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ON THE HYERS—ULAM STABILITY OF SEXTIC FUNCTIONAL EQUATIONS IN β–HOMOGENEOUS PROBABILISTIC MODULAR SPACES

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Abstract. In this paper, we present a fixed point method to prove the generalized Hyers–Ulam stability of the systems of additive-quadratic-cubic functional equations with constant coefficients in β-homogeneous probabilistic modular spaces.

1. Introduction

The *Hyers–Ulam stability problem* of functional equations started with the following question concerning stability of group homomorphisms proposed by S. M. Ulam [67] during a talk before a Mathematical Colloquium at the University of Wisconsin, Madison, in 1940:

Let $(G_1,\cdot)$ be a group and $(G_2,*)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon>0$, does there exist a $\delta>0$ such that, if a mapping $h: G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x\cdot y),h(x)*h(y))<\delta$ for all $x,y\in G_1$, then there exists a homomorphism $H: G_1\longrightarrow G_2$ with $d(h(x),H(x))<\varepsilon$ for all $x\in G_1$?

In 1941, Hyers [28] gave a first affirmative answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by Aoki [2] for additive mappings and by Rassias [59] for linear mappings by considering an unbounded Cauchy difference, respectively.

In 1994, a generalization of Rassias' theorem was obtained by Găvruta [14] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias' approach. For more details about the results concerning such problems, the reader refer to [7, 8, 13, 15, 23, 29, 32, 33, 34, 40, 60] and [53]–[61]. Recently, Sadeghi [63] presented a fixed point method to prove generalized Hyers–Ulam stability of the generalized Jensen functional equation $f(rx+sy)=rg(x)+sh(x)$ in modular spaces.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1}$$

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is related to a symmetric bi–additive function [1, 36]. It is natural that this equation is called a *quadratic functional equation*. In particular, every solution of the quadratic equation (1) is called a *quadratic function*. The Hyers–Ulam stability problem for the quadratic functional equation was solved by Skof [65]. In [8], Czerwik proved the Hyers–Ulam–Rassias stability of the equation (1). Eshaghi Gordji and Khodaei [24] obtained the general solution and the generalized Hyers–Ulam–Rassias stability of the following quadratic functional equation: for all $a,b \in \mathbb{Z}\backslash\{0\}$ with $a \neq \pm 1, \pm b$,

$$f(ax+by)+f(ax-by)=2a^2f(x)+2b^2f(y). \tag{2}$$

Jun and Kim [30] introduced the following cubic functional equation:

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{3}$$

and they established the general solution and the generalized Hyers–Ulam–Rassias stability for the functional equation (3). Jun et al. [31] investigated the solution and the Hyers–Ulam stability for the cubic functional equation:

$$f(ax+by)+f(ax-by)=ab^2(f(x+y)+f(x-y))+2a(a^2-b^2)f(x), \tag{4}$$

where $a,b \in \mathbb{Z}\backslash\{0\}$ with $a \neq \pm 1, \pm b$. For other cubic functional equations, see [49].

Lee et. al. [43] considered the following functional equation:

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{5}$$

In fact, they proved that a function f between two real vector spaces X and Y is a solution of the function equation (5) if and only if there exists a unique symmetric bi-quadratic function $B_2 : X \times X \longrightarrow Y$ such that $f(x) = B_2(x,x)$ for all $x \in X$. The bi-quadratic function B_2 is given by

$$B_2(x,y)=\frac{1}{12}(f(x+y)+f(x-y)-2f(x)-2f(y)).$$

Obviously, the function $f(x) = cx^4$ satisfies the functional equation (5), which is called the *quartic functional equation*. For other quartic functional equations, see [5, 6, 39, 47, 54, 62].

Ebadian et al. [9] considered the generalized Hyers-Ulam stability of the following systems of the additive–quartic functional equations:

$$\begin{cases} f(x_1+x_2,y)=f(x_1,y)+f(x_2,y), \\ f(x,2y_1+y_2)+f(x,2y_1-y_2) \\ \qquad =4f(x,y_1+y_2)+4f(x,y_1-y_2)+24f(x,y_1)-6f(x,y_2) \end{cases} \tag{6}$$

and the quadratic-cubic functional equations:

$$\begin{cases} f(x,y_1+y_2)+f(x,y_1-y_2)=2f(x,y_1)+2f(x,y_2), \\ f(x,2y_1+y_2)+f(x,2y_1-y_2) \\ \qquad =2f(x,y_1+y_2)+2f(x,y_1-y_2)+12f(x,y_1). \end{cases} \tag{7}$$

For more details about the results concerning mixed type functional equations, the readers refer to [18, 19, 21] and [22].

Recently, Ghaemi et. al. [17] investigated the the Hyers– Ulam stability of the following systems of quadratic-cubic functional equations:

$$\begin{cases} f(ax_1+bx_2,y)+f(ax_1-bx_2,y)=2a^2f(x_1,y)+2b^2f(x_2,y), \\ f(x,ay_1+by_2)+f(x,ay_1-by_2) \\ \qquad =ab^2(f(x,y_1+y_2)+f(x,y_1-y_2))+2a(a^2-b^2)f(x,y_1) \end{cases} \tag{8}$$

and additive–quadratic-cubic functional equations:

$$\begin{cases} f(ax_1+bx_2,y,z)+f(ax_1-bx_2,y,z)=2af(x_1,y,z), \\ f(x,ay_1+by_2,z)+f(x,ay_1-by_2,z)=2a^2f(x,y_1,z)+2b^2f(x,y_2,z), \\ f(x,y,az_1+bz_2)+f(x,y,az_1-bz_2) \\ \qquad =ab^2(f(x,y,z_1+z_2)+f(x,y,z_1-z_2))+2a(a^2-b^2)f(x,y,z_1) \end{cases} \tag{9}$$

in PN-spaces, where $a,b\in\mathbb{Z}\backslash\{0\}$ with $a\neq\pm1,\pm b$. The function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ given by $f(x,y)=cx^2y^3$ is a solution of the system (9). In particular, letting $y=x$, we get a quintic function $g:\mathbb{R}\to\mathbb{R}$ in one variable given by $g(x):=f(x,x)=cx^5$. Also, it is easy to see that the function $f:\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ defined by $f(x,y,z)=cxy^2z^3$ is a solution of the system (9). In particular, letting $y=z=x$, we get a sextic function $h:\mathbb{R}\to\mathbb{R}$ in one variable given by $h(x):=f(x,x,x)=cx^6$.

The proof of the following propositions is evident.

PROPOSITION 1. *Let X and Y be real linear spaces. If a function $f:X\times X\times X\longrightarrow Y$ satisfies the system* (9)*, then $f(\lambda x,\mu y,\eta z)=\lambda\mu^2\eta^3f(x,y,z)$ for all $x,y,z\in X$ and rational numbers λ,μ,η.*

In 2003 Cădariu and Radu [4] applied the fixed point method to the investigation of the Jensen functional equation (see also [27, 35]). They could present a short and a simple proof (different from the "*direct method*", which was initiated by Hyers in 1941) for the generalized Hyers–Ulam stability of Jensen functional equation [4].

In this paper, by using some ideas of [11, 63], we investigate the generalized Hyers–Ulame stability of a sextic functional equations from linear spaces into modular spaces.

The theory of modulars on linear spaces and the corresponding theory of modular linear spaces were founded by Nakano [50] and were intensively developed by Koshi and Shimogaki [41], Yamamuro [68] and others. Further, the most complete development of these theories are due to Luxemburg [44], Musielak [48], Orlicz [52], Turpin [66] and their collaborators. Recently, the theory of modulars and modular spaces have been extensively applied, in particular, in the study of various Orlicz spaces [52] and interpolation theory [42, 45], which, in turn, have broad applications [48]. The importance for applications consists in the richness of the structure of modular function spaces, that – besides being Banach spaces (or F–spaces in more general setting) – are equipped with modular equivalent of norm or metric notions.

On the other hand, in 1942, a generalization of the notion of metric space was introduced by Menger [46] under the name of a statistical metric space which is now called a *probabilistic metric space* (write $\mathscr{PM}$-space). Such a probabilistic generalization of metric space appears when there is an uncertainty about the distance between the points and we know only the probabilities of possible values this distance. After the appearance of Menger's paper, the study of probabilistic metric spaces have been performed rapidly by many authors in theory and applications and many concepts and results in classical functional analysis obtained some generalizations and counterparts in probabilistic functional analysis (see [16]). In [12], after introducing the probabilistic modular, authors investigated some basic facts in such spaces and study linear operators defined between them.

DEFINITION 1. Let $\mathscr{X}$ be an arbitrary vector space.

(1) A functional $\rho : \mathscr{X} \to [0,\infty]$ is called a *modular* if, for any $x, y \in \mathscr{X}$,

(a) $\rho(x) = 0$ if and only if $x = 0$;

(b) $\rho(\alpha x) = \rho(x)$ for every scaler α with $|\alpha| = 1$;

(c) $\rho(\alpha x + \beta y) \leqslant \rho(x) + \rho(y)$ if and only if $\alpha + \beta = 1$ and $\alpha, \beta \geqslant 0$.

(2) If (c) is replaced by

(c)$'$ $\rho(\alpha x + \beta y) \leqslant \alpha\rho(x) + \beta\rho(y)$ if and only if $\alpha + \beta = 1$ for all $\alpha, \beta \geqslant 0$,

then we say that ρ is a *convex modular*.

A modular ρ defines a corresponding modular space, i.e., the vector space $\mathscr{X}_\rho$ is given by

$$\mathscr{X}_\rho = \{x \in \mathscr{X} : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

Let ρ be a convex modular. Then the modular space $\mathscr{X}_\rho$ can be equipped with a norm called the *Luxemburg norm* defined by

$$\|x\|_\rho = \inf\left\{\lambda > 0 : \rho\left(\frac{x}{\lambda}\right) \leqslant 1\right\}.$$

A function modular is said to satisfy the Δ_2*–condition* if there exists $\kappa > 0$ such that $\rho(2x) \leqslant \kappa\rho(x)$ for all $x \in \mathscr{X}_\rho$.

DEFINITION 2. Let $\{x_n\}$ and $x \in \mathscr{X}_\rho$.

(1) The sequence $\{x_n\}$ with $x_n \in \mathscr{X}_\rho$ is said to be ρ*–convergent* to x (write $x_n \xrightarrow{\rho} x$) if

$$\rho(x_n - x) \to 0$$

as $n \to \infty$.

(2) The sequence $\{x_n\}$ with $x_n \in \mathscr{X}_\rho$ is called a ρ*–Cauchy sequence* if

$$\rho(x_n - x_m) \to 0$$

as $n, m \to \infty$.

(3) A subset $\mathscr{S}$ of $\mathscr{X}_\rho$ is called ρ*–complete* if every ρ–Cauchy sequence is ρ–convergent to a point in $\mathscr{S}$.

We say that the modular ρ has the *Fatou property* if

$$\rho(x) \leqslant \liminf_{n\to\infty} \rho(x_n)$$

whenever the sequence $\{x_n\}$ is ρ–convergent to x.

REMARK 1. Note that ρ is an increasing function. Suppose $0 < a < b$. Then the property (c) of Definition 1 with $y = 0$ shows that $\rho(ax) = \rho\left(\frac{a}{b}bx\right) \leqslant \rho(bx)$ for all $x \in \mathscr{X}$. Moreover, if ρ is a convex modular on $\mathscr{X}$ and $|\alpha| \leqslant 1$, then $\rho(\alpha x) \leqslant \alpha\rho(x)$ and also $\rho(x) \leqslant \frac{1}{2}\rho(2x)$ for all $x \in \mathscr{X}$.

If a convex function φ defined on the interval $[0,\infty)$ is nondecreasing and continuous for any $\alpha \geqslant 0$ and $\varphi(0) = 0$, $\varphi(\alpha) > 0$ for any $\alpha > 0$, $\varphi(\alpha) \to \infty$ as $\alpha \to \infty$, then φ is called an *Orlicz function*. The Orlicz function φ satisfies the Δ_2–condition if there exists $\kappa > 0$ such that $\varphi(2\alpha) \leqslant \varphi(\alpha)$ for all $\alpha > 0$.

Let (Ω, Σ, ν) be a measure space. Let us consider the space $L^0(\nu)$ consisting of all measurable real–valued (or complex–valued) functions on Ω. For all $f \in L^0(\nu)$, define the *Orlicz modular* $\rho_\varphi(f)$ by the formula

$$\rho_\varphi(f) = \int_\Omega \varphi(|f|)d\nu.$$

The associated modular function space with respect to this modular is called an *Orlicz space*, which is denoted by $L^\varphi(\Omega,\nu)$ or, briefly, L^φ. In other words,

$$L^\varphi = \{f \in L^0(\nu) : \rho_\varphi(\lambda f) \to 0 \text{ as } \lambda \to 0\}$$

or, equivalently,

$$L^\varphi = \{f \in L^0(\nu) : \rho_\varphi(\lambda f) < \infty \text{ for some } \lambda > 0\}.$$

It is known that an Orlicz space L^φ is ρ_φ–complete. Moreover, $(L^\varphi, \|\cdot\|_{\rho_\varphi})$ is a Banach space, where the Luxemburg norm $\|\cdot\|_{\rho_\varphi}$ is defined as follows:

$$\|f\|_{\rho_\varphi} = \inf\left\{\lambda > 0 : \int_\Omega \varphi\left(\frac{|f|}{\lambda}\right)d\nu \leqslant 1\right\}.$$

Moreover, if $\mathfrak{L}$ is the space of sequences $x = \{x_i\}_{i=1}^\infty$ with real or complex terms x_i, $\varphi = \{\varphi_i\}_{i=1}^\infty$, φ_i are Orlicz functions and $\rho_\varphi(x) = \Sigma_{i=1}^\infty \varphi_i(|x_i|)$, then we write ℓ^φ in place of L^φ. The space ℓ^φ is called the *generalized Orlicz sequence space*. The motivation for the study of modular spaces (and Orlicz spaces) and many examples are detailed in [50, 48, 52, 45].

We follow the definition of a probabilistic modular space briefly as given in [12]. In the following, Δ stands for the set of all non-decreasing functions $f : \mathbb{R} \to \mathbb{R}_0^+$ satisfying $\inf_{t\in\mathbb{R}} f(t) = 0$ and $\sup_{t\in\mathbb{R}} f(t) = 1$. We also denote the function min by $\wedge$.

DEFINITION 3. A pair (X,μ) is called a probabilistic modular space ($\mathscr{PM}$-space) if X is a real vector space, μ is a mapping from X into Δ satisfying the following conditions:

(a) $\mu(x)(0)=0$;

(b) $\mu(x)(t)=1$ for all $t>0$ if and only if $x=\theta$ where θ is the null vector in X;

(c) $\mu(-x)(t)=\mu(x)(t)$;

(d) $\mu(\alpha x+\beta y)(s+t)\geqslant\mu(x)(s)\wedge\mu(y)(t)$ for all $x,y\in X$ and $\alpha,\beta,s,t\in\mathbb{R}_0^+$ with $\alpha+\beta=1$.

For example, suppose that X is a real vector space and ρ is a modular on X. Define

$$\mu(x)(t)=\begin{cases}0, & t\leqslant 0,\\ \frac{t}{t+\rho(x)}, & t>0.\end{cases}$$

Then (X,μ) is a probabilistic modular space.

We say that (X,μ) is β*-homogeneous*, where $\beta\in(0,1]$, if

$$\mu(\alpha x)(t)=\mu(x)\left(\frac{t}{|\alpha|^{\beta}}\right)$$

for all $x\in X$, $t>0$ and $\alpha\in\mathbb{R}\setminus\{0\}$.

DEFINITION 4. Let (X,μ) be a $\mathscr{PM}$-space, $\{x_n\}$ be a sequence in X and $x\in X$.

(1) The sequence $\{x_n\}$ with $x_n\in(X,\mu)$ is said to be μ*–convergent* to x (write $x_n\xrightarrow{\mu}x$) if, for any $t>0$ and $\lambda\in(0,1)$, there exists a positive integer n_0 such that

$$\mu(x_n-x)(t)>1-\lambda$$

for all $n\geqslant n_0$.

(2) The sequence $\{x_n\}$ with $x_n\in(X,\mu)$ is called a μ*–Cauchy sequence* if, for any $t>0$ and $\lambda\in(0,1)$, there exists a positive integer n_0 such that

$$\mu(x_n-x_m)(t)>1-\lambda$$

for all $m,n\geqslant n_0$.

By [12], every μ-convergent sequence in a $\mathscr{PM}$-space is a μ-Cauchy sequence. If each μ-Cauchy sequence is μ-convergent in a $\mathscr{PM}$-space (X,μ), then (X,μ) is called a μ*-complete* $\mathscr{PM}$*-space*.

We say that a $\mathscr{PM}$-space (X,μ) possesses the *Fatou property* if, for any sequence $\{x_n\}$ of X μ-converging to a point $x\in X$,

$$\mu(x)(t)\geqslant\limsup_{n\geqslant 1}\mu(x_n)(t)$$

for all $t>0$.

REMARK 2. Note that, for any $x \in X$, $\mu(x)(\cdot)$ is an increasing function since $\mu(x) \in \Delta$. Moreover, if μ is a β-homogeneous probabilistic modular on X and $x,y \in X$, then the property (d) of Definition 3 follows that

$$\mu(x+y)\left(2^{\beta}(s+t)\right) = \mu\left(\frac{1}{2}x+\frac{1}{2}y\right)(s+t) \geqslant \mu(x)(s) \wedge \mu(y)(t).$$

For more details about the $\mathscr{P}\mathscr{M}$-spaces, see [51].

Our aim is based on the following fixed point approach:

THEOREM 1. ([38]) *Let X_ρ be a modular space satisfying the Fatou property. Let $\mathscr{C}$ be a ρ-complete nonempty subset of X_ρ and $T:\mathscr{C}\to\mathscr{C}$ be a quasi-contraction, that is, there exists $K<1$ such that*

$$\begin{aligned}&\rho(T(x)-T(y))\\ &\leqslant K\max\{\rho(x-y),\rho(x-T(x)),\rho(y-T(y)),\rho(x-T(y)),\rho(y-T(x))\}.\end{aligned}$$

Let $x\in\mathscr{C}$ such that

$$\delta_\rho(x) := \sup\{\rho(T^n(x)-T^m(x)) : m,n\in\mathbb{N}\} < \infty.$$

Then $\{T^n(x)\}$ ρ-converges to a point $\omega\in\mathscr{C}$. Moreover, if $\rho(\omega-T(\omega))<\infty$ and $\rho(x-T(\omega))<\infty$, then the ρ-limit of $T^n(x)$ is a fixed point of T. Furthermore, if ω^* is any fixed point of T in $\mathscr{C}$ such that $\rho(\omega-\omega^*)<\infty$, then one has $\omega=\omega^*$.

2. Main results

Throughout this paper, we assume that μ is a probabilistic modular on X with the Fatou property (in the probabilistic modular sense) and (X,μ) is a μ-complete β-homogeneous $\mathscr{P}\mathscr{M}$-space with $\beta\in(0,1]$. In this section, we establish the conditional Hyers– Ulam stability of sextic functional equations in $\mathscr{P}\mathscr{M}$-spaces.

THEOREM 2. *Let $s\in\{-1,1\}$ be fixed. Let E be a linear space and (X,μ) be a μ-complete β-homogeneous $\mathscr{P}\mathscr{M}$-space. Suppose that $f:E\times E\times E\to(X,\mu)$ satisfies the condition $f(x,0,z)=0$ and the inequalities of the form:*

$$\begin{aligned}&\mu\big(f(ax_1+bx_2,y,z)+f(ax_1-bx_2,y,z)-2af(x_1,y,z)\big)(t)\\ &\geqslant \phi(x_1,x_2,y,z)(t), \qquad (10)\\ &\mu\big(f(x,ay_1+by_2,z)+f(x,ay_1-by_2,z)-2a^2f(x,y_1,z)-2b^2f(x,y_2,z)\big)(t)\\ &\geqslant \varphi(x,y_1,y_2,z)(t), \qquad (11)\\ &\mu\big(f(x,y,az_1+bz_2)+f(x,y,az_1-bz_2)-ab^2f(x,y,z_1+z_2)\\ &\quad +f(x,y,z_1-z_2)-2a(a^2-b^2)f(x,y,z_1)\big)(t)\\ &\geqslant \psi(x,y,z_1,z_2)(t), \qquad (12)\end{aligned}$$

where $\phi, \varphi, \psi : E^4 \to \Delta$ *are given functions such that*

$$\lim_{n\to\infty} \phi(a^n x_1, a^n x_2, a^n y, a^n z)(a^{6\beta sn} t) = 1,$$
$$\lim_{n\to\infty} \varphi(a^n x, a^n y_1, a^n y_2, a^n z)(a^{6\beta sn} t) = 1,$$
$$\lim_{n\to\infty} \psi(a^n x, a^n y, a^n z_1, a^n z_2)(a^{6\beta sn} t) = 1$$

for all $x, x_i, y, y_i, z, z_i \in E$, $i = 1, 2$. *Assume that*

$$\begin{aligned}\Phi(x,y,z)(t) := \psi(a^{\frac{s+1}{2}}x, a^{\frac{s+1}{2}}y, a^{\frac{s-1}{2}}z, 0)(a^{(9-3s)\beta} t/2^{\beta+2}) \\ \wedge \varphi(a^{\frac{s+1}{2}}x, a^{\frac{s-1}{2}}y, 0, a^{\frac{s-1}{2}}z)(2a^{(6-3s)\beta} t/2^{\beta+2}) \\ \wedge \phi(a^{\frac{s-1}{2}}x, 0, a^{\frac{s-1}{2}}y, a^{\frac{s-1}{2}}z)(2a^{(4-3s)\beta} t/2)\end{aligned} \tag{13}$$

has the property:

$$\Phi(a^s x, a^s y, a^s z)(a^{6\beta s} L t) \geqslant \Phi(x,y,z)(t)$$

for all $x, y, z \in E$ *and a constant* $0 < L < \frac{1}{2^\beta}$. *Then there exists a unique sextic function* $j : E \times E \times E \to (X, \mu)$ *satisfying the system* (9) *and*

$$\mu(j(x,y,z) - f(x,y,z))\left(\frac{2^\beta}{1-2^\beta L} t\right) \geqslant \Phi(x,y,z)(t) \tag{14}$$

for all $x, y, z \in E$.

Proof. Putting $x_1 = 2x$ and $x_2 = 0$ and replacing y, z by $2y, 2z$ in (10), respectively, we get

$$\mu(2f(2ax, 2y, 2z) - 2af(2x, 2y, 2z))(t) \geqslant \phi(2x, 0, 2y, 2z)(t) \tag{15}$$

for all $x, y, z \in E$. Putting $y_1 = 2y$ and $y_2 = 0$ and replacing x, z by $2ax, 2z$ in (11), respectively, we get

$$\mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z))(t) \geqslant \varphi(2ax, 2y, 0, 2z)(t) \tag{16}$$

for all $x, y, z \in E$. Putting $z_1 = 2z$ and $z_2 = 0$ and replacing x, y by $2ax, 2ay$ in (12), we get

$$\mu(2f(2ax, 2ay, 2az) - 2a^3 f(2ax, 2ay, 2z))(t) \geqslant \psi(2ax, 2ay, 2z, 0)(t) \tag{17}$$

for all $x, y, z \in E$. Since μ is β-homogeneous, it follows from (16) and (17) that

$$\begin{aligned}&\mu(2a^{-3} f(2ax, 2ay, 2az) - 2a^2 f(2ax, 2y, 2z))(2^\beta t) \\ &\geqslant \mu(2a^{-3} f(2ax, 2ay, 2az) - 2f(2ax, 2ay, 2z))(t/2) \\ &\quad \wedge \mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z))(t/2) \\ &= \mu(2f(2ax, 2ay, az) - 2a^3 f(2ax, 2ay, 2z))(a^{3\beta} t/2) \\ &\quad \wedge \mu(2f(2ax, 2ay, 2z) - 2a^2 f(2ax, 2y, 2z))(t/2) \\ &\geqslant \psi(2x, 2ay, 2z, 0)(a^{3\beta} t/2) \wedge \varphi(2ax, 2y, 0, 2z)(t/2)\end{aligned}$$

and hence

$$\begin{aligned}&\mu(2a^{-5}f(2ax,2ay,2az)-2f(2ax,2y,2z))(t)\\&=\mu(2a^{-3}f(2ax,2ay,2az)-2a^2f(2ax,2y,2z))(a^{2\beta}t)\\&\geqslant\psi(2x,2ay,2z,0)(a^{5\beta}t/2^{\beta+1})\wedge\varphi(2ax,2y,0,2z)(a^{2\beta}t/2^{\beta+1})\end{aligned}$$

for all $x,y,z\in E$. By (15) and the last inequality, we get

$$\begin{aligned}&\mu(a^{-5}f(2ax,2ay,2az)-af(2x,2y,2z))(t)\\&\geqslant\mu(2a^{-5}f(2ax,2ay,2az)-2f(2ax,2y,2z))(t/2)\\&\qquad\wedge\mu(2f(2ax,2y,2z)-2af(2x,2y,2z))(t/2)\\&\geqslant\psi(2x,2ay,2z,0)(a^{5\beta}t/2^{\beta+2})\wedge\varphi(2ax,2y,0,2z)(a^{2\beta}t/2^{\beta+2})\\&\qquad\wedge\phi(2x,0,2y,2z)(t/2)\end{aligned}$$

for all $x,y,z\in E$. Therefore, we have

$$\begin{aligned}&\mu(a^{-6}f(2ax,2ay,2az)-f(2x,2y,2z))(t)\\&=\mu(a^{-5}f(2ax,2ay,2az)-af(2x,2y,2z))(a^{\beta}t)\\&\geqslant\psi(2x,2ay,2z,0)(a^{6\beta}t/2^{\beta+2})\wedge\varphi(2ax,2y,0,2z)(a^{3\beta}t/2^{\beta+2})\\&\qquad\wedge\phi(2x,0,2y,2z)(a^{\beta}t/2).\end{aligned}$$

Replacing x, y and z by $\frac{x}{2}$, $\frac{y}{2}$ and $\frac{z}{2}$ in the last inequality, respectively, we have

$$\begin{aligned}&\mu\left(\frac{f(ax,ay,az)}{a^6}-f(x,y,z)\right)(t)\\&\geqslant\psi(x,ay,z,0)(a^{6\beta}t/2^{\beta+2})\wedge\varphi(ax,y,0,z)(a^{3\beta}t/2^{\beta+2})\wedge\phi(x,0,y,z)(a^{\beta}t/2)\end{aligned}\tag{18}$$

for all $x,y,z\in E$. Replacing x,y,z by $a^{-1}x,a^{-1}y,a^{-1}z$ in (18), we get

$$\begin{aligned}&\mu\left(\frac{f(a^{-1}x,a^{-1}y,a^{-1}z)}{a^{-6}}-f(x,y,z)\right)(t)\\&\geqslant\psi(a^{-1}x,y,a^{-1}z,0)(a^{12\beta}t/2^{\beta+2})\wedge\varphi(x,a^{-1}y,0,a^{-1}z)(a^{9\beta}t/2^{\beta+2})\\&\quad\wedge\phi(a^{-1}x,0,a^{-1}y,a^{-1}z)(a^{7\beta}t/2)\end{aligned}$$

and so

$$\mu\left(\frac{f(a^sx,a^sy,a^sz)}{a^{6s}}-f(x,y,z)\right)(t)\geqslant\Phi(x,y,z)(t).\tag{19}$$

Now, we consider the set

$$\mathscr{M}=\{h:E\times E\times E\to X:h(x,0,z)=0\text{ for all }x,z\in E\}$$

and introduce the modular ρ on $\mathscr{M}$ as follows:

$$\rho(h)=\inf\{c>0:\mu(h(x,y,z))(ct)\geqslant\Phi(x,y,z)(t)\}.$$

It is clear that ρ is even and $\rho(0)=0$. If $\rho(h)=0$, then, for each $c>0$,

$$\mu(h(x,y,z))(ct) \geqslant \Phi(x,y,z)(t)$$

for all $t>1$ and $x,y\in E$. Now, if $\varepsilon = ct$ is fixed and $t\to+\infty$, then $\mu(h(x,y,z))(\varepsilon)=1$, which implies that $h=0$. It is sufficient to show that ρ satisfies the following condition:

$$\rho(\alpha g+\beta h) \leqslant \rho(g)+\rho(h)$$

if $\alpha+\beta=1$ for all $\alpha,\beta\geqslant 0$. Let $\varepsilon>0$ be given. Then there exist $c_1>0$ and $c_2>0$ such that

$$c_1 \leqslant \rho(g)+\varepsilon, \quad \mu(g(x,y,z))(c_1t) \geqslant \Phi(x,y,z)(t)$$

and

$$c_2 \leqslant \rho(h)+\varepsilon, \quad \mu(h(x,y,z))(c_2t) \geqslant \Phi(x,y,z)(t).$$

If $\alpha+\beta=1$ for all $\alpha,\beta\geqslant 0$, then we get

$$\begin{aligned}\mu(\alpha g(x,y,z)+\beta h(x,y,z))(c_1t+c_2t) &\geqslant \mu(g(x,y,z))(c_1t)\wedge\mu(h(x,y,z))(c_2t)\\ &\geqslant \Phi(x,y,z)(t)\end{aligned}$$

and

$$\rho(\alpha g+\beta h) \leqslant c_1+c_2 \leqslant \rho(g)+\rho(h)+2\varepsilon$$

and so

$$\rho(\alpha g+\beta h) \leqslant \rho(g)+\rho(h).$$

Now, we show that ρ satisfies the Δ_2–condition with $\kappa=2^\beta$. For any $\varepsilon>0$, there exists $c>0$ such that

$$c \leqslant \rho(h)+\varepsilon, \quad \mu(h(x,y,z))(ct) \leqslant \Phi(x,y,z)(t).$$

Since (X,μ) is a β-homogeneous $\mathscr{P}\mathscr{M}$-space, we have

$$\mu(2h(x,y,z))(2^\beta ct) = \mu(h(x,y,z))(ct) \geqslant \Phi(x,y,z)(t),$$

whence $\rho(2h)\leqslant 2^\beta c \leqslant 2^\beta\rho(h)+2^\beta\varepsilon$ and so $\rho(2h)\leqslant 2^\beta\rho(h)$. Thus ρ satisfies the Δ_2–condition with $\kappa=2^\beta$.

Moreover, ρ satisfies the Fatou property (in the modular sense). Indeed, if the sequence $\{h_n\}$ in $\mathscr{M}$ is ρ-convergent to h, then we can easily see that $h(x,y,z)$ is μ-convergent to $h(x,y,z)$ for any $x,y\in E$. Let $\rho := \liminf_{n\to\infty}\rho(h_n)<\infty$ and $\rho(h)>\rho$. Then we have

$$\mu(h(x,y,z))(\rho t) < \Phi(x,y,z)(t)$$

for all $t>0$. Since μ satisfies the Fatou property (in the probabilistic modular sense), we have

$$\limsup_{n\to\infty}\mu(h_n(x,y,z))(\rho t) \leqslant \mu(h(x,y,z))(\rho t) < \Phi(x,y,z)(t).$$

By the last inequality, we know that there exists a positive integer $n_0\in\mathbb{N}$ such that

$$\mu(h_n(x,y,z))(\rho t) < \Phi(x,y,z)(t)$$

and so $\rho(h_n) > \rho$ for all $n \geqslant n_0$. Thus $\liminf \rho(h_n) > \rho$, which is a contradiction. Therefore, ρ satisfies the Fatou property.

If $\delta > 0$ and $\lambda \in (0,1)$ are given, it follows from $\Phi(x,y,z) \in \Delta$ that there exists $t_0 > 0$ such that $\Phi(x,y,z)(t_0) > 1-\lambda$. Let $\{h_n\}$ be a ρ–Cauchy sequence in $\mathscr{M}_\rho$ and let $\varepsilon < \frac{\delta}{t_0}$ be given. Then there exists a positive integer $n_0 \in \mathbb{N}$ such that $\rho(h_n - h_m) \leqslant \varepsilon$ for all $n,m \geqslant n_0$. Now, by considering the definition of the modular ρ, we see that

$$\begin{aligned} \mu(h_n(x,y,z) - h_m(x,y,z))(\delta) &\geqslant \mu(h_n(x,y,z) - h_m(x,y,z))(\varepsilon t_0) \\ &\geqslant \Phi(x,y,z)(t_0) \\ &> 1-\lambda \end{aligned} \tag{20}$$

for all $x,y,z \in E$ and $n,m \geqslant n_0$. If x,y and z are arbitrary given points of E, then (20) implies that $\{h_n(x,y,z)\}$ is a μ–Cauchy sequence in (X,μ). Since (X,μ) is μ–complete, it follows that $\{h_n(x,y,z)\}$ is μ–convergent in (X,μ) for all $x,y,z \in E$. Hence we can define a function $h : E \times E \to (X,\mu)$ by

$$h(x,y,z) = \lim_{n\to\infty} h_n(x,y,z)$$

for any x,y and $z \in E$. On the other hand, μ has the Fatou property. Then we have

$$\rho(h_n - h) \leqslant \varepsilon$$

for all $n \geqslant n_0$. Thus $\{h_n\}$ is a ρ–convergent sequence in $\mathscr{M}_\rho$. Therefore, $\mathscr{M}_\rho$ is ρ–complete.

Now, we consider the function $\mathscr{T} : \mathscr{M}_\rho \to \mathscr{M}_\rho$ defined by

$$\mathscr{T}h(x,y,z) := a^{-6s}h(a^s x, a^s y, a^s z)$$

for all $h \in \mathscr{M}_\rho$. Let $g,h \in \mathscr{M}_\rho$ and $c \in [0,\infty]$ be an arbitrary constant with $\rho(g-h) \leqslant c$. From the definition of ρ, we have

$$\mu(g(x,y,z) - h(x,y,z))(ct) \geqslant \Phi(x,y,z)$$

for all $x,y,z \in E$. By the assumption and the last inequality, we get

$$\begin{aligned} &\mu\left(\mathscr{T}g(x,y,z) - \mathscr{T}h(x,y,z)\right)(Lct) \\ &\quad = \mu\left(a^{-6s}g(a^s x, a^s y, a^s z) - a^{-6s}h(a^s x, a^s y, a^s z)\right)(Lct) \\ &\quad = \mu(g(a^s x, a^s y, a^s z) - h(a^s x, a^s y, a^s z))(a^{6\beta s}Lct) \\ &\quad \geqslant \Phi(a^s x, a^s y, a^s z)(a^{6\beta s}Lt) \\ &\quad \geqslant \Phi(x,y,z)(t) \end{aligned}$$

for all $x,y,z \in E$ and so $\rho(\mathscr{T}g - \mathscr{T}h) \leqslant L\rho(g-h)$ for all $g,h \in \mathscr{M}_\rho$, that is, $\mathscr{T}$ is a ρ–strict contraction.

Now, we show that the ρ–strict mapping $\mathscr{T}$ satisfies the conditions of Theorem 1. Observe that

$$\mu(a^{-6s}f(a^{2s}x, a^{2s}y, a^{2s}z) - f(a^s x, a^s y, a^s z))(t) \geqslant \Phi(a^s x, a^s y, a^s z)(t)$$

and so

$$\begin{aligned}
\mu(a^{-2(6)s}f(a^{2s}x,a^{2s}y,a^{2s}z)-a^{-6s}f(a^{s}x,a^{s}y,a^{s}z))(Lt)
&= \mu(a^{-6s}f(a^{2s}x,a^{2s}y,a^{2s}z)-f(a^{s}x,a^{s}y,a^{s}z))(a^{6\beta s}Lt)\\
&\geqslant \Phi(a^{s}x,a^{s}y,a^{s}z)(a^{6\beta s}Lt)\\
&\geqslant \Phi(x,y,z)(t).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\mu\left(\frac{f(a^{2s}x,a^{2s}y,a^{2s}z)}{a^{2(6)s}}-f(x,y,z)\right)\left(2^{\beta}(Lt+t)\right)\\
&\geqslant \mu\left(\frac{f(a^{2s}x,a^{2s}y,a^{2s}z)}{a^{2(6)s}}-\frac{f(a^{s}x,a^{s}y,a^{s}z)}{a^{6s}}\right)(Lt)\wedge\mu\left(\frac{f(a^{s}x,a^{s}y,a^{s}z)}{a^{6s}}-f(x,y,z)\right)(t)\\
&\geqslant \Phi(x,y,z)(t) \qquad (21)
\end{aligned}$$

for all $x,y,z\in E$. By replacing x, y and z by $a^{s}x$, $a^{s}y$ and $a^{s}z$ in (21), respectively, we get

$$\begin{aligned}
\mu(a^{-2(6)s}f(a^{3s}x,a^{3s}y,a^{3s}z)-f(a^{s}x,a^{s}y,,a^{s}z))&\left(a^{6\beta s}2^{\beta}(L^{2}t+Lt)\right)\\
&\geqslant \Phi(a^{s}x,a^{s}y,a^{s}z)\left(a^{6\beta s}Lt\right)\\
&\geqslant \Phi(x,y,z)(t)
\end{aligned}$$

and so

$$\mu(a^{-3(6)s}f(a^{3s}x,a^{3s}y,a^{3s}z)-a^{-6s}f(a^{s}x,a^{s}y,a^{s}z))\left(2^{\beta}(L^{2}t+Lt)\right)\geqslant\Phi(x,y,z)(t).$$

Therefore, we have

$$\begin{aligned}
\mu&\left(\frac{f(a^{3s}x,a^{3s}y,a^{3s}z)}{a^{3(6)s}}-f(x,y,z)\right)\left(2^{\beta}\left\{2^{\beta}(L^{2}t+Lt)+t\right\}\right)\\
&\geqslant \mu\left(\frac{f(a^{3s}x,a^{3s}y,,a^{3s}z)}{a^{3(6)s}}-\frac{f(a^{s}x,a^{s}y,a^{s}z)}{a^{6s}}\right)\left(2^{\beta}(L^{2}t+Lt)\right)\\
&\qquad\wedge\mu\left(\frac{f(a^{s}x,a^{s}y,a^{s}z)}{a^{6s}}-f(x,y,z)\right)(t)\\
&\geqslant \Phi(x,y,z)(t)
\end{aligned}$$

for all $x,y,z\in E$. By induction, we can easily see that

$$\mu\left(\frac{f(a^{sn}x,a^{sn}y,a^{sn}z)}{a^{6sn}}-f(x,y,z)\right)\left(\left\{(2^{\beta}L)^{n-1}+2^{\beta}\sum_{i=1}^{n-1}(2^{\beta}L)^{i-1}\right\}t\right)\geqslant\Phi(x,y,z)(t)$$

for all $x,y,z\in E$ and so

$$\rho(\mathscr{T}^{n}f-f)\leqslant(2^{\beta}L)^{n-1}+2^{\beta}\sum_{i=1}^{n-1}(2^{\beta}L)^{i-1}\leqslant 2^{\beta}\sum_{i=1}^{n}(2^{\beta}L)^{i-1}\leqslant\frac{2^{\beta}}{1-2^{\beta}L}. \qquad (22)$$

Next, we assert that $\delta_\rho(f) = \sup\{\rho(\mathscr{T}^n(f) - \mathscr{T}^m(f)) : n,m \in \mathbb{N})\} < \infty$. Since ρ satisfies the Δ_2–condition with $\kappa = 2^\beta$, it follows from the inequality (22) that

$$\begin{aligned}\rho(\mathscr{T}^n f - \mathscr{T}^m f) &\leqslant \frac{1}{2}\rho(2\mathscr{T}^n f - 2f) + \frac{1}{2}\rho(2\mathscr{T}^m f - f)\\ &\leqslant \frac{\kappa}{2}\rho(\mathscr{T}^n f - f) + \frac{\kappa}{2}\rho(\mathscr{T}^m f - f)\\ &\leqslant \frac{2^{2\beta}}{1-2^\beta L}\end{aligned} \tag{23}$$

for all $n,m \in \mathbb{N}$. By the definition of $\delta_\rho(f)$, we have $\delta_\rho(f) < \infty$. Thus Theorem 1 shows that $\{\mathscr{T}^n(f)\}$ is ρ–convergent to a point $j \in \mathscr{M}_\rho$. Since ρ has the Fatou property, the inequality (22) gives $\rho(\mathscr{T}j - f) < \infty$.

If we replace m by $n+1$ in the inequality (23), then we obtain

$$\rho\left(\mathscr{T}^{n+1}f - \mathscr{T}^n f\right) \leqslant \frac{2^{2\beta}}{1-2^\beta L}.$$

Therefore, we have $\rho(\mathscr{T}(j) - j) \leqslant (2^{2\beta}/1 - 2^\beta L) < \infty$. Therefore, it follows from Theorem 1 that ρ–limit of $\{\mathscr{T}^n(f)\}$, $j \in \mathscr{M}_\rho$, is a fixed point of the mapping $\mathscr{T}$.

If we replace x_1, x_2, y and z by $a^n x_1$, $a^n x_2$, $a^n y$ and $a^n z$ in the inequality (10), respectively, then we obtain

$$\begin{aligned}&\mu\Big(\frac{f(a^{sn}(ax_1+bx_2),a^{sn}y,a^{sn}z)}{a^{6sn}} + \frac{f(a^{sn}(ax_1-bx_2),a^{sn}y,a^{sn}z)}{a^{6sn}}\\ &\quad -2a\frac{f(a^{sn}x_1,a^{sn}y,a^{sn}z)}{a^{6sn}}\Big)(t)\\ &\geqslant \mu\big(f(a^{sn}(ax_1+bx_2),a^{sn}y,a^{sn}z) + f(a^{sn}(ax_1-bx_2),a^{sn}y,a^{sn}z)\\ &\quad -2af(a^{sn}x_1,a^{sn}y,a^{sn}z)\big)(a^{6\beta sn}t)\\ &\geqslant \phi(a^{sn}x_1,a^{sn}x_2,a^{sn}y,,a^{sn}z)(a^{6\beta sn}t).\end{aligned} \tag{24}$$

Similarly, by replacing x, y_1, y_2 and z by $a^{sn}x$, $a^{sn}y_1$, $a^{sn}y_2$ and $a^{sn}z$ in the inequality (11), respectively, we have

$$\begin{aligned}&\mu\Big(\frac{f(a^{sn}x,a^{sn}(ay_1+by_2),a^{sn}z)}{a^{6sn}} + \frac{f(a^{sn}x,a^{sn}(ay_1-by_2),a^{sn}z)}{a^{6sn}}\\ &\quad -2a^2\frac{f(a^{sn}x,a^{sn}y_1,a^{sn}z)}{a^{6sn}} - 2b^2\frac{f(a^{sn}x,a^{sn}y_2,a^{sn}z)}{a^{6sn}}\Big)\\ &\geqslant \varphi(a^{sn}x,a^{sn}y_1,a^{sn}y_2,a^{sn}z)(a^{6\beta sn}t)\end{aligned} \tag{25}$$

and, also by replacing x, y, z_1 and z_2 by $a^{sn}x$, $a^{sn}y$, $a^{sn}z_1$ and $a^{sn}z_2$ in the inequality (12), respectively, we get

$$\begin{aligned}
&\mu\Big(\frac{f(a^{sn}x,a^{sn}y,a^{sn}(az_1+bz_2))}{a^{6sn}}+\frac{f(a^{sn}x,a^{sn}y,a^{sn}(az_1-bz_2))}{a^{6sn}} \\
&\quad -ab^2\frac{f(a^{sn}x,a^{sn}y,a^{sn}(z_1+z_2))}{a^{6sn}}+\frac{f(a^{sn}x,a^{sn}y,a^{sn}(z_1-z_2))}{a^{6sn}} \\
&\quad -2a(a^2-b^2)\frac{f(a^{sn}x,a^{sn}y,a^{sn}z_1)}{a^{6sn}}\Big) \\
&\geqslant \psi(a^{sn}x,a^{sn}y,a^{sn}z_1,a^{sn}z_2)(a^{6\beta sn}t)
\end{aligned} \tag{26}$$

for all $x,x_i,y,y_i,z,z_i \in E$, $i=1,2$. Taking $n\to\infty$ in the inequalities (24), (25) and (26), we deduce that j satisfies the system (9), that is, j is sextic. It follows from the inequality (22) that

$$\rho(j-f)\leqslant \frac{2^\beta}{1-2^\beta L}.$$

Hence (14) holds.

If j^* is another fixed point of $\mathscr{T}$, then we have

$$\begin{aligned}
\rho(j-j^*) &\leqslant \frac{1}{2}\rho(2\mathscr{T}(j)-2f)+\frac{1}{2}\rho(2\mathscr{T}(j^*)-2f) \\
&\leqslant \frac{\kappa}{2}\rho(\mathscr{T}(j)-f)+\frac{\kappa}{2}\rho(\mathscr{T}(j^*)-f) \\
&\leqslant \frac{2^{2\beta}}{1-2^\beta L} \\
&<\infty.
\end{aligned}$$

Since $\mathscr{T}$ is ρ–strict contraction, we get

$$\rho(j-j^*)=\rho(\mathscr{T}(j)-\mathscr{T}(j^*))\leqslant L\rho(j-j^*),$$

which implies that $\rho(j-j^*)=0$ or $j=j^*$ since $\rho(j-j^*)<\infty$, which proves the uniqueness of j. This completes the proof. □

Before presenting an example, we firstly introduce some useful concepts:

Fix a real number β with $0<\beta\leqslant 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Let X be a linear space over $\mathbb{K}$. A real-valued function $\|\cdot\|_\beta$ is called a β*-norm* on X if it satisfies the following conditions:

($\beta N1$) $\|x\|_\beta=0$ if and only if $x=0$;

($\beta N2$) $\|\lambda x\|_\beta=|\lambda|^\beta\|x\|_\beta$ for all $\lambda\in\mathbb{K}$ and $x\in X$;

($\beta N3$) $\|x+y\|_\beta\leqslant\|x\|_\beta+\|y\|_\beta$ for all $x,y\in X$.

The pair $(X,\|\cdot\|_\beta)$ is called a β*-normed space* (see [3]). A β*-Banach space* is a complete β-normed space.

EXAMPLE 1. Let E be a linear space and X be a β-Banach space. Define

$$\mu(x)(t)=\begin{cases}0, & t\leqslant 0,\\ \frac{t}{t+\|x\|_\beta}, & t>0\end{cases}$$

for all $x \in X$ and $t \in \mathbb{R}$. Then (X,μ) is a μ-complete β-homogeneous probabilistic modular space. Moreover, let f, ϕ, φ, ψ, Φ and L be same as in the previous theorem. Then there exists a unique quintic function $j : E \times E \times E \to (X,\mu)$ satisfying the system (9) and

$$\frac{2^\beta t}{2^\beta t + (1-2^\beta L)\|j(x,y,z) - f(x,y,z)\|_\beta} \geqslant \Phi(x,y,z)(t)$$

for all $x,y,z \in E$.

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