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ASYMPTOTIC EXPANSIONS OF THE MULTIPLE QUOTIENTS OF GAMMA FUNCTIONS WITH APPLICATIONS

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Abstract. Asymptotic expansions of the multiple quotients of two gamma functions are obtained and analyzed. We apply these results to the hypergeometric function and central multinomial coefficient which leads to the new approximation formulas.

1. Introduction

The central binomial coefficient has the well known asymptotic approximation (see e.g. [9, p. 35]):

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{n\pi}}\left[1-\frac{1}{8n}+\frac{1}{128n^2}+\frac{5}{1024n^3}-\frac{21}{32768n^4}+\dots\right], \quad (n\to\infty). \tag{1.1}$$

This expansion is directly connected to the ratio of two gamma functions. Namely, using duplication formula for the gamma function we have

$$\binom{2n}{n} = \frac{\Gamma(2n+1)}{\Gamma(n+1)^2} = \frac{2}{n}\cdot\frac{2^{2n-1}}{\sqrt{\pi}}\cdot\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)}. \tag{1.2}$$

Therefore, approximation formulas for the central binomial coefficient are easily derived from asymptotic expansions for the quotient of two gamma functions.

In [3, 5], authors obtained the following general asymptotic expansion of the quotient of two gamma functions

$$\frac{\Gamma(x+t)}{\Gamma(x+s)} \sim x^{t-s}\left(\sum_{n=0}^{\infty} P_n(t,s)x^{-n}\right)^{\frac{1}{m}}, \tag{1.3}$$

where polynomials $P_n(t,s)$ are defined by

$$\begin{aligned} P_0(t,s) &= 1, \\ P_n(t,s) &= \frac{m}{n}\sum_{k=1}^{n}(-1)^{k+1}\frac{B_{k+1}(t)-B_{k+1}(t)}{k+1}P_{n-k}(t,s) \end{aligned} \tag{1.4}$$

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and $B_k(t)$ stands for the Bernoulli polynomials. Putting $t = 1/2$, $s = 0$ and applying (1.2), we have

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{n\pi}} \left(\sum_{k=0}^{\infty} \frac{P_k}{n^k} \right)^{1/m} \tag{1.5}$$

where sequence (P_n) is defined by $P_0 = 1$ and

$$P_n = \frac{m}{n} \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} \frac{[2^{-2k} - 1] B_{2k}}{k} P_{n-2k+1}, \qquad n \geqslant 1,$$

B_{2k} are the Bernoulli numbers. Now, for $m = 2$, we get better approximation formula than (1.1):

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{n\pi}} \left(1 - \frac{1}{4n} + \frac{1}{32n^2} + \frac{1}{128n^3} - \frac{5}{2048n^4} + \cdots \right)^{1/2}. \tag{1.6}$$

For details, see [3]. In a recent paper [6], another even better approximation (for bigger values of n) is obtained:

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{n\pi}} \left(1 - \frac{1}{8n^2} + \frac{5}{384n^4} - \frac{13}{5120n^6} + \cdots \right)^{n}. \tag{1.7}$$

But the most natural form for the central binomial coefficient is obtained with shifted variable $n + 1/4$, see [10]. In that case, applying expansion (1.3) with $t = 3/4$ we get

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{N\pi/2}} \left(2 - \frac{2}{N^2} + \frac{21}{N^4} - \frac{671}{N^6} + \cdots \right) \tag{1.8}$$

where $N = 8n + 2$.

The main object of this paper is to obtain asymptotic expansions for the multiple quotients of gamma functions and to derive new aproximation formulas for general central multinomial coefficients

$$\binom{kn}{n, n, \ldots, n}.$$

We will show two approaches. The first one goes through multiplication formula for the gamma function and the second one is based on the properties of asymptotic series. The first approach also leads to the approximation of some hypergeometric functions.

2. Asymptotic expansions of trinomial coefficient

Similar as in (1.2), using multiplication formula for gamma function (see [1]), we have the following connection of central trinomial coefficient with quotients of gamma functions:

$$\binom{3n}{n, n, n} = \frac{\Gamma(3n+1)}{\Gamma(n+1)^3} = \frac{3^{3n+1/2}}{2\pi n} \cdot \frac{\Gamma(n + \frac{1}{3}) \Gamma(n + \frac{2}{3})}{\Gamma(n) \Gamma(n+1)}. \tag{2.1}$$

This has motivated us for the following result:

THEOREM 2.1. *Let $t-s=v-u$ and let $m>0$. The following approximation formula holds true:*

$$F(x,t,s,u,v) := \frac{\Gamma(x+t)\Gamma(x+u)}{\Gamma(x+s)\Gamma(x+v)} \sim \left(\sum_{n=0}^{\infty} \frac{P_n(t,s,u,v)}{x^n}\right)^{1/m} \tag{2.2}$$

where polynomials (P_n) are defined with $P_0=1$ and

$$P_n = \frac{m}{n}\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k+1}[B_{k+1}(t)-B_{k+1}(s)+B_{k+1}(u)-B_{k+1}(v)]P_{n-k}, \tag{2.3}$$

$B_k(t)$ are Bernoulli polynomials.

Proof. Differentiating the logarithm of (2.2)

$$\log\Gamma(x+t)-\log\Gamma(x+s)+\log\Gamma(x+u)-\log\Gamma(x+v) \sim \frac{1}{m}\log\left(\sum_{n=0}^{\infty}\frac{P_n}{x^n}\right)$$

we get

$$\psi(x+t)-\psi(x+s)+\psi(x+u)-\psi(x+v) \sim \frac{1}{m}\left(\sum_{n=0}^{\infty}\frac{P_n}{x^n}\right)^{-1}\left(\sum_{n=1}^{\infty}\frac{-nP_n}{x^{n+1}}\right).$$

Applying known asymptotic expansions of the psi function (see [1]):

$$\psi(x+t) \sim \log x - \sum_{k=1}^{\infty}\frac{(-1)^{k+1}B_k(t)}{kx^k}, \tag{2.4}$$

we have

$$\left(\sum_{n=0}^{\infty}\frac{P_n}{x^n}\right)\left(\sum_{k=1}^{\infty}(-1)^k\frac{B_{k+1}(t)-B_{k+1}(s)+B_{k+1}(u)-B_{k+1}(v)}{(k+1)x^{k+1}}\right) \sim -\frac{1}{m}\sum_{n=1}^{\infty}\frac{nP_n}{x^{n+1}}$$

wherefrom it follows:

$$-nP_n = m\sum_{k=1}^{n}(-1)^k\frac{B_{k+1}(t)-B_{k+1}(s)+B_{k+1}(u)-B_{k+1}(v)}{k+1}P_{n-k}.$$

Theorem is proved. □

Now we can present asymptotic expansion for the central trinomial coefficent. Using (2.1) and putting $t=\frac{1}{3}$, $s=0$, $u=\frac{2}{3}$, $v=1$, $m=1$ we get

$$\binom{3n}{n,n,n} \sim \frac{3^{3n+1/2}}{2\pi n}\cdot\left(1-\frac{2}{9n}+\frac{2}{81n^2}+\frac{14}{2187n^3}-\frac{34}{19683n^4}+\dots\right) \tag{2.5}$$

A natural choice for parameter m is $\frac{1}{m} = t - s = v - u$. In that case we have following asymptotic expansion which gives better approximation results:

$$\binom{3n}{n,n,n} \sim \frac{3^{3n+1/2}}{2\pi n} \cdot \left(1 - \frac{2}{3n} + \frac{2}{9n^2} - \frac{2}{81n^3} - \frac{2}{243n^4} + \dots\right)^{1/3} \tag{2.6}$$

If $\frac{1}{m} = t - s = v - u$, the recursive formula (2.3) for polynomials P_n can be written as

$$P_n = \frac{1}{n} \sum_{k=0}^{n} (-1)^{k+1} [\Delta_k(t,s) - \Delta_k(u,v)] P_{n-k},$$

where Δ_k is Bernoulli quotient function

$$\Delta_k(t,s) = \frac{B_{k+1}(t) - B_{k+1}(s)}{(k+1)(t-s)}.$$

In [3], authors have shown that the Bernoulli quotient has much more natural form if variables t and s are changed by new variables α and β:

$$\alpha = \frac{1}{2}(t+s-1), \qquad \beta = \frac{1}{4}[1-(t-s)^2].$$

In this case, we will define

$$\alpha_1 = \frac{1}{2}(t+s-1), \qquad \alpha_2 = \frac{1}{2}(u+v-1), \tag{2.7}$$

and the variable β is same for the both sequences:

$$\beta = \frac{1}{4}[1-(t-s)^2] = \frac{1}{4}[1-(u-v)^2] = \frac{1}{4}[1-\mu^2]. \tag{2.8}$$

Now we can present a much simpler form of the asymptotic expansion (2.2):

$$F(x,t,s,u,v) \sim \left(\sum_{n=0}^{\infty} \frac{Q_n(\alpha_1,\alpha_2,\beta)}{x^n} \right)^{1/m} \tag{2.9}$$

where polynomials Q_n are defined by $Q_0 = 1$ and

$$Q_n = \frac{1}{n} \sum_{k=0}^{n} (-1)^{k+1} [\Delta_k(\alpha_1,\beta) - \Delta_k(\alpha_2,\beta)] Q_{n-k}. \tag{2.10}$$

For properties and efficient fast algorithms for calculating Δ_k see [3].

Finally, if the choice $n + 1/4$ gives a more natural and simpler formula for the binomial coefficient (1.8), one can wonder what choice is natural for the trinomial coefficient. Unfortunately, this type of formula from Theorem 2.1 cannot give us an answer and we will have to wait for the second approach.

The asymptotic expansion (2.2) can be also used to obtain an expansion of hypergeometric function which appears in various fields of mathematics and has many applications. Hypergeometric function is defined by

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}, \tag{2.11}$$

where $(t)_n = t(t+1)\dots(t+n-1)$ is Pochhammer symbol. The most important case $z=1$ of (2.11) can be written as the quotient of gamma functions (well-known Gauss theorem, see [1]):

$$ {}_2F_1(a,b;c;1) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!(c)_n} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{2.12}$$

This is same as

$$ {}_2F_1(u+s,u+t;x+u;1) = \sum_{n=0}^{\infty} \frac{(u+s)_n(u+t)_n}{n!(x+u)_n} = \frac{\Gamma(x+u)\Gamma(x-u-t-s)}{\Gamma(x-s)\Gamma(x-t)}, \tag{2.13}$$

and now we can apply the asymptotic expansion (2.2).

3. Asymptotic expansions of the multiple quotients

As one can see, the approach presented in the previous section doesn't give much hope for generalization. Since asymptotic expansions of gamma functions are known and manipulation with asymptotic series are justified by properties of asymptotic power series (see [8, § 1.6] or [3] for details), it will be useful to write general multinomial coefficient in a way:

$$ \binom{kn}{n,n,\dots,n} = \frac{(kn)!}{(n!)^k} = \frac{\Gamma(kn+1)}{\Gamma^k(n+1)}. \tag{3.1}$$

To derive an asymptotic expansion of multiple quotient in this form, we need two lemmas about manipulation with asymptotic series.

LEMMA 3.1. *Let function $f(x)$ and $g(x)$ have following asymptotic expansions ($a_0 \neq 0$, $b_0 \neq 0$) as $x \to \infty$:*

$$ f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}, \qquad g(x) \sim \sum_{n=0}^{\infty} b_n x^{-n}.$$

Then asymptotic expansion of their quotient $f(x)/g(x)$ reads as

$$ \frac{f(x)}{g(x)} \sim \sum_{n=0}^{\infty} c_n x^{-n}, \tag{3.2}$$

where coefficients c_n are defined by

$$ c_n = \frac{1}{b_0}\left(a_n - \sum_{k=1}^{n} b_k c_{n-k}\right).$$

Proof. We can write

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^{-n} \cdot g(x)$$

$$\sum_{n=0}^{\infty} a_n x^{-n} \sim \sum_{n=0}^{\infty} c_n x^{-n} \cdot \sum_{n=0}^{\infty} b_n x^{-n}$$

Therefore it follows

$$a_n = \sum_{n=0}^{\infty} b_n c_{n-k} = b_0 c_n + \sum_{n=1}^{\infty} b_n c_{n-k}$$

and lemma is proved. □

LEMMA 3.2. *Let function $f(x)$ has following asymptotic expansion ($a_0 \neq 0$)*

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}.$$

Then asymptotic expansion of function $[f(x)]^p$, $p > 0$, reads as

$$[f(x)]^p \sim \sum_{n=0}^{\infty} c_n x^{-n}, \tag{3.3}$$

where coefficients c_n are defined by $c_0 = a_0^p$ and

$$c_n = \frac{1}{na_0} \sum_{k=1}^{n} [(p+1)k - n] a_k c_{n-k}, \quad n \geqslant 1.$$

Proof. Differentianting we get

$$p[f(x)]^{p-1} f'(x) \sim \left(\sum_{n=0}^{\infty} c_n x^{-n} \right)'$$

$$p \left(\sum_{n=0}^{\infty} c_n x^{-n} \right) \left(-\sum_{n=1}^{\infty} n a_n x^{-n-1} \right) \sim \left(-\sum_{n=1}^{\infty} n c_n x^{-n-1} \right) \left(\sum_{n=0}^{\infty} c_n x^{-n} \right).$$

Equating coefficients on both sides we have

$$\sum_{k=1}^{n} p k a_k c_{n-k} = n a_0 c_n + \sum_{k=1}^{n} (n-k) a_k c_{n-k},$$

and lemma follows. □

In [4], authors have obtained following asymptotic expansion of the gamma function

$$\Gamma(x+t) \sim \sqrt{2\pi} \left(\frac{x}{e} \right)^x x^{t-\frac{1}{2}} \left(\sum_{n=0}^{\infty} P_n(t) x^{-n} \right)^{1/m} \tag{3.4}$$

where polynomials $P_n(t)$ are defined by:

$$P_0(t) = 1$$
$$P_n(t) = \frac{m}{n}\sum_{k=1}^{n}\frac{(-1)^{k+1}B_{k+1}(t)}{k+1}P_{n-k}(t), \qquad n \geqslant 1. \tag{3.5}$$

This expansion is a generalization of many known approximation formulas of Stirlings' type for the factorial function. For example, for $t = 1$ and $m = 1$, we obtain the well-known Laplace formula:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{12n}+\frac{1}{288n^2}-\frac{139}{51840n^3}-\frac{571}{2488320n^4}+\dots\right) \tag{3.6}$$

In [4] it is shown that for bigger m, the numerical precision of the approximation formula is better.

Using (3.4) and applying two lemmas we get the following result:

THEOREM 3.3. *Central multinomial coefficient has the following asymptotic expansion*

$$\binom{kn}{n,n,\dots,n} \sim \left(\sqrt{2\pi n}\right)^{1-k} k^{kn+\frac{1}{2}}\left(\sum_{j=0}^{\infty}c_j n^{-j}\right)^{1/m}, \tag{3.7}$$

where

$$c_0 = 1, \quad c_j = a_j k^{-j} - \sum_{i=1}^{j} b_i c_{j-i}, \quad j \geqslant 1, \tag{3.8}$$

$$b_0 = 1, \quad b_j = \frac{1}{j}\sum_{i=1}^{j}[(k+1)i-j]a_i b_{j-i}, \quad j \geqslant 1, \tag{3.9}$$

$$a_0 = 1, \quad a_j = \frac{m}{j}\sum_{i=1}^{\lfloor (j+1)/2\rfloor}\frac{B_{2i}}{2i}a_{j-2i+1}, \quad j \geqslant 1, \tag{3.10}$$

and B_k are Bernoulli numbers.

Proof. Applying (3.4) we have

$$\Gamma(n+1) \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(\sum_{j=0}^{\infty}a_j n^{-j}\right)^{1/m},$$

$$\Gamma^k(n+1) \sim \left(\sqrt{2\pi n}\right)^k\left(\frac{n}{e}\right)^{kn}\left(\sum_{j=0}^{\infty}b_j n^{-j}\right)^{1/m},$$

$$\Gamma(kn+1) \sim \sqrt{2\pi kn}\left(\frac{kn}{e}\right)^{kn}\left(\sum_{j=0}^{\infty}a_j (kn)^{-j}\right)^{1/m}.$$

a_j is obtained from (3.4) with $t=1$ and b_j is obtained from a_j using Lemma 3.2 with $p=k$. Therefore (3.10) and (3.9) are proved. Rest of the proof follows from (3.1) applying Lemma 3.1. □

Now we can find asymptotic expansion of any central multinomial coefficient. Putting $k=2$ and $m=1$, $m=2$, we get known expansions for binomial coefficient (1.1) and (1.6), and for $k=3$ and $m=1$, $m=3$ we get expansions (2.5) and (2.6). For example, we also present formula for $k=4$ with $m=1$ and $m=4$:

$$\binom{4n}{n,n,n,n} \sim \frac{4^{4n}}{\pi n\sqrt{2\pi n}} \cdot \left(1-\frac{5}{16n}+\frac{25}{512n^2}+\frac{49}{8192n^3}-\frac{1605}{524288n^4}+\dots\right) \tag{3.11}$$

and

$$\binom{4n}{n,n,n,n} \sim \frac{4^{4n}}{\pi n\sqrt{2\pi n}} \cdot \left(1-\frac{5}{4n}+\frac{25}{32n^2}-\frac{9}{32n^3}-\frac{95}{2048n^4}+\dots\right)^{1/4}. \tag{3.12}$$

In [4] authors derived another asymptotic expansion of the gamma function:

$$\Gamma(x+t) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x x^{t-\frac{1}{2}} \left(\sum_{n=0}^{\infty} P_n(t)x^{-n}\right)^x \tag{3.13}$$

where polynomials $P_n(t)$ are defined by $P_0=1$, $P_1=0$ and

$$P_n(t) = \frac{1}{n}\sum_{k=2}^{n} \frac{(-1)^k B_k(t)}{k-1} P_{n-k}(t), \qquad n \geqslant 2. \tag{3.14}$$

From this expansion we can obtain Nemes formula for the factorial function

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1+\frac{1}{12n^2}+\frac{1}{1440n^4}-\frac{239}{362880n^6}+\dots\right)^n \tag{3.15}$$

It is shown that this formula gives better numerical precision than previous ones (for details see [10]).

Applying (3.13) we get a new result:

THEOREM 3.4. *Central multinomial coefficient has the following asymptotic expansion*

$$\binom{kn}{n,n,\dots,n} \sim \left(\sqrt{2\pi n}\right)^{1-k} k^{kn+\frac{1}{2}} \left(\sum_{j=0}^{\infty} c_{2j} n^{-2j}\right)^n, \tag{3.16}$$

where

$$c_0 = 1, \quad c_{2j} = \frac{1}{2j}\sum_{i=1}^{j} [2i(k+1)-2j] b_{2i} c_{2j-2i}, \quad j \geqslant 1, \tag{3.17}$$

$$b_0 = 1, \quad b_{2j} = a_{2j}k^{-2j} - \sum_{i=1}^{j} a_{2i} b_{2j-2i}, \quad j \geqslant 1, \tag{3.18}$$

$$a_0 = 1, \quad a_{2j} = \frac{1}{2j}\sum_{i=1}^{j} \frac{B_{2i}}{2i-1} a_{2j-2i}, \quad j \geqslant 1, \tag{3.19}$$

and B_k are Bernoulli numbers.

Proof. Similar as in previous theorem, applying (3.13) we get

$$\Gamma(kn+1) \sim \sqrt{2\pi kn}\left(\frac{kn}{e}\right)^{kn}\left(\sum_{j=0}^{\infty} a_j (kn)^{-j}\right)^{kn},$$

$$\Gamma^k(n+1) \sim \left(\sqrt{2\pi n}\right)^k \left(\frac{n}{e}\right)^{kn}\left(\sum_{j=0}^{\infty} a_j n^{-j}\right)^{kn},$$

$$\frac{\Gamma(nk+1)}{\Gamma^k(n+1)} \sim \left(\sqrt{2\pi n}\right)^{1-k} k^{kn+\frac{1}{2}}\left(\sum_{j=0}^{\infty} b_j n^{-j}\right)^{kn},$$

where b_j is obtained using Lemma 3.1. Finally, (3.17) follows from Lemma 3.2 with $p=k$. □

For example, $k=2$ gives a known expansion for binomial coefficient (1.7), and for $k=3$ and $k=4$ we have the following new asymptotic expansions:

$$\binom{3n}{n,n,n} \sim \frac{3^{3n+1/2}}{2\pi n}\left(1-\frac{2}{9n^2}+\frac{8}{243n^4}-\frac{22}{3645n^6}+\ldots\right)^n \tag{3.20}$$

$$\binom{4n}{n,n,n,n} \sim \frac{4^{4n}}{\pi n\sqrt{2\pi n}}\left(1-\frac{5}{16n^2}+\frac{23}{384n^4}-\frac{3}{256n^6}+\ldots\right)^n \tag{3.21}$$

which are better approximation results than expansions (2.6) and (3.12). These formulas are new and using (3.16) similar expansions can be derived for other values of k.

4. Asymptotic expansions with shifted variable

As we mentioned before, the most natural form of expansion for central binomial coefficient is the special case of (1.8) with variable $N=8n+2$ and the first approach didn't lead us to the general shifted formula for the central multinomial coefficient.

It is natural to assume that $n+1/k^2$ would be the best choice for shifted variable in a general case. This is indeed so, and applying the second approach with proved lemmas we present the following recursive formulas.

THEOREM 4.1. *Central multinomial coefficient has the following asymptotic expansion through shifted variable $N=k^2n+1$:*

$$\binom{kn}{n,n,\ldots,n} \sim \left(\sqrt{2\pi N}\right)^{1-k} k^{k-\frac{1}{2}+N/k}\left(\sum_{j=0}^{\infty}\frac{c_j k^{2j}}{N^j}\right)^{1/m}, \tag{4.1}$$

where

$$c_0 = 1, \quad c_j = d_j k^{-j} - \sum_{i=1}^{j} b_i c_{j-i}, \quad j \geqslant 1, \tag{4.2}$$

$$b_0 = 1, \quad b_j = \frac{1}{j}\sum_{i=1}^{j}[(k+1)i - j]a_i b_{j-i}, \quad j \geqslant 1, \tag{4.3}$$

$$a_0 = 1, \quad a_j = \frac{m}{j}\sum_{i=1}^{j}\frac{(-1)^{i+1}B_{i+1}(1-1/k^2)}{i+1}a_{j-i}, \quad j \geqslant 1, \tag{4.4}$$

$$d_0 = 1, \quad d_j = \frac{m}{j}\sum_{i=1}^{j}\frac{(-1)^{i+1}B_{i+1}(1-1/k)}{i+1}d_{j-i}, \quad j \geqslant 1, \tag{4.5}$$

$$\tag{4.6}$$

and B_k are Bernoulli polynomials.

Proof. Applying (3.4) with shifted variable $n+1/k^2$ we get

$$\Gamma(kn+1) = \Gamma(k(n+1/k^2)+1-1/k) \sim \sqrt{2\pi k(n+1/k^2)}\left(\frac{k(n+1/k^2)}{e}\right)^{k(n+1/k^2)}\left(\sum_{j=0}^{\infty}\frac{d_j k^{-j}}{(n+1/k^2)^j}\right)^{1/m},$$

$$\Gamma^k(n+1) = \Gamma^k(n+1/k^2+1-1/k^2) \sim \left(\sqrt{2\pi(n+1/k^2)}\right)^k\left(\frac{n+1/k^2}{e}\right)^{k(n+1/k^2)}\left(\sum_{j=0}^{\infty}\frac{b_j}{(n+1/k^2)^j}\right)^{1/m},$$

$$\frac{\Gamma(nk+1)}{\Gamma^k(n+1)} \sim \left(\sqrt{2\pi(n+1/k^2)}\right)^{1-k} k^{k(n+1/k^2)+\frac{1}{2}}\left(\sum_{j=0}^{\infty}\frac{c_j}{(n+1/k^2)^j}\right)^{1/m}.$$

Therefore, (4.5) is obtained from (3.4) with $t = 1-1/k$, b_j is obtained putting $t = 1-1/k^2$ and applying Lemma 3.2 with $p = k$ and (3.17) follows from Lemma 3.1. Final form of expansion follows by modifying variable to $N = k^2 n + 1$. □

Of course, $m = 1$ and $k = 2$ leads to the known expansion (1.8) and for example, $k = 3$ and $k = 4$ lead to the following shifted expansions for central coefficient:

$$\binom{3n}{n,n,n} \sim \frac{3^{N/3+5/2}}{2\pi N}\cdot\left(1 - \frac{1}{N} - \frac{1}{N^2} + \frac{17}{3N^3} + \frac{37}{3N^4} + \dots\right) \tag{4.7}$$

and

$$\binom{4n}{n,n,n,n} \sim \frac{4^{N/4+3}}{\pi N\sqrt{2\pi N}}\cdot\left(1 - \frac{7}{2N} + \frac{15}{8N^2} + \frac{777}{16N^3} - \frac{2853}{128N^4} + \dots\right), \tag{4.8}$$

where $N = 9n+1$ and $N = 16n+1$, respectively.

If we want to get even simpler coefficients which correspond to (1.8) we can easily modify these variables to $M = k(k^2n+1)$ and get:

$$\binom{3n}{n,n,n} \sim \frac{3^{M/9+7/2}}{2\pi M} \cdot \left(1 - \frac{3}{M} - \frac{9}{M^2} + \frac{153}{M^3} + \frac{999}{M^4} + \dots\right) \tag{4.9}$$

and

$$\binom{4n}{n,n,n,n} \sim \frac{4^{M/16+9/2}}{\pi M\sqrt{2\pi M}} \cdot \left(1 - \frac{14}{M} + \frac{30}{M^2} + \frac{3108}{M^3} - \frac{5706}{M^4} + \dots\right), \tag{4.10}$$

where are $M = 27n+3$ and $M = 64n+4$, respectively.

It is interesting that this additional modification has to be done only for k which are multiple of 2 and 3, but for prime $k \geqslant 5$ (and power of prime) simple formula is obtained already for $N = k^2n+1$. For example, here is expansion (4.1) for $k=5$ and $k=7$:

$$\binom{5n}{n,n,n,n,n} \sim \frac{5^{N/5+9/2}}{(2\pi N)^2} \cdot \left(1 - \frac{8}{N} - \frac{23}{N^2} + \frac{194}{N^3} - \frac{1095}{N^4} + \dots\right) \tag{4.11}$$

and

$$\binom{7n}{n,n,n,n,n,n,n} \sim \frac{7^{N/7+13/2}}{(2\pi N)^3} \cdot \left(1 - \frac{25}{N} + \frac{286}{N^2} + \frac{318}{N^3} - \frac{41313}{N^4} + \dots\right) \tag{4.12}$$

where $N = 25n+1$ and $N = 49n+1$, respectively.

As we mentioned before, the most natural choice for parametar m is exactly k which gives better approximation results. If we apply (4.1) with $m=k$ we don't need to do additional modification and immediately with $N = k^2n+1$ we get the simplest coefficients. Here are examples for $k=3$ and $k=4$ (with $N = 9n+1$ and $N = 16n+1$ respectively):

$$\binom{3n}{n,n,n} \sim \frac{3^{N/3+5/2}}{2\pi N} \cdot \left(1 - \frac{3}{N} + \frac{22}{N^3} + \frac{3}{N^4} - \frac{477}{N^5} \dots\right)^{1/3} \tag{4.13}$$

and

$$\binom{4n}{n,n,n,n} \sim \frac{4^{N/4+3}}{\pi N\sqrt{2\pi N}} \cdot \left(1 - \frac{14}{N} + \frac{81}{N^2} - \frac{56}{N^3} - \frac{1682}{N^4} + \dots\right)^{1/4}. \tag{4.14}$$

Only for central binomial coefficient we need to use $N = 8n+2$ to get the simplest formula for $m=k=2$:

$$\binom{2n}{n} \sim \frac{2^{2n+2}}{\sqrt{\pi N}} \left(1 - \frac{2}{N^2} + \frac{22}{N^4} - \frac{692}{N^6} + \dots\right)^{1/2}. \tag{4.15}$$

Expansion (4.1) and formulas derived from it are new and they are not known in the literature. If we compare them with all of the formulas mentioned before, we truly see that they have the most natural form for the central multinomial coefficient, especially the final one with $m = k$.

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