\to-\ln y_i$ and $g_i(x)\to-\ln g_i(x)$ in $\mu_{1,1}(\mathbf{x},\mathbf{y},\Phi_1,\Omega_3)$ and $\nu_{1,1}(\cdot,\Phi_i,\tilde{\Omega}_3)$ $(i=1,2,3,4,5)$ we will get means for $\mathbf{x}$ (and $\mathbf{y}$) or for $\mathbf{g}(x)$:

$$M_{1,1}(\mathbf{x},\mathbf{y},\Phi_1,\Omega_3)=\mu_{1,1}(-\ln\mathbf{x},-\ln\mathbf{y},\Phi_1,\Omega_3)\ \text{ and}$$

$$N_{1,1}(\cdot,\Phi_i,\tilde{\Omega}_3)=\nu_{1,1}(-\ln(\cdot),\Phi_i,\tilde{\Omega}_3)\ ,i=1,2,3,4,5.$$

EXAMPLE 4. Consider a family of functions

$$\Omega_4=\{k_s:(0,\infty)\to\mathbb{R}:s\in(0,\infty)\}$$

defined by

$$k_s(x)=\frac{e^{-x\sqrt{s}}}{(-\sqrt{s})^{n+1}}.$$

Since $\frac{d^{n+1}k_s}{dx^{n+1}}(x)=e^{-x\sqrt{s}}$ is the Laplace transform of a non-negative function (see [17]) it is exponentially convex. Obviously k_s are $(n+1)$-convex functions for every $s>0$.

For this family of functions, $\mu_{s,q}(\mathbf{x},\mathbf{y},\Phi_1,\Omega_4)$, in this case for $x_j > 0$ and $y_j > 0$, $j = 0,\dots,n$, from (17) becomes

$$\mu_{s,q}(\mathbf{x},\mathbf{y},\Phi_1,\Omega_4) = \begin{cases} \left(\frac{\Phi_1(\mathbf{x},\mathbf{y},k_s)}{\Phi_1(\mathbf{x},\mathbf{y},k_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_1(\mathbf{x},\mathbf{y},id\cdot k_s)}{2\sqrt{s}\Phi_1(\mathbf{x},\mathbf{y},k_s)} - \frac{n+1}{2s}\right), & s = q. \end{cases}$$

It is monotonous function in parameters s and q by (16).

For a family of functions

$$\tilde{\Omega}_4 = \{\tilde{k}_s : (0,\infty) \to \mathbb{R} : s \in (0,\infty)\}$$

defined by

$$\tilde{k}_s(x) = \frac{e^{-x\sqrt{s}}}{(-\sqrt{s})^{n+2}}.$$

using Theorem 8, Corollary 4 and Corollary 6, analogous as above we get that

$$\nu_{s,q}(\cdot,\Phi_i,\tilde{\Omega}_4) = \begin{cases} \left(\frac{\Phi_i(\cdot,\tilde{k}_s)}{\Phi_i(\cdot,\tilde{k}_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(\cdot,id\cdot\tilde{k}_s)}{2\sqrt{s}\Phi_i(\cdot,\tilde{k}_s)} - \frac{n+2}{2s}\right), & s = q. \end{cases}$$

These are monotonous functions in parameters s and q.

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