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OPPENHEIM'S PROBLEM AND RELATED INEQUALITIES FOR DUNKL KERNELS

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Abstract. In this paper, we establish some inequalities related to Oppenheim's problem for Dunkl kernels. In order to prove our main results, we present new inequalities involving modified Bessel functions of the first kind. Refinements of inequalities for modified Bessel functions are also given.

1. Introduction

We consider the Oppenheim's problem: What are the best possible constants $l_1, l_2, r_1, r_2 \in \mathbb{R}$ such that

$$l_1 \cosh x + l_2 \leqslant \frac{\sinh x}{x} \leqslant r_1 \cosh x + r_2$$

hold for all $x \in \mathbb{R} \setminus \{0\}$?

Since the hyperbolic cosine and hyperbolic sine functions are particular cases of modified Bessel functions, then it is natural to generalize some formulas and inequalities involving these elementary functions to modified Bessel functions. The extension of the Oppenheim's problem to Bessel and modified Bessel functions was first considered by Á. Baricz in [1]. L. Zhu solved completely this problem for trigonometric functions, see [15].

Our aim is to solve the analogues of the Oppenheim's problem for Dunkl kernels ψ^{α}_{-i}: What are, for $\alpha \geqslant -\frac{1}{2}$, the best possible constants l_1, l_2, r_1, $r_2 \in \mathbb{R}$ such that

$$l_1 \psi^{\alpha}_{-i}(x) + l_2 \leqslant \psi^{\alpha+1}_{-i}(x) \leqslant r_1 \psi^{\alpha}_{-i}(x) + r_2$$

hold for all $x \in I$; $I = \mathbb{R}$, $[a,b]$, a, $b \in \mathbb{R}$; $a < b$, $[a,+\infty[$, $a \in \mathbb{R}$, $]-\infty,b]$, $b \in \mathbb{R}$?

In the beginning, we present some new inequalities related to this problem for hyperbolic functions. These inequalities and Sonine integral formula for modified Bessel functions allow us to get a new version of the solution of this type of problem for modified Bessel functions $\mathscr{I}_\alpha$, $\alpha \geqslant -\frac{1}{2}$. Next, by using Sonine integral formula for Dunkl

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kernels, we solve the Oppenheim's problem for Dunkl kernels ψ^{α}_{-i}, $\alpha \geqslant -\frac{1}{2}$. At the end of this paper, we give refinements of inequalities for modified Bessel functions $\mathscr{I}_{\alpha}$, $\alpha \geqslant -\frac{1}{2}$. More precisely, in view of the inequalities given in [3], we prove that for all $\alpha \geqslant -\frac{1}{2}$ and $x \in \mathbb{R}\setminus\{0\}$, we have

$$\mathscr{I}_{\alpha+1}(x) \leqslant \frac{(\alpha+1)\mathscr{I}_{\alpha}(x)+1}{\alpha+2} < \frac{2(\alpha+1)a_1\mathscr{I}_{\alpha}(x)+1}{(2\alpha+1)a_1+a_2},$$

where $a_1 > \frac{1}{2}$, $a_2 := a_1+1 > \frac{3}{2}$.

2. Dunkl kernels

In this section, we take $\alpha > -1$.

DEFINITION 2.1. Let $\lambda \in \mathbb{C}$. We call Dunkl kernel the function ψ^{α}_{λ} defined by

$$\psi^{\alpha}_{\lambda}(z) := \mathscr{J}_{\alpha}(\lambda z) + \frac{i\lambda z}{2(\alpha+1)}\mathscr{J}_{\alpha+1}(\lambda z), \quad z \in \mathbb{C},$$

where $\mathscr{J}_{\alpha}$ is the normalized Bessel function of index α given by

$$\mathscr{J}_{\alpha}(z) := \Gamma(\alpha+1)\sum_{n=0}^{+\infty}\frac{(-1)^n z^{2n}}{2^{2n}n!\,\Gamma(\alpha+n+1)}.$$

For more details of these functions, we can see [2, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

PROPOSITION 2.2. *For all λ, $z \in \mathbb{C}$, $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have*

1. $\psi^{\alpha}_{\lambda}(z) = \psi^{\alpha}_{z}(\lambda) = \psi^{\alpha}_{\lambda z}(1)$.
2. $\psi^{\alpha}_{-\lambda}(z) = \psi^{\alpha}_{\lambda}(-z)$, $\psi^{\alpha}_{-\lambda}(-z) = \psi^{\alpha}_{\lambda}(z)$, $\overline{\psi^{\alpha}_{\lambda}(z)} = \psi^{\alpha}_{\overline{\lambda}}(-\overline{z})$.
3. $\psi^{\alpha}_{\lambda}(0) = 1$.
4. $\psi^{\alpha}_{\lambda}(z) = \Gamma(\alpha+1)\left(\sum_{n=0}^{+\infty}\frac{(-1)^n(\lambda z)^{2n}}{2^{2n}n!\,\Gamma(\alpha+n+1)} + i\sum_{n=0}^{+\infty}\frac{(-1)^n(\lambda z)^{2n+1}}{2^{2n+1}n!\,\Gamma(\alpha+n+2)}\right)$.
5. $\psi^{\alpha}_{\lambda}(z) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)}\int_{-1}^{1}(1-t)^{\alpha+\frac{1}{2}}(1+t)^{\alpha-\frac{1}{2}}e^{-i\lambda zt}\,dt$.
6. $|\psi^{\alpha}_{\lambda}(z)| \leqslant e^{|\Im(\lambda z)|}$.
7. $\left|\frac{d^n}{dx^n}(\psi^{\alpha}_{\lambda})(x)\right| \leqslant |\lambda|^n \psi^{\alpha}_{i\Im\lambda}(x)$. *In particular, if $\lambda \in \mathbb{R}$ then* $|\psi^{\alpha}_{\lambda}(x)| \leqslant 1$.

8. $\lim_{\lambda\to\pm\infty} \psi_\lambda^\alpha(x) = 0,\ x \neq 0.$

Proof. We denote by $a_\alpha := \dfrac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}$.

3) It suffices to use $\mathscr{J}_\alpha(0) = 1$.

5) Let $\lambda, z \in \mathbb{C}$. We have

$$\begin{aligned} \mathscr{J}_\alpha(\lambda z) &= a_\alpha \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} e^{-i\lambda z t}\, dt, \\ \mathscr{J}_{\alpha+1}(\lambda z) &= a_\alpha \frac{\alpha+1}{\alpha+\frac{1}{2}} \int_{-1}^{1} (1-t^2)^{\alpha+\frac{1}{2}} e^{-i\lambda z t}\, dt. \end{aligned}$$

An integration by parts shows that

$$\frac{i\lambda z}{2(\alpha+1)} \mathscr{J}_{\alpha+1}(\lambda z) = -a_\alpha \int_{-1}^{1} t(1-t^2)^{\alpha-\frac{1}{2}} e^{-i\lambda z t}\, dt.$$

Hence

$$\psi_\lambda^\alpha(z) = a_\alpha \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}} (1-t) e^{-i\lambda z t}\, dt = a_\alpha \int_{-1}^{1} (1-t)^{\alpha+\frac{1}{2}} (1+t)^{\alpha-\frac{1}{2}} e^{-i\lambda z t}\, dt.$$

7) Let $\lambda \in \mathbb{C}$. For all $x \in \mathbb{R}$, we have

$$\psi_\lambda^\alpha(x) = a_\alpha \int_{-1}^{1} (1-t)^{\alpha+\frac{1}{2}} (1+t)^{\alpha-\frac{1}{2}} e^{-i\lambda x t}\, dt.$$

For all $t \in]-1,1[$, the function $x \longmapsto a_\alpha (1-t)^{\alpha+\frac{1}{2}} (1+t)^{\alpha-\frac{1}{2}} e^{-i\lambda x t}$ is C^∞ on $\mathbb{R}$. For all $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $t \in]-1,1[$, we have

$$\left| (1-t)^{\alpha+\frac{1}{2}} (1+t)^{\alpha-\frac{1}{2}} (-i\lambda t)^n e^{-i\lambda x t} \right| \leqslant |\lambda^n| e^{|\Im\lambda||x|} (1-t)^{\alpha+\frac{1}{2}} (1+t)^{\alpha-\frac{1}{2}}$$

and

$$a_\alpha \int_{-1}^{1} (1-t)^{\alpha+\frac{1}{2}} (1+t)^{\alpha-\frac{1}{2}}\, dt = \psi_\lambda^\alpha(0) = 1.$$

Then, from the derivation theorem under the sign integral, ψ_λ^α is C^∞ on $\mathbb{R}$ and for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$, we have

$$\frac{d^n}{dx^n}(\psi_\lambda^\alpha)(x) = a_\alpha \int_{-1}^{1} (1-t)^{\alpha+\frac{1}{2}} (1+t)^{\alpha-\frac{1}{2}} (-i\lambda t)^n e^{-i\lambda x t}\, dt.$$

8) Let $x > 0$. Remark that the function f_x defined by

$$f_x(t) := \frac{a_\alpha}{x} \left(1 - \frac{t}{x}\right)^{\alpha+\frac{1}{2}} \left(1 + \frac{t}{x}\right)^{\alpha-\frac{1}{2}} \mathbf{1}_{]-x,x[}(t)$$

is integrable on $\mathbb{R}$ and its Fourier transformation is the function $\lambda \longmapsto \psi_\lambda^\alpha(x)$.

By the same way we get the result in the case $x < 0$. $\square$

PROPOSITION 2.3. *The Dunkl kernel ψ_λ^α, $\lambda \in \mathbb{C}$, is the unique entire solution on $\mathbb{R}$ of the following equation:*

$$\begin{cases} \Lambda_\alpha u = i\lambda u, \\ u(0) = \ \ 1, \end{cases}$$

where Λ_α is the Dunkl operator on $\mathbb{R}$ of index $\left(\alpha + \frac{1}{2}\right)$ associated to the reflexions groupe $\mathbb{Z}_2$ and given by

$$\Lambda_\alpha f(x) := \frac{d}{dx} f(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right), \quad f \in C^1(\mathbb{R}),\ \alpha > -\frac{1}{2}.$$

Proof. Let $x \in \mathbb{R}$, $\psi_0^\alpha(x) = 1$ and $\Lambda_\alpha \psi_0^\alpha = 0 = i \times 0 \times \psi_0^\alpha$. Let $\lambda \in \mathbb{C}\setminus\{0\}$. We have

$$\psi_\lambda^\alpha(0) = 1, \quad \psi_\lambda^\alpha(x) = \mathscr{I}_\alpha(\lambda x) + \frac{i\lambda x}{2(\alpha+1)} \mathscr{I}_{\alpha+1}(\lambda x)$$

and

$$\frac{d}{dx}\mathscr{I}_\alpha(x) = -\frac{x}{2(\alpha+1)}\mathscr{I}_{\alpha+1}(x).$$

Hence

$$\begin{aligned}
\Lambda_\alpha \psi_\lambda^\alpha(x) &= \frac{d}{dx}\psi_\lambda^\alpha(x) + \frac{2\alpha+1}{x}\left(\frac{\psi_\lambda^\alpha(x)-\psi_\lambda^\alpha(-x)}{2}\right) \\
&= \frac{d}{dx}\left[\mathscr{I}_\alpha(\lambda x) + \frac{i\lambda x}{2(\alpha+1)}\mathscr{I}_{\alpha+1}(\lambda x)\right] + \frac{2\alpha+1}{x}\frac{i\lambda x}{2(\alpha+1)}\mathscr{I}_{\alpha+1}(\lambda x) \\
&= \frac{d}{dx}[\mathscr{I}_\alpha(\lambda x)] - \frac{i}{\lambda}\frac{d^2}{dx^2}[\mathscr{I}_\alpha(\lambda x)] - \frac{i}{\lambda}\frac{2\alpha+1}{x}\frac{d}{dx}[\mathscr{I}_\alpha(\lambda x)] \\
&= i\lambda\frac{i\lambda x}{2(\alpha+1)}\mathscr{I}_{\alpha+1}(\lambda x) - \frac{i}{\lambda}L_\alpha[\mathscr{I}_\alpha(\lambda x)] \\
&= i\lambda\left(\frac{i\lambda x}{2(\alpha+1)}\mathscr{I}_{\alpha+1}(\lambda x) + \mathscr{I}_\alpha(\lambda x)\right) \\
&= i\lambda\psi_\lambda^\alpha(x),
\end{aligned}$$

where L_α is the Bessel operator on $\mathbb{R}$ given by

$$L_\alpha f(x) := \frac{d^2}{dx^2}f(x) + \frac{2\alpha+1}{x}\frac{d}{dx}f(x), \quad f \in C^2(\mathbb{R}),\ \alpha > -\frac{1}{2}. \tag{1}$$

Unicity: Let $\lambda \in \mathbb{C}$, $x \in \mathbb{R}$ and u an entire function on $\mathbb{R}$ such that

$$\begin{cases} \Lambda_\alpha u = i\lambda u, \\ u(0) = \ \ 0. \end{cases}$$

Writting $u(x) = \sum_{n=0}^{+\infty} a_n x^n$, where a_n are complex numbers, we get $i\lambda u(x) = \sum_{n=0}^{+\infty} i\lambda a_n x^n$ and

$$\Lambda_\alpha u(x) = \sum_{n=1}^{+\infty} a_n \Lambda_\alpha(x^n) = \sum_{n=1}^{+\infty} a_n b_n x^{n-1} = \sum_{n=0}^{+\infty} a_{n+1} b_{n+1} x^n,$$

with $b_n = n+(2\alpha+1)\dfrac{1-(-1)^n}{2}$. By identification we obtain for all $n \in \mathbb{N}$, $a_{n+1}b_{n+1} = i\lambda a_n$. Since $a_0 = 0$ and for all $n \in \mathbb{N}\setminus\{0\}$, we have $b_n > 0$, then for all $n \in \mathbb{N}$, we get $a_n = 0$. □

REMARKS 2.4. If $f \in C^1(\mathbb{R})$, then we have

1. $\Lambda_\alpha f(0) = 2(\alpha+1)\dfrac{d}{dx}f(0)$,
2. $(\Lambda_\alpha f)_e = \Lambda_\alpha(f_o) = \dfrac{d}{dx}f_o + (2\alpha+1)\dfrac{f_o}{x}$,
3. $(\Lambda_\alpha f)_o = \Lambda_\alpha(f_e) = \dfrac{d}{dx}f_e$,
4. $\Lambda_\alpha f(x) = \dfrac{d}{dy}f(x) + \left(\alpha+\dfrac{1}{2}\right)\displaystyle\int_{-1}^{1}\dfrac{d}{dy}f(xt)dt, \quad x \in \mathbb{R}$,

where f_e (resp. f_o) is the even (resp. odd) part of f.

For all $z \in \mathbb{C}$, we have the following relation:

$$\psi_{-i}^{\alpha}(z) = \mathscr{I}_\alpha(z) + \frac{z}{2(\alpha+1)}\mathscr{I}_{\alpha+1}(z),$$

where $\mathscr{I}_\alpha$ is the normalized modified Bessel function of index α given by

$$\mathscr{I}_\alpha(z) := \sum_{n=0}^{+\infty} c_n(\alpha) z^{2n},$$

with for all $n \in \mathbb{N}$, we have

$$c_n(\alpha) := \frac{\Gamma(\alpha+1)}{2^{2n} n!\,\Gamma(\alpha+n+1)}. \tag{2}$$

PROPOSITION 2.5. *For all $z \in \mathbb{C}$, $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have*

1. $\psi_{-i}^{\alpha}(0) = 1$.
2. $\psi_{-i}^{\alpha}(z) = \displaystyle\sum_{k=0}^{+\infty} a_k(\alpha) z^k$, *with for all $k \in \mathbb{N}$, we have*

$$a_{2k}(\alpha) = c_k(\alpha), \quad a_{2k+1}(\alpha) = \frac{c_k(\alpha)}{2(k+\alpha+1)}, \tag{3}$$

where $c_k(\alpha)$ is given by (2).

3. $\psi_{-i}^{\alpha}(z) = \dfrac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)}\displaystyle\int_{-1}^{1}(1-t)^{\alpha+\frac{1}{2}}(1+t)^{\alpha-\frac{1}{2}}e^{-zt}\,dt$.

4. $|\psi^{\alpha}_{-i}(z)| \leqslant e^{|\Re z|}$.

5. $0 < e^{-|x|} \leqslant \psi^{\alpha}_{-i}(x) \leqslant e^{|x|}$.

6.
$$\begin{aligned}\frac{d^n}{dx^n}(\psi^{\alpha}_{-i})(x) &= \sum_{k=0}^{+\infty} \frac{(n+k)!}{k!} a_{n+k}(\alpha) x^k \\ &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \int_{-1}^{1} (-t)^n (1-t)^{\alpha+\frac{1}{2}} (1+t)^{\alpha-\frac{1}{2}} e^{-xt}\, dt.\end{aligned}$$

7. $\left|\frac{d^n}{dx^n}(\psi^{\alpha}_{-i})(x)\right| \leqslant \psi^{\alpha}_{-i}(x)$.

8. *The function* ψ^{α}_{-i} *is the unique entire solution on* $\mathbb{R}$ *of the following equation:*

$$\begin{cases} \Lambda_\alpha u = u, \\ u(0) = 1. \end{cases}$$

EXAMPLES 2.6. For all $x \in \mathbb{R} \setminus \{0\}$, we have

1. $\mathscr{J}_{-\frac{1}{2}}(x) = \cos x, \quad \mathscr{I}_{-\frac{1}{2}}(x) = \cosh x, \quad \psi^{-\frac{1}{2}}_{-i}(x) = e^x, \quad x \in \mathbb{R}.$

2. $\mathscr{J}_{\frac{1}{2}}(x) = \frac{\sin x}{x}, \quad \mathscr{I}_{\frac{1}{2}}(x) = \frac{\sinh x}{x}, \quad \psi^{\frac{1}{2}}_{-i}(x) = \frac{x\cosh x + (x-1)\sinh x}{x^2}.$

3. $\mathscr{J}_{\frac{3}{2}}(x) = 3\frac{\sin x - x\cos x}{x^3}, \quad \mathscr{I}_{\frac{3}{2}}(x) = 3\frac{x\cosh x - \sinh x}{x^3},$

   $\psi^{\frac{3}{2}}_{-i}(x) = \frac{x(x-3)\cosh x + (x^2 - x + 3)\sinh x}{x^4}.$

REMARKS 2.7.

1. The function $\mathscr{J}_\alpha$ is the unique entire solution on $\mathbb{R}$ of the following equation:

$$\begin{cases} L_\alpha u &= -u, \\ u(0) &= 1, \\ \frac{d}{dx}u(0) &= 0. \end{cases}$$

2. The function $\mathscr{I}_\alpha$ is the unique entire solution on $\mathbb{R}$ of the following equation:

$$\begin{cases} L_\alpha u &= u, \\ u(0) &= 1, \\ \frac{d}{dx}u(0) &= 0, \end{cases}$$

where L_α is the Bessel operator on $\mathbb{R}$ given by (1).

PROPOSITION 2.8. *For all* $\alpha > 0$ *and* $x \in \mathbb{R}$, *we have*

1. $\dfrac{d}{dx}\mathscr{J}_\alpha(x) = -\dfrac{x}{2(\alpha+1)}\mathscr{J}_{\alpha+1}(x)$.

2. $\mathscr{J}_{\alpha-1}(x) = \mathscr{J}_\alpha(x) - \dfrac{x^2}{4\alpha(\alpha+1)}\mathscr{J}_{\alpha+1}(x)$.

3. $\dfrac{d}{dx}\mathscr{I}_\alpha(x) = \dfrac{x}{2(\alpha+1)}\mathscr{I}_{\alpha+1}(x)$.

4. $\mathscr{I}_{\alpha-1}(x) = \mathscr{I}_\alpha(x) + \dfrac{x^2}{4\alpha(\alpha+1)}\mathscr{I}_{\alpha+1}(x)$.

PROPOSITION 2.9. *For all* $k,\ n \in \mathbb{N}$ *and* $x \in \mathbb{R}$, *we have*

1. $\dfrac{d^{2k}}{dx^{2k}}(\psi^\alpha_{-i})(x) > 0$.

2. *If* $x \geqslant -1$, *then* $\dfrac{d^{2k+1}}{dx^{2k+1}}(\psi^\alpha_{-i})(x) > 0$.

3. *If* $x \geqslant -1$, *then* $\dfrac{d^{n}}{dx^{n}}(\psi^\alpha_{-i})(x) > 0$.

Proof. Let $k \in \mathbb{N}$ and $x \in \mathbb{R}$. We have

$$\frac{d^{2k}}{dx^{2k}}(\psi^\alpha_{-i})(x) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)}\int_{-1}^{1} t^{2k}(1-t)^{\alpha+\frac{1}{2}}(1+t)^{\alpha-\frac{1}{2}}e^{-xt}\,dt > 0$$

and

$$\begin{aligned}\frac{d^{2k+1}}{dx^{2k+1}}(\psi^\alpha_{-i})(x) &= -\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)}\int_{-1}^{1} t^{2k+1}(1-t)^{\alpha+\frac{1}{2}}(1+t)^{\alpha-\frac{1}{2}}e^{-xt}\,dt\\ &= \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)}\int_{0}^{1} t^{2k+1}(1-t^2)^{\alpha-\frac{1}{2}}(t\cosh(xt)+\sinh(xt))\,dt.\end{aligned}$$

Let $t \in]0,1[$ and $x \geqslant -1$. We have $\dfrac{1}{1-t^2} + x \geqslant \dfrac{1}{1-t^2} - 1 = \dfrac{t^2}{1-t^2} > 0$. Then $\dfrac{1}{2}\ln\left(\dfrac{1+t}{1-t}\right) + xt > 0$ and $t + \tanh(xt) > 0$. Hence for all $t \in]0,1[$ and $x \geqslant -1$, we have $t\cosh(xt) + \sinh(xt) > 0$. We deduce that for all $x \geqslant -1$, we have $\dfrac{d^{2k+1}}{dx^{2k+1}}(\psi^\alpha_{-i})(x) > 0$. □

REMARKS 2.10.

1. For all $k \in \mathbb{N}$ and $x \in \mathbb{R}$, we have

$$\frac{d^{2k}}{dx^{2k}}(\psi_{-i}^{\alpha})(x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \int_0^1 t^{2k}(1-t^2)^{\alpha-\frac{1}{2}}(\cosh(xt)+t\sinh(xt))\,dt > 0.$$

In fact, if $t \in [-1,1]\setminus\{0\}$ and $x \in \mathbb{R}$, then $1+t\tanh(xt) > 1-|t| \geqslant 0$. Morever, $1+0.\tanh(x.0) = 1 > 0$. Hence, for all $t \in [-1,1]$ and $x \in \mathbb{R}$, we have $\cosh(xt)+t\sinh(xt) > 0$.

2. For all $n \in \mathbb{N}$ and $x \geqslant 0$, we have

$$\frac{d^n}{dx^n}(\psi_{-i}^{\alpha})(x) = \sum_{k=0}^{+\infty} \frac{(n+k)!}{k!} a_{n+k}(\alpha)x^k > 0,$$

where $a_k(\alpha)$ is given by (3).

3. Main results

3.1. Oppenheim's problem for modified Bessel functions

We begin to find the best possible constants l_1, l_2, r_1, $r_2 \in \mathbb{R}$ such that

$$l_1\mathscr{I}_{-\frac{1}{2}}(x)+l_2 \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant r_1\mathscr{I}_{-\frac{1}{2}}(x)+r_2 \tag{4}$$

hold for all $x \in I$; $I = \mathbb{R}$, $[-b,b]$, $b > 0$.

The solution of this problem can be stated in the following theorem:

THEOREM 3.1.

1. *For all $x \in \mathbb{R}$, we have*

$$1 \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant \frac{1}{3}\mathscr{I}_{-\frac{1}{2}}(x)+\frac{2}{3} \leqslant \mathscr{I}_{-\frac{1}{2}}(x).$$

2. *For all $x > 0$, we have $x\cosh x - \sinh x > 0$.*

3. *For all $x < 0$, we have $x\cosh x - \sinh x < 0$.*

4. *For all $b > 0$ and $x \in \mathbb{R}$, we have*

$$1 \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant u_1(b)\mathscr{I}_{-\frac{1}{2}}(x)+u_4(b).$$

5. *For all $b > 0$ and $x \in [-b,b]$, we have*

$$\sup\left\{u_1(b)\mathscr{I}_{-\frac{1}{2}}(x)+1-u_1(b), \frac{3\sinh b-2b}{3b\cosh b}\mathscr{I}_{-\frac{1}{2}}(x)+\frac{2}{3}\right\} \leqslant \mathscr{I}_{\frac{1}{2}}(x)$$

$$\leqslant \inf\left\{\frac{1}{3}\mathscr{I}_{-\frac{1}{2}}(x)+\frac{2}{3}, u_1(b)\mathscr{I}_{-\frac{1}{2}}(x)+u_4(b)\right\},$$

where u_1 and u_4 are respectively given by (6) *and* (7).

With the aid of Sonine integral formula for modified Bessel functions we get the following theorem:

THEOREM 3.2. *Let $\alpha \geqslant -\frac{1}{2}$, $b > 0$ and $x \in \mathbb{R}$. We have*

1. $\mathscr{I}_{\alpha+1}(x) \leqslant \inf\left\{\frac{(\alpha+1)\mathscr{I}_\alpha(x)+1}{\alpha+2}, \frac{2(\alpha+1)u_1(b)\mathscr{I}_\alpha(x)+u_4(b)}{(2\alpha+1)u_1(b)+1}\right\}.$

2. *If $x \in [-b,b]$, then*

$$\sup\left\{\frac{2(\alpha+1)u_1(b)\mathscr{I}_\alpha(x)+1-u_1(b)}{(2\alpha+1)u_1(b)+1}, \frac{2(\alpha+1)(3\sinh b-2b)\mathscr{I}_\alpha(x)+2b\cosh b}{(2\alpha+1)(3\sinh b-2b)+3b\cosh b}\right\}$$
$$\leqslant \mathscr{I}_{\alpha+1}(x) \leqslant \inf\left\{\frac{(\alpha+1)\mathscr{I}_\alpha(x)+1}{\alpha+2}, \frac{2(\alpha+1)u_1(b)\mathscr{I}_\alpha(x)+u_4(b)}{(2\alpha+1)u_1(b)+1}\right\},$$

where u_1 and u_4 are respectively given by (6) *and* (7).

3.2. Oppenheim's problem for Dunkl kernels

Now, we are going to find the best possible constants l_1, l_2, r_1, $r_2 \in \mathbb{R}$ such that

$$l_1 \psi_{-i}^{-\frac{1}{2}}(x) + l_2 \leqslant \psi_{-i}^{\frac{1}{2}}(x) \leqslant r_1 \psi_{-i}^{-\frac{1}{2}}(x) + r_2 \tag{5}$$

hold for all $x \in I$; $I = \mathbb{R}$, $[a,b]$, a, $b \in \mathbb{R}$; $a < b$, $[a,+\infty[$, $a \in \mathbb{R}$, $]-\infty,b]$, $b \in \mathbb{R}$.
The solution of this problem can be stated in the following theorem:

THEOREM 3.3.

1. *For all $a \leqslant a_0$ and $x \in \mathbb{R}$, we have*

$$\theta_1(a)\psi_{-i}^{-\frac{1}{2}}(x) + \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(a) \leqslant \psi_{-i}^{\frac{1}{2}}(x).$$

2. *For all a, $b \in \mathbb{R}$; $b \geqslant 0$, $a < b$ and $x \in [a,b]$, we have*

$$e^{-b}\left(\psi_{-i}^{\frac{1}{2}}(b) - \frac{2}{3}\right)\psi_{-i}^{-\frac{1}{2}}(x) + \frac{2}{3} \leqslant \psi_{-i}^{\frac{1}{2}}(x) \leqslant \frac{1}{3}\psi_{-i}^{-\frac{1}{2}}(x) + \psi_{-i}^{\frac{1}{2}}(a) - \frac{e^a}{3}.$$

3. *For all a, $b \in \mathbb{R}$; $a < b \leqslant 0$ and $x \in [a,b]$, we have*

$$\frac{\psi_{-i}^{\frac{1}{2}}(b)}{e^b+2}\left(\psi_{-i}^{-\frac{1}{2}}(x) + 2\right) \leqslant \psi_{-i}^{\frac{1}{2}}(x) \leqslant \frac{1}{3}\psi_{-i}^{-\frac{1}{2}}(x) + \psi_{-i}^{\frac{1}{2}}(a) - \frac{e^a}{3}.$$

4. *For all* $a \leqslant 0$ *and* $x \geqslant a$, *we have*

$$\psi_{-i}^{\frac{1}{2}}(x) \leqslant \theta_1(-a)\psi_{-i}^{-\frac{1}{2}}(x) + \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(a).$$

5. *For all* $b \geqslant 0$ *and* $x \leqslant b$, *we have*

$$\theta_1(-b)\psi_{-i}^{-\frac{1}{2}}(x) + \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(b) \leqslant \psi_{-i}^{\frac{1}{2}}(x),$$

where a_0 *and* θ_1 *are respectively given by* (10) *and* (19).

By using Sonine integral formula for Dunkl kernels, the following theorem gives the solution of the problem: Find, for $\alpha \geqslant -\frac{1}{2}$, the best possible constants l_1, l_2, r_1, $r_2 \in \mathbb{R}$ such that

$$l_1\psi_{-i}^{\alpha}(x) + l_2 \leqslant \psi_{-i}^{\alpha+1}(x) \leqslant r_1\psi_{-i}^{\alpha}(x) + r_2$$

hold for all $x \in I$; $I = \mathbb{R}$, $[a,b]$, a, $b \in \mathbb{R}$; $a < b$, $[a,+\infty[$, $a \in \mathbb{R}$, $]-\infty,b]$, $b \in \mathbb{R}$.

THEOREM 3.4. *Let* $\alpha \geqslant -\frac{1}{2}$.

1. *For all* $a \in]a_3, a_0]$ *and* $x \in \mathbb{R}$, *we have*

$$\frac{2[3(\alpha+1)\theta_1(a)\psi_{-i}^{\alpha}(x) + \mathscr{I}_{\frac{3}{2}}(a)]}{3[(2\alpha+1)\theta_1(a)+1]} \leqslant \psi_{-i}^{\alpha+1}(x).$$

2. *For all* a, $b \in \mathbb{R}$; $a \leqslant 0 \leqslant b$, $a < b$ *and* $x \in [a,b]$, *we have*

$$\frac{2(\alpha+1)e^{-b}\left(3\psi_{-i}^{\frac{1}{2}}(b) - 2\right)\psi_{-i}^{\alpha}(x) + 2}{(2\alpha+1)e^{-b}\left(3\psi_{-i}^{\frac{1}{2}}(b) - 2\right) + 3}$$

$$\leqslant \psi_{-i}^{\alpha+1}(x) \leqslant \frac{2(\alpha+1)\psi_{-i}^{\alpha}(x) + 3\psi_{-i}^{\frac{1}{2}}(a) - e^{a}}{2(\alpha+2)}.$$

3. *For all* $a \leqslant 0$ *and* $x \geqslant a$, *we have*

$$\psi_{-i}^{\alpha+1}(x) \leqslant \frac{2[3(\alpha+1)\theta_1(-a)\psi_{-i}^{\alpha}(x) + \mathscr{I}_{\frac{3}{2}}(a)]}{3[(2\alpha+1)\theta_1(-a)+1]}.$$

4. *For all* $b \in [0, -a_3[$ *and* $x \leqslant b$, *we have*

$$\frac{2[3(\alpha+1)\theta_1(-b)\psi_{-i}^{\alpha}(x) + \mathscr{I}_{\frac{3}{2}}(b)]}{3[(2\alpha+1)\theta_1(-b)+1]} \leqslant \psi_{-i}^{\alpha+1}(x),$$

where a_0, θ_1 *and* a_3 *are respectively given by* (10), (19) *and* $\theta_1(a_3) := -\dfrac{1}{2\alpha+1}$.

COROLLARY 3.5. *Let* $\alpha \geqslant -\dfrac{1}{2}$ *and* $x \in \mathbb{R}$. *We have*

1. $b_0 \leqslant \psi_{-i}^{\alpha+1}(x)$.
2. *If* $x \geqslant 0$, *then* $\psi_{-i}^{\alpha+1}(x) \leqslant \dfrac{(\alpha+1)\psi_{-i}^{\alpha}(x)+1}{\alpha+2}$.
3. *If* $x \leqslant 0$, *then* $\dfrac{(\alpha+1)\psi_{-i}^{\alpha}(x)+1}{\alpha+2} \leqslant \psi_{-i}^{\alpha+1}(x)$,

where b_0 *is given by* (11).

COROLLARY 3.6. *Let* $\alpha \geqslant -\dfrac{1}{2}$, $\lambda,\ z \in \mathbb{C};\ \Re(\lambda z) = 0$. *We have*

1. $b_0 \leqslant \psi_{\lambda}^{\alpha+1}(z)$.
2. *If* $\Im(\lambda z) \leqslant 0$, *then* $\psi_{\lambda}^{\alpha+1}(z) \leqslant \dfrac{(\alpha+1)\psi_{\lambda}^{\alpha}(z)+1}{\alpha+2}$.
3. *If* $\Im(\lambda z) \geqslant 0$, *then* $\dfrac{(\alpha+1)\psi_{\lambda}^{\alpha}(z)+1}{\alpha+2} \leqslant \psi_{\lambda}^{\alpha+1}(z)$,

where b_0 *is given by* (11).

Now, we give the following interesting result.

THEOREM 3.7. *Let* $\alpha \geqslant -\dfrac{1}{2}$.

1. *For all* $b \in]0,1[$ *and* $x \in \mathbb{R}$, *we have*

$$\psi_{-i}^{\alpha}(x) \leqslant \inf\left\{2\mathscr{I}_{\alpha}(x), \frac{2}{1-b^2}\left(\mathscr{I}_{\alpha}(x)-b\right)\right\}.$$

2. *For all* $b > 1$ *and* $x \in \mathbb{R}$, *we have*

$$-\frac{2}{b^2-1}\left(\mathscr{I}_{\alpha}(x)-b\right) \leqslant \psi_{-i}^{\alpha}(x).$$

3. *For all* $a,\ b \in \mathbb{R};\ a < 0 \leqslant b$ *and* $x \in [a,b]$, *we have*

$$-\frac{2}{e^{-2a}-1}\left(\mathscr{I}_{\alpha}(x)-e^{-a}\right) \leqslant \psi_{-i}^{\alpha}(x).$$

4. *For all* $a,\ b\in\mathbb{R};\ a\leqslant 0<b$ *and* $x\in[a,b]$, *we have*

$$\sup\left\{\frac{1}{1-e^{-2b}}\left(2\mathscr{I}_\alpha(x)-e^{-a}-e^{a-2b}\right),\frac{(e^a\sinh b+1)\mathscr{I}_\alpha(x)-\cosh a}{\cosh a\sinh b}\right\}$$
$$\leqslant\psi^\alpha_{-i}(x)\leqslant\frac{2}{1-e^{-2b}}\left(\mathscr{I}_\alpha(x)-e^{-b}\right).$$

5. *For all* $a,\ b\in\mathbb{R};\ -\ln(\cosh b)<a\leqslant 0\leqslant b,\ a<b$ *and* $x\in[a,b]$, *we have*

$$\frac{2\left(\mathscr{I}_\alpha(x)-e^{-a}\right)}{e^{-2b}-2e^{-(a+b)}+1}\leqslant\psi^\alpha_{-i}(x).$$

6. *For all* $a,\ b\in\mathbb{R};\ a<-\ln(\cosh b)\leqslant 0\leqslant b$ *and* $x\in[a,b]$, *we have*

$$\psi^\alpha_{-i}(x)\leqslant\frac{2\left(\mathscr{I}_\alpha(x)-e^{-a}\right)}{e^{-2b}-2e^{-(a+b)}+1}.$$

7. *For all* $a\leqslant 0$ *and* $x\geqslant a$, *we have*

$$2\mathscr{I}_\alpha(x)-e^{-a}\leqslant\psi^\alpha_{-i}(x)\leqslant\left(\sqrt{e^{-2a}+1}+1\right)\mathscr{I}_\alpha(x)-e^{-a}.$$

8. *For all* $a\in]-\ln 2,0]$ *and* $x\geqslant a$, *we have*

$$\psi^\alpha_{-i}(x)\leqslant\frac{4}{4-e^{-2a}}\left(2\mathscr{I}_\alpha(x)-e^{-a}\right).$$

9. *For all* $a<-\ln 2$ *and* $x\geqslant a$, *we have*

$$-\frac{4}{e^{-2a}-4}\left(2\mathscr{I}_\alpha(x)-e^{-a}\right)\leqslant\psi^\alpha_{-i}(x).$$

10. *For all* $b\geqslant 0$ *and* $x\leqslant b$, *we have*

$$-\left(\sqrt{e^{2b}+1}-1\right)\mathscr{I}_\alpha(x)+e^b\leqslant\psi^\alpha_{-i}(x)\leqslant 2\mathscr{I}_\alpha(x)-e^{-b}.$$

COROLLARY 3.8. *Let* $\alpha\geqslant-\frac{1}{2}$.

1. *For all* $x\geqslant 0$, *we have*

$$2\mathscr{I}_\alpha(x)-1\leqslant\psi^\alpha_{-i}(x)\leqslant\inf\left\{\left(\sqrt{2}+1\right)\mathscr{I}_\alpha(x)-1,\frac{4}{3}\left(2\mathscr{I}_\alpha(x)-1\right)\right\}.$$

2. *For all* $x\leqslant 0$, *we have*

$$-\left(\sqrt{2}-1\right)\mathscr{I}_\alpha(x)+1\leqslant\psi^\alpha_{-i}(x)\leqslant 2\mathscr{I}_\alpha(x)-1.$$

4. Preliminary results

In order to solve Problem (4), we present the following propositions:

PROPOSITION 4.1. *Let* $b > 0,\ 0 < a < b$ *and* $x \in [-b, b]$. *We have*

1. $1 \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant \dfrac{\sinh b}{b}$.
2. $\dfrac{1}{3}\mathscr{I}_{-\frac{1}{2}}(x) + \dfrac{3\sinh b - b\cosh b}{3b} \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant \dfrac{1}{3}\mathscr{I}_{-\frac{1}{2}}(x) + \dfrac{2}{3}$.
3. $u_1(b)\mathscr{I}_{-\frac{1}{2}}(x) + 1 - u_1(b) \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant u_1(b)\mathscr{I}_{-\frac{1}{2}}(x) + u_4(b)$.
4. *If* $x \in [-b, -a] \cup [a, b]$, *then*

$$u_1(a)\mathscr{I}_{-\frac{1}{2}}(x) + \frac{\sinh b}{b} - u_1(a)\cosh b \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant u_1(a)\mathscr{I}_{-\frac{1}{2}}(x) + u_4(a),$$

where

$$u_1(x) := \frac{\mathscr{I}_{\frac{3}{2}}(x)}{3\mathscr{I}_{\frac{1}{2}}(x)} = \frac{x\cosh x - \sinh x}{x^2 \sinh x}, \quad x \neq 0, \tag{6}$$

$$u_4(x) := \mathscr{I}_{\frac{1}{2}}(x) - \cosh x \quad u_1(x) = \frac{\sinh(2x) - 2x}{2x^2 \sinh x}, \quad x \neq 0. \tag{7}$$

PROPOSITION 4.2. *Let* $b > 0,\ 0 < a < b$ *and* $x \in [-b, b]$. *We have*

1. $\dfrac{3\sinh b - 2b}{3b\cosh b}\mathscr{I}_{-\frac{1}{2}}(x) + \dfrac{2}{3} \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant \dfrac{1}{3}\mathscr{I}_{-\frac{1}{2}}(x) + \dfrac{2}{3}$.
2. $(1 - u_4(b))\mathscr{I}_{-\frac{1}{2}}(x) + u_4(b) \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant u_1(b)\mathscr{I}_{-\frac{1}{2}}(x) + u_4(b)$.
3. *If* $x \in [-b, -a] \cup [a, b]$, *then*

$$\frac{\sinh b - u_4(a)b}{b\cosh b}\mathscr{I}_{-\frac{1}{2}}(x) + u_4(a) \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant u_1(a)\mathscr{I}_{-\frac{1}{2}}(x) + u_4(a),$$

where u_1 *and* u_4 *are respectively given by* (6) *and* (7).

PROPOSITION 4.3. *Let* $b > 0,\ 0 < a < b$ *and* $x \in [-b, b]$. *We have*

1. $\dfrac{\sinh b}{b(\cosh b + 2)}\left(\mathscr{I}_{-\frac{1}{2}}(x) + 2\right) \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant \dfrac{1}{3}\mathscr{I}_{-\frac{1}{2}}(x) + \dfrac{2}{3}$.
2. $\dfrac{1}{1 - w_1(b)}\left(\mathscr{I}_{-\frac{1}{2}}(x) - w_1(b)\right) \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant u_1(b)\mathscr{I}_{-\frac{1}{2}}(x) + u_4(b)$.
3. *If* $x \in [-b, -a] \cup [a, b]$, *then*

$$\frac{\sinh b}{b(\cosh b - w_1(a))}\left(\mathscr{I}_{-\frac{1}{2}}(x) - w_1(a)\right) \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant u_1(a)\mathscr{I}_{-\frac{1}{2}}(x) + u_4(a),$$

where u_1, u_4 and w_1 are respectively given by (6), (7) *and*

$$w_1(x) := -\frac{u_4(x)}{u_1(x)} = -3\frac{\mathscr{I}_{\frac{1}{2}}(2x)-1}{x^2\mathscr{I}_{\frac{3}{2}}(x)} = -\frac{\sinh(2x)-2x}{2(x\cosh x-\sinh x)}, \quad x \neq 0. \tag{8}$$

We are going to find the best possible constants $l_1, l_2, r_1, r_2 \in \mathbb{R}$ such that

$$l_1\psi_{-i}^{-\frac{1}{2}}(x)+l_2 \leqslant \mathscr{I}_{-\frac{1}{2}}(x) \leqslant r_1\psi_{-i}^{-\frac{1}{2}}(x)+r_2 \tag{9}$$

hold for all $x \in I$; $I=\mathbb{R}$, $[a,b]$, $a, b \in \mathbb{R}$; $a<b$, $[a,+\infty[$, $a \in \mathbb{R}$, $]-\infty,b]$, $b \in \mathbb{R}$.

In order to solve this problem, we present the following propositions:

PROPOSITION 4.4. *Let $a, b \in \mathbb{R}$; $a<b$ and $x \in [a,b]$. We have*

1. $\frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-b}}{2} \leqslant \mathscr{I}_{-\frac{1}{2}}(x) \leqslant \frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-a}}{2}$.
2. $\frac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-b} \leqslant \mathscr{I}_{-\frac{1}{2}}(x) \leqslant \frac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-a}+e^{a-2b}}{2}$.
3. $\frac{1-e^{-2a}}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-a} \leqslant \mathscr{I}_{-\frac{1}{2}}(x) \leqslant \frac{1-e^{-2a}}{2}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-b}+e^{b-2a}}{2}$.

PROPOSITION 4.5. *Let $a, b \in \mathbb{R}$; $a<b$ and $x \in [a,b]$. We have*

1. $\frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x) < \frac{1+e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x) \leqslant \mathscr{I}_{-\frac{1}{2}}(x) \leqslant \frac{1+e^{-2a}}{2}\psi_{-i}^{-\frac{1}{2}}(x)$.
2. $\frac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-b} \leqslant \mathscr{I}_{-\frac{1}{2}}(x) \leqslant \frac{e^{-2a}-2e^{-(a+b)}+1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-b}$.
3. $$\frac{1-e^{-2a}}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-a} \leqslant \mathscr{I}_{-\frac{1}{2}}(x) \leqslant \frac{e^{-2b}-2e^{-(a+b)}+1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-a} < \frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-a}.$$

PROPOSITION 4.6. *Let $a, b \in \mathbb{R}$; $a<b$ and $x \in [a,b]$. We have*

1. $\frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x) < \frac{1+e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x) \leqslant \mathscr{I}_{-\frac{1}{2}}(x) \leqslant \frac{1+e^{-2a}}{2}\psi_{-i}^{-\frac{1}{2}}(x)$.
2. $\frac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-b} \leqslant \mathscr{I}_{-\frac{1}{2}}(x) \leqslant \frac{\cosh a}{e^a\sinh b+1}\left(\sinh b\,\psi_{-i}^{-\frac{1}{2}}(x)+1\right)$.
3. *If $a>0$, then*

$$\frac{1-e^{-2a}}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-a} \leqslant \mathscr{I}_{-\frac{1}{2}}(x) \leqslant \frac{\cosh b}{\sinh a\, e^b+1}\left(\sinh a\,\psi_{-i}^{-\frac{1}{2}}(x)+1\right) < \frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{1}{2\sinh a}.$$

4. *If $a<0$, then*

$$-\frac{2}{e^{-2a}-1}\mathscr{I}_{-\frac{1}{2}}(x)-\frac{1}{\sinh a}\leqslant \psi_{-i}^{-\frac{1}{2}}(x)\leqslant \frac{\sinh ae^{b}+1}{\sinh a\cosh b}\mathscr{I}_{-\frac{1}{2}}(x)-\frac{1}{\sinh a} < 2\mathscr{I}_{-\frac{1}{2}}(x)-\frac{1}{\sinh a}.$$

The solution of Problem (9) can be stated as follows.

COROLLARY 4.7. *Let $a,\ b\in\mathbb{R}$; $a<b$ and $x\in\mathbb{R}$. We have*

1. $\frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x)<\mathscr{I}_{-\frac{1}{2}}(x)$.
2. *If $b>0$, then* $\frac{1-b^2}{2}\psi_{-i}^{-\frac{1}{2}}(x)+b\leqslant\mathscr{I}_{-\frac{1}{2}}(x)$.
3. *If $x\in[a,b]$, then*

$$\sup\left\{\frac{1-e^{-2a}}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-a},\frac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-b}\right\}\leqslant\mathscr{I}_{-\frac{1}{2}}(x)$$

$$\leqslant\inf\left\{\frac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-a}+e^{a-2b}}{2},\frac{e^{-2b}-2e^{-(a+b)}+1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-a},\right.$$
$$\left.\frac{\cosh a}{e^{a}\sinh b+1}\left(\sinh b\psi_{-i}^{-\frac{1}{2}}(x)+1\right)\right\}.$$

4. (a) i. *If $x>a$, $x\neq a+\ln 2$, then*

$$\frac{4-e^{-2a}}{8}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-a}}{2}<\mathscr{I}_{-\frac{1}{2}}(x)<\frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-a}}{2}.$$

ii. *If $x=a+\ln 2$, then*

$$\frac{4-e^{-2a}}{8}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-a}}{2}=\mathscr{I}_{-\frac{1}{2}}(x)<\frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-a}}{2}.$$

iii. *If $x=a$, then* $\frac{4-e^{-2a}}{8}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-a}}{2}<\mathscr{I}_{-\frac{1}{2}}(x)=\frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-a}}{2}$.

iv. *If $x<a$, then* $\frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-a}}{2}<\mathscr{I}_{-\frac{1}{2}}(x)$.

(b) i. *If $x>a$, $x\neq a+\ln\left(1+\sqrt{e^{-2a}+1}\right)$, then*

$$\left(\sqrt{e^{2a}+1}-e^{a}\right)\left(e^{a}\psi_{-i}^{-\frac{1}{2}}(x)+1\right)<\mathscr{I}_{-\frac{1}{2}}(x)<\frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-a}}{2}.$$

ii. If $x=a+\ln\left(1+\sqrt{e^{-2a}+1}\right)$, *then*

$$\left(\sqrt{e^{2a}+1}-e^{a}\right)\left(e^{a}\psi_{-i}^{-\frac{1}{2}}(x)+1\right)=\mathscr{I}_{-\frac{1}{2}}(x)<\frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-a}}{2}.$$

iii. If $x=a$, *then*

$$\left(\sqrt{e^{2a}+1}-e^{a}\right)\left(e^{a}\psi_{-i}^{-\frac{1}{2}}(x)+1\right)<\mathscr{I}_{-\frac{1}{2}}(x)=\frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-a}}{2}.$$

iv. If $x<a$, *then* $\frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-a}}{2}<\mathscr{I}_{-\frac{1}{2}}(x)$.

(c) *i. If* $x\leqslant a,\ x\neq a-\ln\left(1+\sqrt{e^{2a}+1}\right)$, *then*

$$-\left(e^{-a}+\sqrt{e^{-2a}+1}\right)\left(e^{-a}\psi_{-i}^{-\frac{1}{2}}(x)-1\right)<\mathscr{I}_{-\frac{1}{2}}(x).$$

ii. If $x=a-\ln\left(1+\sqrt{e^{2a}+1}\right)$, *then*

$$-\left(e^{-a}+\sqrt{e^{-2a}+1}\right)\left(e^{-a}\psi_{-i}^{-\frac{1}{2}}(x)-1\right)=\mathscr{I}_{-\frac{1}{2}}(x).$$

iii. If $x>a$, *then* $\frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x)-\frac{e^{a}}{2}<\mathscr{I}_{-\frac{1}{2}}(x)$.

The study of the modified Bessel function $\mathscr{I}_{\frac{3}{2}}$ gives the following proposition:

PROPOSITION 4.8. *The function* $\mathscr{I}_{\frac{3}{2}}$ *is even on* $\mathbb{R}$, *strictly decreasing on* $]-\infty,0]$, *strictly increasing on* $[0,+\infty[$, *and satisfies*

1. $\lim_{x\to-\infty}\mathscr{I}_{\frac{3}{2}}(x)=+\infty,\ \mathscr{I}_{\frac{3}{2}}(0)=1,\ \lim_{x\to+\infty}\mathscr{I}_{\frac{3}{2}}(x)=+\infty.$
2. *For all* $x\in\mathbb{R}$, *we have* $\frac{x^2}{3}\mathscr{I}_{\frac{3}{2}}(x)=\mathscr{I}_{-\frac{1}{2}}(x)-\mathscr{I}_{\frac{1}{2}}(x)$.
3. *For all* $x\in\mathbb{R}$, *we have* $\mathscr{I}_{\frac{1}{2}}(x)+\frac{x^2}{3}\leqslant\mathscr{I}_{-\frac{1}{2}}(x)$.

The study of the Dunkl kernel $\psi_{-i}^{\frac{1}{2}}$ gives the following proposition:

PROPOSITION 4.9. *Let* a_0 *be the root of the equation*

$$-(x+2)e^{-2x}+2x^2-3x+2=0,\quad -1,4<a_0<-1,3, \tag{10}$$

$$b_0:=\psi_{-i}^{\frac{1}{2}}(a_0)=\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(a_0)=\frac{2e^{a_0}}{a_0+2}\in]0,778,0,822[, \tag{11}$$

and $x\in\mathbb{R}$. *The function* $\psi_{-i}^{\frac{1}{2}}$ *is strictly decreasing on* $]-\infty,a_0]$, *strictly increasing on* $[a_0,+\infty[$, *and satisfies*

1. $\lim_{x\to-\infty}\psi^{\frac{1}{2}}_{-i}(x)=+\infty, \quad \psi^{\frac{1}{2}}_{-i}(0)=1, \quad \lim_{x\to+\infty}\psi^{\frac{1}{2}}_{-i}(x)=+\infty.$

2. $x\psi^{\frac{1}{2}}_{-i}(x)=\psi^{-\frac{1}{2}}_{-i}(x)-\mathscr{I}_{\frac{1}{2}}(x).$

3. $\frac{d}{dx}\psi^{\frac{1}{2}}_{-i}(x)=\psi^{\frac{1}{2}}_{-i}(x)-\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(x).$

4. *$\psi^{\frac{1}{2}}_{-i}$ is strictly convex on $\mathbb{R}$.*

5. $b_0\leqslant\psi^{\frac{1}{2}}_{-i}(x)\leqslant\frac{1}{3}\psi^{-\frac{1}{2}}_{-i}(x)+\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(x).$

6. *If $x>a_0$, then* $\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(x)<\psi^{\frac{1}{2}}_{-i}(x)\leqslant\frac{1}{3}\psi^{-\frac{1}{2}}_{-i}(x)+\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(x).$

7. *If $x<a_0$, then* $\psi^{\frac{1}{2}}_{-i}(x)<\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(x).$

8. *If $x>0$, then* $1<\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(x)+\frac{1}{3}<\psi^{\frac{1}{2}}_{-i}(x).$

9. *If $x<0$, then* $\psi^{\frac{1}{2}}_{-i}(x)<\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(x)+\frac{1}{3}.$

We are going to find the best possible constants l_1, l_2, r_1, $r_2\in\mathbb{R}$ such that

$$l_1\psi^{-\frac{1}{2}}_{-i}(x)+l_2\leqslant\mathscr{I}_{\frac{1}{2}}(x)\leqslant r_1\psi^{-\frac{1}{2}}_{-i}(x)+r_2 \tag{12}$$

hold for all $x\in I$; $I=\mathbb{R}$, $[a,b]$, a, $b\in\mathbb{R}$; $a<b$.

In order to solve this problem, we present the following propositions:

PROPOSITION 4.10. *Let a, $b\in\mathbb{R}$; $a<b$ and $x\in[a,b]$. We have*

1. $b_1\psi^{-\frac{1}{2}}_{-i}(x)+\frac{(-2b_1b+1)e^b-e^{-b}}{2b}\leqslant\mathscr{I}_{\frac{1}{2}}(x)\leqslant b_1\psi^{-\frac{1}{2}}_{-i}(x)+\frac{(-2b_1a+1)e^a-e^{-a}}{2a}.$

2. *If $b<-a_0$, then*

$$k_1(b)\psi^{-\frac{1}{2}}_{-i}(x)+\psi^{\frac{1}{2}}_{-i}(-b)\leqslant\mathscr{I}_{\frac{1}{2}}(x)\leqslant k_1(b)\psi^{-\frac{1}{2}}_{-i}(x)+\frac{(-2k_1(b)a+1)e^a-e^{-a}}{2a}.$$

3. *If $a\leqslant 0$, then*

$$k_1(a)\psi^{-\frac{1}{2}}_{-i}(x)+\psi^{\frac{1}{2}}_{-i}(-a)\leqslant\mathscr{I}_{\frac{1}{2}}(x)\leqslant k_1(a)\psi^{-\frac{1}{2}}_{-i}(x)+\frac{(-2k_1(a)b+1)e^b-e^{-b}}{2b}.$$

4. *If $0<a<-a_0<b$; $k_1(a)=k_1(b)$, then*

$$k_1(a)\psi^{-\frac{1}{2}}_{-i}(x)+\psi^{\frac{1}{2}}_{-i}(-a)\leqslant\mathscr{I}_{\frac{1}{2}}(x)\leqslant k_1(b)\psi^{-\frac{1}{2}}_{-i}(x)+\psi^{\frac{1}{2}}_{-i}(-b).$$

5. *If $a > -a_0$, then*

$$k_1(a)\psi_{-i}^{-\frac{1}{2}}(x) + \frac{(-2k_1(a)b+1)e^b - e^{-b}}{2b} \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant k_1(a)\psi_{-i}^{-\frac{1}{2}}(x) + \psi_{-i}^{\frac{1}{2}}(-a),$$

where a_0 is given by (10),

$$k_1(x) := \frac{xe^{-x}}{3}\mathscr{I}_{\frac{3}{2}}(x) = \frac{(x+1)e^{-2x}+x-1}{2x^2}, \quad x \neq 0, \tag{13}$$

and

$$b_1 := k_1(-a_0) = -\frac{a_0 e^{2a_0}}{a_0+2} \in]0,137, 0,142[. \tag{14}$$

PROPOSITION 4.11. *Let $a,\ b \in \mathbb{R}$; $a < b$ and $x \in [a,b]$. We have*

1. $$\frac{-e^{-2b}-2b_0be^{-b}+1}{2b}\psi_{-i}^{-\frac{1}{2}}(x)+b_0 \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant \frac{-e^{-2a}-2b_0ae^{-a}+1}{2a}\psi_{-i}^{-\frac{1}{2}}(x)+b_0.$$
2. *If $b < -a_0$, then*
$$k_1(b)\psi_{-i}^{-\frac{1}{2}}(x)+\psi_{-i}^{\frac{1}{2}}(-b) \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant \frac{-e^{-2a}-2ae^{-a}\psi_{-i}^{\frac{1}{2}}(-b)+1}{2a}\psi_{-i}^{-\frac{1}{2}}(x)+\psi_{-i}^{\frac{1}{2}}(-b).$$
3. *If $a < -a_0 < b$; $\psi_{-i}^{\frac{1}{2}}(-a) = \psi_{-i}^{\frac{1}{2}}(-b)$, then*
$$k_1(a)\psi_{-i}^{-\frac{1}{2}}(x) + \psi_{-i}^{\frac{1}{2}}(-a) \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant k_1(b)\psi_{-i}^{-\frac{1}{2}}(x) + \psi_{-i}^{\frac{1}{2}}(-b).$$
4. *If $a > -a_0$, then*
$$\frac{-e^{-2b}-2be^{-b}\psi_{-i}^{\frac{1}{2}}(-a)+1}{2b}\psi_{-i}^{-\frac{1}{2}}(x)+\psi_{-i}^{\frac{1}{2}}(-a) \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant k_1(a)\psi_{-i}^{-\frac{1}{2}}(x)+\psi_{-i}^{\frac{1}{2}}(-a),$$

where a_0, b_0 and k_1 are respectively given by (10), (11) *and* (13).

PROPOSITION 4.12. *Let $a,\ b \in \mathbb{R}$; $a < b$ and $x \in [a,b]$. We have*

1. $$\frac{\sinh b}{b(e^b - b_2)}\left(\psi_{-i}^{-\frac{1}{2}}(x) - b_2\right) \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant \frac{\sinh a}{a(e^a - b_2)}\left(\psi_{-i}^{-\frac{1}{2}}(x) - b_2\right).$$
2. *If $b < 0$, then*
$$\frac{b\cosh b - \sinh b}{b^2 e^b}\left(\psi_{-i}^{-\frac{1}{2}}(x) - q_1(b)\right) \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant \frac{\sinh a}{a(e^a - q_1(b))}\left(\psi_{-i}^{-\frac{1}{2}}(x) - q_1(b)\right).$$
3. *If $a < 0$, then*
$$\frac{a^2e^a}{a\cosh a - \sinh a}\mathscr{I}_{\frac{1}{2}}(x) + q_1(a) \leqslant \psi_{-i}^{-\frac{1}{2}}(x) \leqslant \frac{b(e^b - q_1(a))}{\sinh b}\mathscr{I}_{\frac{1}{2}}(x) + q_1(a).$$

4. *If* $0 < b < -a_0$, *then*

$$\frac{b\cosh b - \sinh b}{b^2 e^b}\left(\psi_{-i}^{-\frac{1}{2}}(x) - q_1(b)\right) \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant \frac{\sinh a}{a(e^a - q_1(b))}\left(\psi_{-i}^{-\frac{1}{2}}(x) - q_1(b)\right).$$

5. *If* $0 < a < -a_0 < b$; $q_1(a) = q_1(b)$, *then*

$$\frac{a\cosh a - \sinh a}{a^2 e^a}\left(\psi_{-i}^{-\frac{1}{2}}(x) - q_1(a)\right) \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant \frac{b\cosh b - \sinh b}{b^2 e^b}\left(\psi_{-i}^{-\frac{1}{2}}(x) - q_1(b)\right).$$

6. *If* $a > -a_0$, *then*

$$\frac{\sinh b}{b(e^b - q_1(a))}\left(\psi_{-i}^{-\frac{1}{2}}(x) - q_1(a)\right) \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant \frac{a\cosh a - \sinh a}{a^2 e^a}\left(\psi_{-i}^{-\frac{1}{2}}(x) - q_1(a)\right),$$

where a_0, q_1 *and* b_2 *are respectively given by* (10),

$$q_1(x) := -3\frac{e^x \psi_{-i}^{\frac{1}{2}}(-x)}{x\mathscr{I}_{\frac{3}{2}}(x)} = \frac{-e^{2x} + 2x + 1}{(x-1)e^x + (x+1)e^{-x}}, \quad x \neq 0, \tag{15}$$

and

$$b_2 := q_1(-a_0) = 2\frac{e^{-a_0}}{a_0} \in]-5,794, -5,645[. \tag{16}$$

The solution of Problem (12) can be stated as follows.

COROLLARY 4.13. *Let* a, $b \in \mathbb{R}$; $a < b$ *and* $x \in \mathbb{R}$.

1. *If* $a \leqslant 0$, *then*

$$k_1(a)\psi_{-i}^{-\frac{1}{2}}(x) + \psi_{-i}^{\frac{1}{2}}(-a) \leqslant \mathscr{I}_{\frac{1}{2}}(x).$$

2. *If* $a < 0$, *then*

$$\frac{a\cosh a - \sinh a}{a^2 e^a}\left(\psi_{-i}^{-\frac{1}{2}}(x) - q_1(a)\right) \leqslant \mathscr{I}_{\frac{1}{2}}(x).$$

3. *If* $a \leqslant 0$ *and* $x \in [a,b]$, *then*

$$\sup\left\{b_1\psi_{-i}^{-\frac{1}{2}}(x) + \frac{(-2b_1 b + 1)e^b - e^{-b}}{2b}, k_1(a)\psi_{-i}^{-\frac{1}{2}}(x) + \psi_{-i}^{\frac{1}{2}}(-a),\right.$$

$$\left.\frac{-e^{-2b} - 2b_0 b e^{-b} + 1}{2b}\psi_{-i}^{-\frac{1}{2}}(x) + b_0, \frac{\sinh b}{b(e^b - b_2)}\left(\psi_{-i}^{-\frac{1}{2}}(x) - b_2\right)\right\} \leqslant \mathscr{I}_{\frac{1}{2}}(x)$$

$$\leqslant \inf\left\{b_1\psi_{-i}^{-\frac{1}{2}}(x) + \frac{(-2b_1 a + 1)e^a - e^{-a}}{2a}, k_1(a)\psi_{-i}^{-\frac{1}{2}}(x) + \frac{(-2k_1(a)b + 1)e^b - e^{-b}}{2b}\right\},$$

where k_1, q_1, b_1, b_0 *and* b_2 *are respectively given by* (13), (15), (14), (11) *and* (16).

In order to solve Problem (5), we present the following propositions:

PROPOSITION 4.14. *Let* $a,\ b \in \mathbb{R}$*;* $a < b$ *and* $x \in [a,b]$*. We have*

1. $\dfrac{1}{3}\psi_{-i}^{-\frac{1}{2}}(x) + \psi_{-i}^{\frac{1}{2}}(b) - \dfrac{e^b}{3} \leqslant \psi_{-i}^{\frac{1}{2}}(x) \leqslant \dfrac{1}{3}\psi_{-i}^{-\frac{1}{2}}(x) + \psi_{-i}^{\frac{1}{2}}(a) - \dfrac{e^a}{3}.$
2. *If* $b < 0$*, then*
$$\theta_1(b)\psi_{-i}^{-\frac{1}{2}}(x) + \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(b) \leqslant \psi_{-i}^{\frac{1}{2}}(x) \leqslant \theta_1(b)\psi_{-i}^{-\frac{1}{2}}(x) + \psi_{-i}^{\frac{1}{2}}(a) - e^a\theta_1(b).$$
3. *If* $a_0 < a < 0 < b$*;* $\theta_1(a) = \theta_1(b)$*, then*
$$\theta_1(a)\psi_{-i}^{-\frac{1}{2}}(x) + \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(a) \leqslant \psi_{-i}^{\frac{1}{2}}(x) \leqslant \theta_1(b)\psi_{-i}^{-\frac{1}{2}}(x) + \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(b).$$
4. *If* $a > 0$*, then*
$$\theta_1(a)\psi_{-i}^{-\frac{1}{2}}(x) + \psi_{-i}^{\frac{1}{2}}(b) - \theta_1(a)e^b \leqslant \psi_{-i}^{\frac{1}{2}}(x) \leqslant \theta_1(a)\psi_{-i}^{-\frac{1}{2}}(x) + \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(a).$$
5. *If* $a \leqslant a_0$*, then*
$$\theta_1(a)\psi_{-i}^{-\frac{1}{2}}(x) + \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(a) \leqslant \psi_{-i}^{\frac{1}{2}}(x) \leqslant \theta_1(a)\psi_{-i}^{-\frac{1}{2}}(x) + \psi_{-i}^{\frac{1}{2}}(b) - \theta_1(a)e^b,$$

where a_0 *and* θ_1 *are respectively given by* (10) *and* (19).

PROPOSITION 4.15. *Let* $a,\ b \in \mathbb{R}$*;* $a < b$ *and* $x \in [a,b]$*. We have*

1. $e^{-b}\left(\psi_{-i}^{\frac{1}{2}}(b) - \dfrac{2}{3}\right)\psi_{-i}^{-\frac{1}{2}}(x) + \dfrac{2}{3} \leqslant \psi_{-i}^{\frac{1}{2}}(x) \leqslant e^{-a}\left(\psi_{-i}^{\frac{1}{2}}(a) - \dfrac{2}{3}\right)\psi_{-i}^{-\frac{1}{2}}(x) + \dfrac{2}{3}.$
2. *If* $b < 0$*, then*
$$\theta_1(b)\psi_{-i}^{-\frac{1}{2}}(x) + \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(b) \leqslant \psi_{-i}^{\frac{1}{2}}(x) \leqslant e^{-a}\left(\psi_{-i}^{\frac{1}{2}}(a) - \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(b)\right)\psi_{-i}^{-\frac{1}{2}}(x) + \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(b).$$
3. *If* $b > 0$ *and* $a = -b$*, then*
$$\theta_1(-b)\psi_{-i}^{-\frac{1}{2}}(x) + \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(b) \leqslant \psi_{-i}^{\frac{1}{2}}(x) \leqslant \theta_1(b)\psi_{-i}^{-\frac{1}{2}}(x) + \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(b).$$
4. *If* $a > 0$*, then*
$$e^{-b}\left(\psi_{-i}^{\frac{1}{2}}(b) - \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(a)\right)\psi_{-i}^{-\frac{1}{2}}(x) + \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(a) \leqslant \psi_{-i}^{\frac{1}{2}}(x) \leqslant \theta_1(a)\psi_{-i}^{-\frac{1}{2}}(x) + \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(a),$$

where θ_1 *is given by* (19).

PROPOSITION 4.16. *Let* $a,\ b\in\mathbb{R}$*;* $a<b$ *and* $x\in[a,b]$*. We have*

1. $\dfrac{\psi^{\frac{1}{2}}_{-i}(b)}{e^b+2}\left(\psi^{-\frac{1}{2}}_{-i}(x)+2\right)\leqslant\psi^{\frac{1}{2}}_{-i}(x)\leqslant\dfrac{\psi^{\frac{1}{2}}_{-i}(a)}{e^a+2}\left(\psi^{-\frac{1}{2}}_{-i}(x)+2\right).$

2. *If* $b<0$*, then*

$$\theta_1(b)\psi^{-\frac{1}{2}}_{-i}(x)+\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(b)\leqslant\psi^{\frac{1}{2}}_{-i}(x)\leqslant\frac{\psi^{\frac{1}{2}}_{-i}(a)}{e^ah_1(b)+1}\left(h_1(b)\psi^{-\frac{1}{2}}_{-i}(x)+1\right).$$

3. *If* $a<a_0$*, then*

$$\frac{1}{\theta_1(a)}\psi^{\frac{1}{2}}_{-i}(x)-\frac{1}{h_1(a)}\leqslant\psi^{-\frac{1}{2}}_{-i}(x)\leqslant\frac{h_1(a)e^b+1}{h_1(a)\psi^{\frac{1}{2}}_{-i}(b)}\psi^{\frac{1}{2}}_{-i}(x)-\frac{1}{h_1(a)}.$$

4. *If* $a_0<a<0<b$*;* $h_1(a)=h_1(b)$*, then*

$$\theta_1(a)\psi^{-\frac{1}{2}}_{-i}(x)+\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(a)\leqslant\psi^{\frac{1}{2}}_{-i}(x)\leqslant\theta_1(b)\psi^{-\frac{1}{2}}_{-i}(x)+\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(b).$$

5. *If* $a>0$*, then*

$$\frac{\psi^{\frac{1}{2}}_{-i}(b)}{h_1(a)e^b+1}\left(h_1(a)\psi^{-\frac{1}{2}}_{-i}(x)+1\right)\leqslant\psi^{\frac{1}{2}}_{-i}(x)\leqslant\theta_1(a)\psi^{-\frac{1}{2}}_{-i}(x)+\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(a),$$

where a_0*,* θ_1 *and* h_1 *are respectively given by* (10)*,* (19) *and*

$$h_1(x):=\frac{3\theta_1(x)}{2\mathscr{I}_{\frac{3}{2}}(x)}=\frac{-(x+2)e^{-2x}+2x^2-3x+2}{2[(x-1)e^x+(x+1)e^{-x}]},\quad x\neq 0. \tag{17}$$

In view of [14] we deduce the following Sonine integral formulas:

PROPOSITION 4.17. *For all* $\alpha>\beta>-1$ *and* $x\in\mathbb{R}$*, we have*

1. $\mathscr{I}_\alpha(x)=\dfrac{2}{B(\alpha-\beta,\beta+1)}\displaystyle\int_0^1\mathscr{I}_\beta(xt)t^{2\beta+1}(1-t^2)^{\alpha-\beta-1}\,dt$

$$=\frac{2}{B(\alpha-\beta,\beta+1)}\int_0^{\frac{\pi}{2}}\mathscr{I}_\beta(x\sin\theta)(\sin\theta)^{2\beta+1}(\cos\theta)^{2(\alpha-\beta)-1}\,d\theta.$$

2. $$\psi^{\alpha}_{-i}(x) = \frac{1}{B(\alpha-\beta,\beta+1)}\int_{-1}^{1}\psi^{\beta}_{-i}(xt)|t|^{2\beta+1}(1+t)(1-t^2)^{\alpha-\beta-1}\,dt$$
$$= \frac{1}{B(\alpha-\beta,\beta+1)}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\psi^{\beta}_{-i}(x\sin\theta)(1+\sin\theta)(\sin(|\theta|))^{2\beta+1}(\cos\theta)^{2(\alpha-\beta)-1}\,d\theta$$
$$= \frac{2}{B(\alpha-\beta,\beta+1)}\int_{0}^{1}\left[\mathscr{I}_{\beta}(xt)+\frac{xt^2}{2(\beta+1)}\mathscr{I}_{\beta+1}(xt)\right]t^{2\beta+1}(1-t^2)^{\alpha-\beta-1}\,dt$$
$$= \frac{2}{B(\alpha-\beta,\beta+1)}\int_{0}^{\frac{\pi}{2}}\left[\mathscr{I}_{\beta}(x\sin\theta)+\frac{x(\sin\theta)^2}{2(\beta+1)}\mathscr{I}_{\beta+1}(x\sin\theta)\right]$$
$$\times(\sin\theta)^{2\beta+1}(\cos\theta)^{2(\alpha-\beta)-1}\,d\theta.$$

5. Concluding remarks

REMARKS 5.1. Let $b>0$, $0<a<b$ and $x\in\mathbb{R}$. We have

1. (a) $1 < \dfrac{3\sinh(2b)-4b^2\sinh b-6b}{-6b\cosh b+2b^2\sinh b+6b} < \cosh b.$

   (b) If $|x|<x_0$, then $\dfrac{1}{3}\mathscr{I}_{-\frac{1}{2}}(x)+\dfrac{2}{3} < u_1(b)\mathscr{I}_{-\frac{1}{2}}(x)+u_4(b).$

   (c) If $x_0<|x|$, then $u_1(b)\mathscr{I}_{-\frac{1}{2}}(x)+u_4(b) < \dfrac{1}{3}\mathscr{I}_{-\frac{1}{2}}(x)+\dfrac{2}{3}.$

   (d) If $|x|=x_0$, then $\dfrac{1}{3}\mathscr{I}_{-\frac{1}{2}}(x)+\dfrac{2}{3} = u_1(b)\mathscr{I}_{-\frac{1}{2}}(x)+u_4(b),$

   where $\cosh(x_0) := \dfrac{3\sinh(2b)-4b^2\sinh b-6b}{-6b\cosh b+2b^2\sinh b+6b}$, $0<x_0<b.$

2. $(1-u_4(b))\mathscr{I}_{-\frac{1}{2}}(x)+u_4(b) \leqslant u_1(b)\mathscr{I}_{-\frac{1}{2}}(x)+1-u_1(b).$

3. $\dfrac{1}{1-w_1(b)}\left(\mathscr{I}_{-\frac{1}{2}}(x)-w_1(b)\right) \leqslant u_1(b)\mathscr{I}_{-\frac{1}{2}}(x)+1-u_1(b).$

4. If $x\in[-b,b]$, then $\dfrac{1}{3}\mathscr{I}_{-\frac{1}{2}}(x)+\dfrac{3\sinh b-b\cosh b}{3b} \leqslant \dfrac{3\sinh b-2b}{3b\cosh b}\mathscr{I}_{-\frac{1}{2}}(x)+\dfrac{2}{3}.$

5. If $x\in[-b,b]$, then $\dfrac{\sinh b}{b(\cosh b+2)}\left(\mathscr{I}_{-\frac{1}{2}}(x)+2\right) \leqslant \dfrac{3\sinh b-2b}{3b\cosh b}\mathscr{I}_{-\frac{1}{2}}(x)+\dfrac{2}{3}.$

6. (a) $1 < \dfrac{(b^2+3)\sinh(2b)-6b(\cosh b)^2}{3\sinh(2b)-4b^2\sinh b-6b} < \cosh b.$

   (b) If $|x|<x_1$, then $\dfrac{3\sinh b-2b}{3b\cosh b}\mathscr{I}_{-\frac{1}{2}}(x)+\dfrac{2}{3} < u_1(b)\mathscr{I}_{-\frac{1}{2}}(x)+1-u_1(b).$

   (c) If $x_1<|x|\leqslant b$, then $u_1(b)\mathscr{I}_{-\frac{1}{2}}(x)+1-u_1(b) < \dfrac{3\sinh b-2b}{3b\cosh b}\mathscr{I}_{-\frac{1}{2}}(x)+\dfrac{2}{3}.$

(d) If $|x| = x_1$, then $\dfrac{3\sinh b - 2b}{3b\cosh b}\mathscr{I}_{-\frac{1}{2}}(x) + \dfrac{2}{3} = u_1(b)\mathscr{I}_{-\frac{1}{2}}(x) + 1 - u_1(b)$,

where $\cosh(x_1) := \dfrac{(b^2+3)\sinh(2b) - 6b(\cosh b)^2}{3\sinh(2b) - 4b^2\sinh b - 6b}$, $0 < x_1 < b$.

7. If $x \in [-b,-a] \cup [a,b]$, then

$$u_1(a)\mathscr{I}_{-\frac{1}{2}}(x) + \frac{\sinh b}{b} - u_1(a)\cosh b \leqslant \frac{\sinh b - u_4(a)b}{b\cosh b}\mathscr{I}_{-\frac{1}{2}}(x) + u_4(a).$$

8. If $x \in [-b,-a] \cup [a,b]$, then

$$\frac{\sinh b}{b(\cosh b - w_1(a))}\left(\mathscr{I}_{-\frac{1}{2}}(x) - w_1(a)\right) \leqslant \frac{\sinh b - u_4(a)b}{b\cosh b}\mathscr{I}_{-\frac{1}{2}}(x) + u_4(a).$$

9. If $x \in [-b,-a] \cup [a,b]$, then

$$\frac{3\sinh b - 2b}{3b\cosh b}\mathscr{I}_{-\frac{1}{2}}(x) + \frac{2}{3} \leqslant \frac{\sinh b - u_4(a)b}{b\cosh b}\mathscr{I}_{-\frac{1}{2}}(x) + u_4(a).$$

REMARKS 5.2. Let $a,\ b \in \mathbb{R}$; $a < b$ and $x \in [a,b]$. We have

1. $\dfrac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x) + \dfrac{e^{-a}+e^{a-2b}}{2} \leqslant \dfrac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x) + \dfrac{e^{-a}}{2}$.

2. $\dfrac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x) + \dfrac{e^{-a}+e^{a-2b}}{2} \leqslant \dfrac{1+e^{-2a}}{2}\psi_{-i}^{-\frac{1}{2}}(x)$.

3. $\dfrac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x) + \dfrac{e^{-a}+e^{a-2b}}{2} \leqslant \dfrac{e^{-2a}-2e^{-(a+b)}+1}{2}\psi_{-i}^{-\frac{1}{2}}(x) + e^{-b}$.

4. If $b \geqslant 0$, then

$$\frac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x) + \frac{e^{-a}+e^{a-2b}}{2} \leqslant \frac{\cosh a}{e^a\sinh b + 1}\left(\sinh b\,\psi_{-i}^{-\frac{1}{2}}(x) + 1\right).$$

5. If $b < 0$, then

$$\frac{\cosh a}{e^a\sinh b + 1}\left(\sinh b\,\psi_{-i}^{-\frac{1}{2}}(x) + 1\right) \leqslant \frac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x) + \frac{e^{-a}+e^{a-2b}}{2}.$$

6. $\dfrac{e^{-2b}-2e^{-(a+b)}+1}{2}\psi_{-i}^{-\frac{1}{2}}(x) + e^{-a} \leqslant \dfrac{1-e^{-2a}}{2}\psi_{-i}^{-\frac{1}{2}}(x) + \dfrac{e^{-b}+e^{b-2a}}{2}$.

7. If $0 < a$ or $(a < 0$ and $b < -\ln(-\sinh a))$, then

$$\frac{e^{-2b}-2e^{-(a+b)}+1}{2}\psi_{-i}^{-\frac{1}{2}}(x) + e^{-a} \leqslant \frac{\cosh b}{\sinh a e^b + 1}\left(\sinh a\,\psi_{-i}^{-\frac{1}{2}}(x) + 1\right).$$

8. $\frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-b}}{2}\leqslant\frac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-b}$.

9. $\frac{1+e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x)\leqslant\frac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-b}$.

10. If $a<0$ and $-\ln(-\sinh a)<b$, then

$$\frac{\cosh b}{\sinh ae^{b}+1}\left(\sinh a\psi_{-i}^{-\frac{1}{2}}(x)+1\right)\leqslant\frac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-b}.$$

11. (a) $a<\frac{a+b}{2}<\frac{a+b}{2}+\ln\left(\cosh\left(\frac{b-a}{2}\right)\right)<b$.

(b) If $a\leqslant x<\frac{a+b}{2}+\ln\left(\cosh\left(\frac{b-a}{2}\right)\right)$, then

$$\frac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-a}+e^{a-2b}}{2}<\frac{e^{-2b}-2e^{-(a+b)}+1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-a}.$$

(c) If $\frac{a+b}{2}+\ln\left(\cosh\left(\frac{b-a}{2}\right)\right)<x\leqslant b$, then

$$\frac{e^{-2b}-2e^{-(a+b)}+1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-a}<\frac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-a}+e^{a-2b}}{2}.$$

(d) If $x=\frac{a+b}{2}+\ln\left(\cosh\left(\frac{b-a}{2}\right)\right)$, then

$$\frac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x)+\frac{e^{-a}+e^{a-2b}}{2}=\frac{e^{-2b}-2e^{-(a+b)}+1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-a}.$$

12. (a) If $a\leqslant x<x_2$, then

$$\frac{\cosh a}{e^{a}\sinh b+1}\left(\sinh b\psi_{-i}^{-\frac{1}{2}}(x)+1\right)<\frac{e^{-2b}-2e^{-(a+b)}+1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-a}.$$

(b) If $x_2<x\leqslant b$, then

$$\frac{e^{-2b}-2e^{-(a+b)}+1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-a}<\frac{\cosh a}{e^{a}\sinh b+1}\left(\sinh b\psi_{-i}^{-\frac{1}{2}}(x)+1\right).$$

(c) If $x=x_2$, then

$$\frac{e^{-2b}-2e^{-(a+b)}+1}{2}\psi_{-i}^{-\frac{1}{2}}(x)+e^{-a}=\frac{\cosh a}{e^{a}\sinh b+1}\left(\sinh b\psi_{-i}^{-\frac{1}{2}}(x)+1\right),$$

where $e^{x_2}:=\frac{4\cosh\left(\frac{a+b}{2}\right)}{2\cosh(b-a)+e^{\frac{a-5b}{2}}(3e^{b-a}-1)}$.

13. (a) $a < \frac{a+b}{2} - \ln\left(\cosh\left(\frac{b-a}{2}\right)\right) < \frac{a+b}{2} < b.$

(b) If $a \leqslant x < \frac{a+b}{2} - \ln\left(\cosh\left(\frac{b-a}{2}\right)\right)$, then

$$\frac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x) + e^{-b} < \frac{1-e^{-2a}}{2}\psi_{-i}^{-\frac{1}{2}}(x) + e^{-a}.$$

(c) If $\frac{a+b}{2} - \ln\left(\cosh\left(\frac{b-a}{2}\right)\right) < x \leqslant b$, then

$$\frac{1-e^{-2a}}{2}\psi_{-i}^{-\frac{1}{2}}(x) + e^{-a} < \frac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x) + e^{-b}.$$

(d) If $x = \frac{a+b}{2} - \ln\left(\cosh\left(\frac{b-a}{2}\right)\right)$, then

$$\frac{1-e^{-2a}}{2}\psi_{-i}^{-\frac{1}{2}}(x) + e^{-a} = \frac{1-e^{-2b}}{2}\psi_{-i}^{-\frac{1}{2}}(x) + e^{-b}.$$

REMARK 5.3. Let $b > 0$.

1. If $x < \ln 2 - \ln b$, $x \neq -\ln b$, then $\frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x) < \frac{1-b^2}{2}\psi_{-i}^{-\frac{1}{2}}(x) + b < \mathscr{I}_{-\frac{1}{2}}(x)$.
2. If $x = -\ln b$, then $\frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x) < \frac{1-b^2}{2}\psi_{-i}^{-\frac{1}{2}}(x) + b = \mathscr{I}_{-\frac{1}{2}}(x)$.
3. If $x = \ln 2 - \ln b$, then $\frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x) = \frac{1-b^2}{2}\psi_{-i}^{-\frac{1}{2}}(x) + b < \mathscr{I}_{-\frac{1}{2}}(x)$.
4. If $x > \ln 2 - \ln b$, then $\frac{1-b^2}{2}\psi_{-i}^{-\frac{1}{2}}(x) + b < \frac{1}{2}\psi_{-i}^{-\frac{1}{2}}(x) < \mathscr{I}_{-\frac{1}{2}}(x)$.

REMARKS 5.4. Let $a,\ b \in \mathbb{R}$; $a < b$ and $x \in [a,b]$.

1. If $b < -a_0$, then

$$\frac{-e^{-2a} - 2ae^{-a}\psi_{-i}^{\frac{1}{2}}(-b) + 1}{2a}\psi_{-i}^{-\frac{1}{2}}(x) + \psi_{-i}^{\frac{1}{2}}(-b) \leqslant \frac{-e^{-2a} - 2b_0ae^{-a} + 1}{2a}\psi_{-i}^{-\frac{1}{2}}(x) + b_0.$$

2. If $0 < b < -a_0$, then

$$\frac{\sinh a}{a(e^a - q_1(b))}\left(\psi_{-i}^{-\frac{1}{2}}(x) - q_1(b)\right) \leqslant \frac{\sinh a}{a(e^a - b_2)}\left(\psi_{-i}^{-\frac{1}{2}}(x) - b_2\right).$$

3. If $a \leqslant 0$, then

$$b_1\psi_{-i}^{-\frac{1}{2}}(x) + \frac{(-2b_1a+1)e^a - e^{-a}}{2a} \leqslant \frac{-e^{-2a} - 2b_0ae^{-a} + 1}{2a}\psi_{-i}^{-\frac{1}{2}}(x) + b_0.$$

4. If $a \leqslant 0$, then $b_1\psi_{-i}^{-\frac{1}{2}}(x)+\dfrac{(-2b_1a+1)e^a-e^{-a}}{2a} \leqslant \dfrac{\sinh a}{a(e^a-b_2)}\left(\psi_{-i}^{-\frac{1}{2}}(x)-b_2\right)$.

5. If $b<-a_0$, then $b_1\psi_{-i}^{-\frac{1}{2}}(x)+\dfrac{(-2b_1b+1)e^b-e^{-b}}{2b} \leqslant k_1(b)\psi_{-i}^{-\frac{1}{2}}(x)+\psi_{-i}^{\frac{1}{2}}(-b)$,

where a_0, b_0, q_1, b_2, b_1 and k_1 are respectively given by (10), (11), (15), (16), (14) and (13).

REMARKS 5.5. Let $a,\, b\in\mathbb{R}$; $a<b$ and $x\in[a,b]$. We have

1. $\dfrac{1}{3}\psi_{-i}^{-\frac{1}{2}}(x)+\psi_{-i}^{\frac{1}{2}}(a)-\dfrac{e^a}{3} \leqslant e^{-a}\left(\psi_{-i}^{\frac{1}{2}}(a)-\dfrac{2}{3}\right)\psi_{-i}^{-\frac{1}{2}}(x)+\dfrac{2}{3}$.

2. $\dfrac{1}{3}\psi_{-i}^{-\frac{1}{2}}(x)+\psi_{-i}^{\frac{1}{2}}(a)-\dfrac{e^a}{3} \leqslant \dfrac{\psi_{-i}^{\frac{1}{2}}(a)}{e^a+2}\left(\psi_{-i}^{-\frac{1}{2}}(x)+2\right)$.

3. $\dfrac{1}{3}\psi_{-i}^{-\frac{1}{2}}(x)+\psi_{-i}^{\frac{1}{2}}(b)-\dfrac{e^b}{3} \leqslant e^{-b}\left(\psi_{-i}^{\frac{1}{2}}(b)-\dfrac{2}{3}\right)\psi_{-i}^{-\frac{1}{2}}(x)+\dfrac{2}{3}$.

4. $\dfrac{1}{3}\psi_{-i}^{-\frac{1}{2}}(x)+\psi_{-i}^{\frac{1}{2}}(b)-\dfrac{e^b}{3} \leqslant \dfrac{\psi_{-i}^{\frac{1}{2}}(b)}{e^b+2}\left(\psi_{-i}^{-\frac{1}{2}}(x)+2\right)$.

5. If $b\geqslant 0$, then $\dfrac{\psi_{-i}^{\frac{1}{2}}(b)}{e^b+2}\left(\psi_{-i}^{-\frac{1}{2}}(x)+2\right) \leqslant e^{-b}\left(\psi_{-i}^{\frac{1}{2}}(b)-\dfrac{2}{3}\right)\psi_{-i}^{-\frac{1}{2}}(x)+\dfrac{2}{3}$.

6. If $b\leqslant 0$, then $e^{-b}\left(\psi_{-i}^{\frac{1}{2}}(b)-\dfrac{2}{3}\right)\psi_{-i}^{-\frac{1}{2}}(x)+\dfrac{2}{3} \leqslant \dfrac{\psi_{-i}^{\frac{1}{2}}(b)}{e^b+2}\left(\psi_{-i}^{-\frac{1}{2}}(x)+2\right)$.

7. If $0<a$, then

$$\theta_1(a)\psi_{-i}^{-\frac{1}{2}}(x)+\psi_{-i}^{\frac{1}{2}}(b)-\theta_1(a)e^b \leqslant e^{-b}\left(\psi_{-i}^{\frac{1}{2}}(b)-\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(a)\right)\psi_{-i}^{-\frac{1}{2}}(x)+\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(a).$$

8. If $0<a$, then

$$\frac{\psi_{-i}^{\frac{1}{2}}(b)}{h_1(a)e^b+1}\left(h_1(a)\psi_{-i}^{-\frac{1}{2}}(x)+1\right) \leqslant e^{-b}\left(\psi_{-i}^{\frac{1}{2}}(b)-\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(a)\right)\psi_{-i}^{-\frac{1}{2}}(x)+\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(a).$$

9. If $b<0$, then

$$\theta_1(b)\psi_{-i}^{-\frac{1}{2}}(x)+\psi_{-i}^{\frac{1}{2}}(a)-e^a\theta_1(b) \leqslant e^{-a}\left(\psi_{-i}^{\frac{1}{2}}(a)-\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(b)\right)\psi_{-i}^{-\frac{1}{2}}(x)+\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(b).$$

10. If $a_0\leqslant b<0$, then

$$\theta_1(b)\psi_{-i}^{-\frac{1}{2}}(x)+\psi_{-i}^{\frac{1}{2}}(a)-e^a\theta_1(b) \leqslant \frac{\psi_{-i}^{\frac{1}{2}}(a)}{e^a h_1(b)+1}\left(h_1(b)\psi_{-i}^{-\frac{1}{2}}(x)+1\right).$$

11. If $b \leqslant a_0$, then

$$\frac{\psi_{-i}^{\frac{1}{2}}(a)}{e^a h_1(b)+1}\left(h_1(b)\psi_{-i}^{-\frac{1}{2}}(x)+1\right) \leqslant \theta_1(b)\psi_{-i}^{-\frac{1}{2}}(x)+\psi_{-i}^{\frac{1}{2}}(a)-e^a\theta_1(b).$$

12. If $a \leqslant a_0$ and $b=-a$, then

$$\theta_1(-a)\psi_{-i}^{-\frac{1}{2}}(x)+\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(a) \leqslant \theta_1(a)\psi_{-i}^{-\frac{1}{2}}(x)+\psi_{-i}^{\frac{1}{2}}(-a)-\theta_1(a)e^{-a}.$$

13. If $a_1 < a < a_0$ and $b=-a$, then

$$\theta_1(-a)\psi_{-i}^{-\frac{1}{2}}(x)+\frac{2}{3}\mathscr{I}_{\frac{3}{2}}(a) \leqslant \frac{\psi_{-i}^{\frac{1}{2}}(-a)}{h_1(a)e^{-a}+1}\left(h_1(a)\psi_{-i}^{-\frac{1}{2}}(x)+1\right),$$

where $e^{-a_1}h_1(a_1) := -1$, $a_1 < a_0$.

REMARKS 5.6. From [3], for all $\alpha \geqslant -\frac{1}{2}$ and $x \in \mathbb{R}$, we have

$$\mathscr{I}_{\alpha+1}(x) \leqslant \frac{2(\alpha+1)a_1\mathscr{I}_\alpha(x)+1}{(2\alpha+1)a_1+a_2},$$

where $\left(0 < a_1 < \frac{1}{2} \text{ and } a_2 := 4a_1(1-a_1^2)\right)$ or $\left(a_1 \geqslant \frac{1}{2} \text{ and } a_2 := a_1+1\right)$.
We can show the refinements of these inequalities as follows.

1. Let $\alpha \geqslant -\frac{1}{2}$, $a_1 > \frac{1}{2}$, $a_2 := a_1+1 > \frac{3}{2}$ and $x \in \mathbb{R}\setminus\{0\}$. We have
   (a) $\frac{a_1}{a_2} > \frac{1}{3}$, $\frac{1}{a_2} < \frac{2}{3}$, $\frac{2(\alpha+1)a_1}{(2\alpha+1)a_1+a_2} > \frac{\alpha+1}{\alpha+2}$, $\frac{1}{(2\alpha+1)a_1+a_2} < \frac{1}{\alpha+2}$.
   (b) $\frac{(\alpha+1)\mathscr{I}_\alpha(x)+1}{\alpha+2} < \frac{2(\alpha+1)a_1\mathscr{I}_\alpha(x)+1}{(2\alpha+1)a_1+a_2}$.
   (c) $\frac{(\alpha+1)\mathscr{I}_\alpha(0)+1}{\alpha+2} = \frac{2(\alpha+1)a_1\mathscr{I}_\alpha(0)+1}{(2\alpha+1)a_1+a_2} = 1$.

2. Let $a_1 \geqslant \frac{1}{2}$, $a_2 := a_1+1 \geqslant \frac{3}{2}$ and $b > 0$. We have
   (a) $\frac{a_1}{a_2} \geqslant \frac{1}{3} > u_1(b)$, $\frac{1}{a_2} \leqslant \frac{2}{3} < u_4(b)$.
   (b) If $|x| < x_3$, then $\frac{a_1}{a_2}\mathscr{I}_{-\frac{1}{2}}(x)+\frac{1}{a_2} < u_1(b)\mathscr{I}_{-\frac{1}{2}}(x)+u_4(b)$.
   (c) If $|x| > x_3$, then $u_1(b)\mathscr{I}_{-\frac{1}{2}}(x)+u_4(b) < \frac{a_1}{a_2}\mathscr{I}_{-\frac{1}{2}}(x)+\frac{1}{a_2}$.

(d) If $|x| = x_3$, then $u_1(b)\mathscr{I}_{-\frac{1}{2}}(x) + u_4(b) = \dfrac{a_1}{a_2}\mathscr{I}_{-\frac{1}{2}}(x) + \dfrac{1}{a_2}$,

where $\cosh(x_3) := \dfrac{a_2 u_4(b) - 1}{a_1 - a_2 u_1(b)}$, $0 < x_3 < b$.

3. Let $a_1 \in \left]0, \frac{1}{2}\right[$ and $a_2 := 4a_1(1 - a_1^2) \in \left]0, \frac{3}{2}\right[$. We have

(a) $\dfrac{1}{4} < \dfrac{a_1}{a_2} < \dfrac{1}{3}$, $\dfrac{1}{a_2} > \dfrac{2}{3}$, $\dfrac{8a_1^3 - 8a_1 + 3}{a_1(1 - 4a_1^2)} > 1$.

(b) If $|x| < x_4$, then $\dfrac{1}{3}\mathscr{I}_{-\frac{1}{2}}(x) + \dfrac{2}{3} < \dfrac{a_1}{a_2}\mathscr{I}_{-\frac{1}{2}}(x) + \dfrac{1}{a_2}$.

(c) If $|x| > x_4$, then $\dfrac{a_1}{a_2}\mathscr{I}_{-\frac{1}{2}}(x) + \dfrac{1}{a_2} < \dfrac{1}{3}\mathscr{I}_{-\frac{1}{2}}(x) + \dfrac{2}{3}$.

(d) If $|x| = x_4$, then $\dfrac{1}{3}\mathscr{I}_{-\frac{1}{2}}(x) + \dfrac{2}{3} = \dfrac{a_1}{a_2}\mathscr{I}_{-\frac{1}{2}}(x) + \dfrac{1}{a_2}$,

where $\cosh(x_4) := \dfrac{8a_1^3 - 8a_1 + 3}{a_1(1 - 4a_1^2)}$, $x_4 > 0$.

6. Proofs

Proof of Proposition 4.1. Let $x > 0$, $c \in \mathbb{R}$ and

$$u(x) := \mathscr{I}_{\frac{1}{2}}(x) - c\mathscr{I}_{-\frac{1}{2}}(x) = \frac{\sinh x - cx\cosh x}{x} = -\frac{(cx-1)e^x + (cx+1)e^{-x}}{2x}.$$

We have $u'(x) = \dfrac{e^{-x}}{2x^2}[-(cx^2 - x + 1)e^{2x} + cx^2 + x + 1]$.

$$u'(x) = 0 \text{ if and only if } c = u_1(x) := \frac{\mathscr{I}_{\frac{3}{2}}(x)}{3\mathscr{I}_{\frac{1}{2}}(x)} = \frac{x\cosh x - \sinh x}{x^2 \sinh x} = \frac{(x-1)e^{2x} + x + 1}{x^2(e^{2x} - 1)}.$$

$$u_1'(x) = \frac{u_2(x)}{x^3(e^{2x} - 1)^2}, \quad u_2(x) := -(x-2)e^{4x} - 4(x^2+1)e^{2x} + x + 2,$$
$$u_2'(x) = -(4x-7)e^{4x} - 8(x^2+x+1)e^{2x} + 1, \quad u_2''(x) = -8e^{4x}u_3(x),$$
$$u_3(x) := (2x^2+4x+3)e^{-2x} + 2x - 3, \quad u_3'(x) = -2[(2x^2+2x+1)e^{-2x} - 1],$$
$$u_3''(x) = 8x^2 e^{-2x}.$$

u_1 is strictly decreasing on $[0, +\infty[$,

$$u_1(0) = \frac{1}{3}, \quad \lim_{x \to +\infty} u_1(x) = 0.$$

If $c \leqslant 0$, then u is strictly increasing on $[0,+\infty[$,

$$u(0) = 1 - c \geqslant 1, \quad \lim_{x\to+\infty} u(x) = +\infty.$$

If $c \geqslant \frac{1}{3}$, then u is strictly decreasing on $[0,+\infty[$,

$$u(0) = 1 - c, \quad \lim_{x\to+\infty} u(x) = -\infty.$$

If $c \in \left]0, \frac{1}{3}\right[$, then there exists $x_1 > 0$ such that $u_1(x_1) = c$. u is strictly increasing on $[0, x_1]$ and strictly decreasing on $[x_1, +\infty[$,

$$\frac{2}{3} < u(0) = 1 - c < u(x_1) = u_4(x_1), \quad \lim_{x\to+\infty} u(x) = -\infty,$$

where

$$u_4(x) := \frac{\sinh(2x) - 2x}{2x^2 \sinh x}.$$
$$u_4'(x) = \frac{e^{-3x} u_5(x)}{2x^3 (e^x - e^{-x})^2},$$
$$u_5(x) := (x-2)e^{6x} + (4x^2 + x + 2)e^{4x} + (4x^2 - x + 2)e^{2x} - (x+2),$$
$$u_5'(x) = (6x - 11)e^{6x} + (16x^2 + 12x + 9)e^{4x} + (8x^2 + 6x + 3)e^{2x} - 1,$$
$$u_5''(x) = 4e^{2x} u_6(x), \quad u_6(x) := 3(3x-5)e^{4x} + 4(4x^2 + 5x + 3)e^{2x} + 4x^2 + 7x + 3,$$
$$u_6'(x) = 3(12x - 17)e^{4x} + 4(8x^2 + 18x + 11)e^{2x} + 8x + 7,$$
$$u_6''(x) = 8[3(6x-7)e^{4x} + 2(4x^2 + 13x + 10)e^{2x} + 1], \quad u_6'''(x) = 16e^{2x} u_7(x),$$
$$u_7(x) := 3(12x - 11)e^{2x} + 8x^2 + 34x + 33, \quad u_7'(x) = 2[3(12x-5)e^{2x} + 8x + 17],$$
$$u_7''(x) = 4[3(12x+1)e^{2x} + 4] > 0.$$

u_4 is strictly increasing on $[0,+\infty[$,

$$u_4(0) = \frac{2}{3}, \quad \lim_{x\to+\infty} u_4(x) = +\infty. \quad \square$$

Proof of Proposition 4.2. Let $x > 0$, $c \in \mathbb{R}$ and

$$v(x) := \frac{\mathscr{I}_{\frac{1}{2}}(x) - c}{\mathscr{I}_{-\frac{1}{2}}(x)} = \frac{\sinh x - cx}{x \cosh x} = \frac{e^x - e^{-x} - 2cx}{x(e^x + e^{-x})}.$$

We have $v'(x) = \dfrac{-e^{2x} + e^{-2x} + 4x + 2cx^2(e^x - e^{-x})}{x^2 (e^x + e^{-x})^2}$.

$$v'(x) = 0 \quad \text{if and only if} \quad c = u_4(x),$$

where u_4 is given by (7).

If $c \leqslant \frac{2}{3}$, then v is strictly decreasing on $[0,+\infty[$,

$$v(0) = 1 - c > 0, \quad \lim_{x\to+\infty} v(x) = 0.$$

If $c > \frac{2}{3}$, then there exists $x_1 > 0$ such that $u_4(x_1) = c$. v is strictly increasing on $[0,x_1]$ and strictly decreasing on $[x_1,+\infty[$,

$$v(0) = 1 - c, \quad v(x_1) = u_1(x_1) > 0, \quad \lim_{x\to+\infty} v(x) = 0,$$

where u_1 is given by (6). □

Proof of Proposition 4.3. Let $x > 0$, $c \in \mathbb{R}$ and

$$w(x) := \frac{\mathscr{I}_{-\frac{1}{2}}(x) - c}{\mathscr{I}_{\frac{1}{2}}(x)} = \frac{x(\cosh x - c)}{\sinh x} = \frac{x(e^x + e^{-x} - 2c)}{e^x - e^{-x}}.$$

We have $w'(x) = \dfrac{e^{2x} - e^{-2x} - 4x + 2c[(x-1)e^x + (x+1)e^{-x}]}{(e^x - e^{-x})^2}$.

$w'(x) = 0$ if and only if $c = w_1(x) := -\dfrac{\sinh(2x) - 2x}{2(x\cosh x - \sinh x)} = -\dfrac{e^{2x} - e^{-2x} - 4x}{2[(x-1)e^x + (x+1)e^{-x}]}$.

$$w_1'(x) = \frac{e^{-x} w_2(x)}{2[(x-1)e^x + (x+1)e^{-x}]^2},$$

$$w_2(x) := -(x-2)e^{4x} - (4x^2 - x + 6)e^{2x} - (x+2)e^{-2x} + 4x^2 + x + 6,$$

$$w_2'(x) = -(4x-7)e^{4x} - (8x^2 + 6x + 11)e^{2x} + (2x+3)e^{-2x} + 8x + 1,$$

$$w_2''(x) = -4[2(2x-3)e^{4x} + (4x^2 + 7x + 7)e^{2x} + (x+1)e^{-2x} - 2,$$

$$w_2'''(x) = -4e^{-2x} w_3(x),$$

$$w_3(x) := 4(4x-5)e^{6x} + (8x^2 + 22x + 21)e^{4x} - 2x - 1,$$

$$w_3'(x) := 2[4(12x-13)e^{6x} + (16x^2 + 52x + 53)e^{4x} - 1], \quad w_3''(x) = 16e^{4x} w_4(x),$$

$$w_4(x) := 3(12x-11)e^{2x} + 8x^2 + 30x + 33, \quad w_4'(x) = 2[3(12x-5)e^{2x} + 8x + 15],$$

$$w_4''(x) = 4[3(12x+1)e^{2x} + 4] > 0.$$

w_1 is strictly decreasing on $[0,+\infty[$,

$$w_1(0) = -2, \quad \lim_{x\to+\infty} w_1(x) = -\infty.$$

If $c \geqslant -2$, then w is strictly increasing on $[0,+\infty[$,

$$w(0) = 1 - c, \quad \lim_{x\to+\infty} w(x) = +\infty.$$

If $c<-2$, then there exists $x_1>0$ such that $w_1(x_1)=c$. w is strictly decreasing on $[0,x_1]$ and strictly increasing on $[x_1,+\infty[$,

$$3<w(x_1)=\frac{1}{u_1(x_1)}<w(0)=1-c,\quad \lim_{x\longrightarrow+\infty}w(x)=+\infty,$$

where u_1 is given by (6). □

Proof of Theorem 3.1. We get the result from Propositions 4.1, 4.2, 4.3 and Remarks 5.1. □

Proof of Proposition 4.4. Let x, $c\in\mathbb{R}$ and

$$r(x):=\mathscr{I}_{-\frac{1}{2}}(x)-c\psi_{-i}^{-\frac{1}{2}}(x)=\cosh x-ce^x=\frac{(1-2c)e^x+e^{-x}}{2}.$$

We have $r'(x)=\dfrac{e^x(-e^{-2x}+1-2c)}{2}$.

$$r'(x)=0\quad\text{if and only if}\quad c=\frac{1-e^{-2x}}{2}.$$

If $c=\dfrac{1}{2}$, then r is strictly decreasing on $\mathbb{R}$,

$$\lim_{x\longrightarrow-\infty}r(x)=+\infty,\quad r(0)=\frac{1}{2},\quad \lim_{x\longrightarrow+\infty}r(x)=0.$$

If $c>\dfrac{1}{2}$, then r is strictly decreasing on $\mathbb{R}$,

$$\lim_{x\longrightarrow-\infty}r(x)=+\infty,\quad r(0)=1-c,\quad \lim_{x\longrightarrow+\infty}r(x)=-\infty.$$

If $c<\dfrac{1}{2}$, then r is strictly decreasing on $\left]-\infty,-\ln\left(\sqrt{1-2c}\right)\right]$ and strictly increasing on $\left[-\ln\left(\sqrt{1-2c}\right),+\infty\right[$,

$$r(0)=1-c,\quad r\left(-\ln\left(\sqrt{1-2c}\right)\right)=\sqrt{1-2c},\quad \lim_{x\longrightarrow-\infty}r(x)=\lim_{x\longrightarrow+\infty}r(x)=+\infty.\quad\square$$

Proof of Proposition 4.5. Let x, $c\in\mathbb{R}$ and

$$s(x):=\frac{\mathscr{I}_{-\frac{1}{2}}(x)-c}{\psi_{-i}^{-\frac{1}{2}}(x)}=e^{-x}(\cosh x-c)=\frac{e^{-2x}-2ce^{-x}+1}{2}.$$

We have $s'(x)=e^{-x}(c-e^{-x})$.

If $c \leqslant 0$, then s is strictly decreasing on $\mathbb{R}$,

$$\lim_{x\to-\infty} s(x) = +\infty, \quad s(0) = 1-c \leqslant 1, \quad \lim_{x\to+\infty} s(x) = \frac{1}{2}.$$

If $c > 0$, then s is strictly decreasing on $]-\infty, -\ln c]$ and strictly increasing on $[-\ln c, +\infty[$,

$$\lim_{x\to-\infty} s(x) = +\infty, \quad s(0) = 1-c, \quad s(-\ln c) = \frac{1-c^2}{2}, \quad \lim_{x\to+\infty} s(x) = \frac{1}{2}. \quad \square$$

Proof of Proposition 4.6. Let x, $c \in \mathbb{R}$ and

$$t(x) := \frac{\psi_{-i}^{-\frac{1}{2}}(x) - c}{\mathscr{I}_{-\frac{1}{2}}(x)} = \frac{2(e^x - c)}{e^x + e^{-x}}.$$

We have $t'(x) = \dfrac{c\sinh x + 1}{(\cosh x)^2}$.

If $c = 0$, then t is strictly increasing on $\mathbb{R}$,

$$\lim_{x\to-\infty} t(x) = 0, \quad t(0) = 1, \quad \lim_{x\to+\infty} t(x) = 2.$$

If $c < 0$, then there exists $x_1 > 0$ such that $\sinh x_1 = -\dfrac{1}{c}$. t is strictly increasing on $]-\infty, x_1]$ and strictly decreasing on $[x_1, +\infty[$,

$$\lim_{x\to-\infty} t(x) = 0, \quad t(0) = 1-c > 1, \quad t(x_1) = \frac{2}{1-e^{-2x_1}} > 2, \quad \lim_{x\to+\infty} t(x) = 2.$$

If $c > 0$, then there exists $x_2 < 0$ such that $\sinh x_2 = -\dfrac{1}{c}$. t is strictly decreasing on $]-\infty, x_2]$ and strictly increasing on $[x_2, +\infty[$,

$$\lim_{x\to-\infty} t(x) = 0, \quad t(0) = 1-c, \quad t(x_2) = -\frac{2}{e^{-2x_2}-1} < 0, \quad \lim_{x\to+\infty} t(x) = 2. \quad \square$$

Proof of Proposition 4.8. Let $x \in \mathbb{R} \setminus \{0\}$ and

$$d(x) := \mathscr{I}_{\frac{3}{2}}(x) = 3\frac{(x-1)e^x + (x+1)e^{-x}}{2x^3}.$$

We have $d'(x) = \dfrac{3e^x}{2x^4} d_1(x)$, where

$$d_1(x) := -(x^2+3x+3)e^{-2x} + x^2 - 3x + 3, \tag{18}$$

$$d_1'(x) = (2x^2+4x+3)e^{-2x} + 2x - 3, \quad d_1''(x) = -2[(2x^2+2x+1)e^{-2x} - 1],$$

$d_1'''(x) = 8x^2e^{-2x} > 0$. d_1 is strictly increasing on $\mathbb{R}$,

$$\lim_{x\to+\infty} d_1(x) = -\infty, \quad d_1(0) = 0, \quad \lim_{x\to+\infty} d_1(x) = +\infty.$$

d is even on $\mathbb{R}$ and strictly increasing on $[0,+\infty[$,

$$d(0) = 1, \quad \lim_{x\to+\infty} d(x) = +\infty. \quad \square$$

Proof of Proposition 4.9. Let $x \in \mathbb{R}\setminus\{0\}$ and

$$\theta(x) := \psi_{-i}^{\frac{1}{2}}(x) = \frac{(2x-1)e^x + e^{-x}}{2x^2}.$$

We have $\theta'(x) = e^x\theta_1(x)$, where

$$\theta_1(x) := e^{-x}\left(\psi_{-i}^{\frac{1}{2}}(x) - \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(x)\right) = \frac{-(x+2)e^{-2x} + 2x^2 - 3x + 2}{2x^3}, \tag{19}$$

$\theta_1'(x) = -\dfrac{d_1(x)}{x^4}$, where d_1 is given by (18). θ_1 is strictly increasing on $]-\infty,0]$ and strictly decreasing on $[0,+\infty[$,

$$\lim_{x\to-\infty} \theta_1(x) = -\infty, \quad \theta_1(0) = \frac{1}{3}, \quad \lim_{x\to+\infty} \theta_1(x) = 0.$$

There exists $a_0 < 0$ such that $-1,4 < a_0 < -1,3$, $\theta_1(a_0) = 0$. θ is strictly decreasing on $]-\infty,a_0]$ and strictly increasing on $[a_0,+\infty[$,

$$\lim_{x\to-\infty} \theta(x) = +\infty, \quad \theta(0) = 1, \quad \lim_{x\to+\infty} \theta(x) = +\infty.$$

$$\theta''(x) = \frac{e^x}{2x^4}\theta_2(x), \quad \theta_2(x) := (x^2+4x+6)e^{-2x} + 2x^3 - 5x^2 + 8x - 6,$$

$$\theta_2'(x) = 2[-(x^2+3x+4)e^{-2x} + 3x^2 - 5x + 4], \quad \theta_2''(x) = 2[(2x^2+4x+5)e^{-2x} + 6x - 5],$$

$$\theta_2'''(x) = 4[-(2x^2+2x+3)e^{-2x} + 3], \quad \theta_2^{(4)}(x) = 16(x^2+1)e^{-2x} > 0.$$

$\theta' = \psi_{-i}^{\frac{1}{2}} - \dfrac{2}{3}\mathscr{I}_{\frac{3}{2}}$ is strictly increasing on $\mathbb{R}$,

$$\lim_{x\to-\infty} \theta'(x) = -\infty, \quad \theta'(a_0) = 0, \quad \theta'(0) = \frac{1}{3}, \quad \lim_{x\to+\infty} \theta'(x) = +\infty.$$

$$x\theta(x) = x\mathscr{I}_{\frac{1}{2}}(x) + \frac{x^2}{3}\mathscr{I}_{\frac{3}{2}}(x) = \psi_{-i}^{-\frac{1}{2}}(x) - \mathscr{I}_{-\frac{1}{2}}(x) + \frac{x^2}{3}\mathscr{I}_{\frac{3}{2}}(x) = \psi_{-i}^{-\frac{1}{2}}(x) - \mathscr{I}_{\frac{1}{2}}(x). \quad \square$$

Proof of Proposition 4.10. Let $x \in \mathbb{R}\setminus\{0\}$, $c \in \mathbb{R}$ and

$$k(x) := \mathscr{I}_{\frac{1}{2}}(x) - c\psi_{-i}^{-\frac{1}{2}}(x) = \frac{(-2cx+1)e^x - e^{-x}}{2x}.$$

We have $k'(x) = \dfrac{(-2cx^2+x-1)e^x+(x+1)e^{-x}}{2x^2}$.

$$k'(x)=0 \quad \text{if and only if} \quad c = k_1(x) := \frac{xe^{-x}}{3}\mathscr{I}_{\frac{3}{2}}(x) = \frac{(x+1)e^{-2x}+x-1}{2x^2}.$$

$$k_1'(x) = \frac{k_2(x)}{2x^3}, \quad k_2(x) := -(2x^2+3x+2)e^{-2x}-x+2, \quad k_2'(x) = (4x^2+2x+1)e^{-2x}-1,$$

$k_2''(x) = -4x(2x-1)e^{-2x}$. k_2' is strictly decreasing on $]-\infty,0]$ and $\left[\frac{1}{2},+\infty\right[$, and strictly increasing on $\left[0,\frac{1}{2}\right]$,

$$\lim_{x\to-\infty} k_2'(x) = +\infty, \quad k_2'(0)=0, \quad k_2'\left(\frac{1}{2}\right) = 3e^{-1}-1>0, \quad \lim_{x\to+\infty} k_2'(x) = -1.$$

There exists $x_1 \in \left]\frac{1}{2},1\right[$ such that $e^{2x_1} = 4x_1^2+2x_1+1$. k_2 is strictly increasing on $]-\infty,x_1]$ and strictly decreasing on $[x_1,+\infty[$,

$$k_2(0)=k_2(-a_0)=0, \quad k_2(x_1) = 4\frac{-x_1^3+x_1^2+1}{4x_1^2+2x_1+1}>0, \quad \lim_{x\to-\infty} k_2(x) = \lim_{x\to+\infty} k_2(x) = -\infty,$$

where a_0 is given by (10). k_1 is strictly increasing on $]-\infty,-a_0]$ and strictly decreasing on $[-a_0,+\infty[$,

$$\lim_{x\to-\infty} k_1(x) = -\infty, \quad k_1(0)=0, \quad k_1(-a_0) = b_1 := -\frac{a_0e^{2a_0}}{a_0+2}>0, \quad \lim_{x\to+\infty} k_1(x) = 0.$$

If $c \geqslant b_1$, then k is strictly decreasing on $\mathbb{R}$,

$$\lim_{x\to-\infty} k(x) = +\infty, \quad k(0) = 1-c, \quad \lim_{x\to+\infty} k(x) = -\infty.$$

If $c \leqslant 0$, then there exists $x_2 \leqslant 0$ such that $k_1(x_2) = c$. k is strictly decreasing on $]-\infty,x_2]$ and strictly increasing on $[x_2,+\infty[$,

$$k(0) = 1-c \geqslant 1, \quad k(x_2) = \psi_{-i}^{\frac{1}{2}}(-x_2)>0, \quad \lim_{x\to-\infty} k(x) = \lim_{x\to+\infty} k(x) = +\infty.$$

If $c \in]0,b_1[$, then there exist $x_3,x_4 \in \mathbb{R}$ such that $0<x_3<-a_0<x_4$, $k_1(x_3) = k_1(x_4) = c$.

k is strictly decreasing on $]-\infty,x_3]$ and $[x_4,+\infty[$, and strictly increasing on $[x_3,x_4]$,

$$0<k(x_3) = \psi_{-i}^{\frac{1}{2}}(-x_3) < k(-a_0) = -\frac{(2ca_0+1)e^{-a_0}-e^{a_0}}{2a_0} < k(x_4) = \psi_{-i}^{\frac{1}{2}}(-x_4),$$

$$k(0) = 1-c>0, \quad \lim_{x\to-\infty} k(x) = +\infty, \quad \lim_{x\to+\infty} k(x) = -\infty. \quad \square$$

Proof of Proposition 4.11. Let $x \in \mathbb{R} \setminus \{0\}$, $c \in \mathbb{R}$ and

$$p(x) := \frac{\mathscr{I}_{\frac{1}{2}}(x) - c}{\psi_{-i}^{-\frac{1}{2}}(x)} = \frac{-e^{-2x} - 2cxe^{-x} + 1}{2x}.$$

We have $p'(x) = \dfrac{(2x+1)e^{-2x} + 2cx^2e^{-x} - 1}{2x^2}$.

$$p'(x) = 0 \quad \text{if and only if} \quad c = \psi_{-i}^{\frac{1}{2}}(-x).$$

If $c \leqslant b_0$, where b_0 is given by (11), then p is strictly decreasing on $\mathbb{R}$,

$$\lim_{x \to -\infty} p(x) = +\infty, \quad p(0) = 1 - c > 0, \quad \lim_{x \to +\infty} p(x) = 0.$$

If $c > b_0$, then there exist x_1, $x_2 \in \mathbb{R}$ such that $x_1 < -a_0 < x_2$, $\psi_{-i}^{\frac{1}{2}}(-x_1) = \psi_{-i}^{\frac{1}{2}}(-x_2) = c$, where a_0 is given by (10). p is strictly decreasing on $]-\infty, x_1]$ and $[x_2, +\infty[$, and strictly increasing on $[x_1, x_2]$,

$$\lim_{x \to -\infty} p(x) = +\infty, \quad p(0) = 1 - c, \quad p(x_1) = k_1(x_1),$$
$$p(x_2) = k_1(x_2) > 0, \quad \lim_{x \to +\infty} p(x) = 0. \quad \square$$

Proof of Proposition 4.12. Let $x \in \mathbb{R} \setminus \{0\}$, $c \in \mathbb{R}$ and

$$q(x) := \frac{\psi_{-i}^{-\frac{1}{2}}(x) - c}{\mathscr{I}_{\frac{1}{2}}(x)} = \frac{2x(e^x - c)}{e^x - e^{-x}}.$$

We have $q'(x) = 2\dfrac{c[(x-1)e^x + (x+1)e^{-x}] + e^{2x} - (2x+1)}{(e^x - e^{-x})^2}$.

$$q'(x) = 0 \quad \text{if and only if} \quad c = q_1(x) := \frac{-e^{2x} + 2x + 1}{(x-1)e^x + (x+1)e^{-x}} = -\frac{\psi_{-i}^{\frac{1}{2}}(-x)}{k_1(x)},$$

where k_1 is given by (13). $q_1'(x) = \dfrac{e^{3x}}{[(x-1)e^x + (x+1)e^{-x}]^2} q_2(x)$, where

$$q_2(x) := (2x^2 + 3x + 2)e^{-4x} - 2(x^2 + x + 2)e^{-2x} - x + 2,$$

$$q_2'(x) = -(8x^2 + 8x + 5)e^{-4x} + 2(2x^2 + 3)e^{-2x} - 1, \quad q_2''(x) = 4e^{-2x} q_3(x),$$
$$q_3(x) := (8x^2 + 4x + 3)e^{-2x} - (2x^2 - 2x + 3), \quad q_3'(x) = -2q_4(x),$$
$$q_4(x) := (8x^2 - 4x + 1)e^{-2x} + 2x - 1, \quad q_4'(x) = 2[-(8x^2 - 12x + 3)e^{-2x} + 1],$$

$$q_4''(x) = 4(8x^2 - 20x + 9)e^{-2x},$$

$$q_4'(0) = -4, \quad q_4'\left(\frac{5+\sqrt{7}}{4}\right) = 2\left[1 - 2\left(2+\sqrt{7}\right)e^{-\frac{5+\sqrt{7}}{2}}\right] > 0,$$

$$q_4(0) = q_3(0) = q_2'(0) = q_2(0) = q_2(-a_1) = 0.$$

q_1 is strictly increasing on $]-\infty, 0[$ and $]0, -a_0]$, and strictly decreasing on $[-a_0, +\infty[$, where a_0 is given by (10),

$$\lim_{x\to-\infty} q_1(x) = 0, \quad \lim_{x\to 0^-} q_1(x) = +\infty, \quad \lim_{x\to 0^+} q_1(x) = -\infty, \quad \lim_{x\to+\infty} q_1(x) = -\infty,$$

$$q_1(-a_0) = b_2 := 2\frac{e^{-a_0}}{a_0}.$$

If $c \in [b_2, 0]$, then q is strictly increasing on $\mathbb{R}$,

$$\lim_{x\to-\infty} q(x) = 0, \quad q(0) = 1 - c \geqslant 1, \quad \lim_{x\to+\infty} q(x) = +\infty.$$

If $c > 0$, then there exists $x_1 < 0$ such that $q_1(x_1) = c$. q is strictly decreasing on $]-\infty, x_1]$ and strictly increasing on $[x_1, +\infty[$,

$$\lim_{x\to-\infty} q(x) = 0, \quad q(x_1) = \frac{x_1^2 e^{x_1}}{x_1 \cosh x_1 - \sinh x_1} < 0, \quad q(0) = 1 - c, \quad \lim_{x\to+\infty} q(x) = +\infty.$$

If $c < b_2$, then there exist x_2, $x_3 \in \mathbb{R}$ such that $0 < x_2 < -a_0 < x_3$, $q_1(x_2) = q_1(x_3) = c$.

q is strictly increasing on $]-\infty, x_2]$ and $[x_3, +\infty[$, and strictly decreasing on $[x_2, x_3]$,

$$q(0) = 1 - c > 1, \quad 0 < q(x_3) = \frac{x_3^2 e^{x_3}}{x_3 \cosh x_3 - \sinh x_3} < q(x_2) = \frac{x_2^2 e^{x_2}}{x_2 \cosh x_2 - \sinh x_2},$$

$$\lim_{x\to-\infty} q(x) = 0, \quad \lim_{x\to+\infty} q(x) = +\infty. \quad \square$$

Proof of Proposition 4.14. Let $x \in \mathbb{R} \setminus \{0\}$, $c \in \mathbb{R}$ and

$$f(x) := \psi_{-i}^{\frac{1}{2}}(x) - c\psi_{-i}^{-\frac{1}{2}}(x) = \frac{(-2cx^2 + 2x - 1)e^x + e^{-x}}{2x^2}.$$

We have $f'(x) = e^x(\theta_1(x) - c)$, where θ_1 is given by (19).

If $c \geqslant \frac{1}{3}$, then f is strictly decreasing on $\mathbb{R}$,

$$\lim_{x\to-\infty} f(x) = +\infty, \quad f(0) = 1 - c, \quad \lim_{x\to+\infty} f(x) = -\infty.$$

If $0 < c < \dfrac{1}{3}$, then there exist $x_1,\ x_2 \in \mathbb{R}$ such that $a_0 < x_1 < 0 < -x_1 < x_2$, $\theta_1(x_1) = \theta_1(x_2) = c$. f is strictly decreasing on $]-\infty, x_1]$ and $[x_2, +\infty[$, and strictly increasing on $[x_1, x_2]$,

$$\frac{2}{3} < f(x_1) = \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(x_1) < f(0) = 1 - c < f(x_2) = \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(x_2), \quad f(x_1) < b_0,$$

$$\lim_{x\to-\infty} f(x) = +\infty, \quad \lim_{x\to+\infty} f(x) = -\infty,$$

where b_0 is given by (11).

If $c \leqslant 0$, then there exists $x_3 \leqslant a_0$ such that $\theta_1(x_3) = c$. f is strictly decreasing on $]-\infty, x_3]$ and strictly increasing on $[x_3, +\infty[$,

$$b_0 \leqslant f(x_3) = \frac{2}{3}\mathscr{I}_{\frac{3}{2}}(x_3) < f(0) = 1 - c, \quad \lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = +\infty. \quad \square$$

Proof of Proposition 4.15. Let $x \in \mathbb{R}\setminus\{0\}$, $c \in \mathbb{R}$ and

$$g(x) := \frac{\psi_{-i}^{\frac{1}{2}}(x) - c}{\psi_{-i}^{-\frac{1}{2}}(x)} = \frac{e^{-2x} - 2cx^2 e^{-x} + 2x - 1}{2x^2}.$$

We have $g'(x) = \dfrac{e^{-x}}{3}\left(3c - 2\mathscr{I}_{\frac{3}{2}}(x)\right)$.

If $c \leqslant \dfrac{2}{3}$, then g is strictly decreasing on $\mathbb{R}$,

$$\lim_{x\to-\infty} g(x) = +\infty, \quad g(0) = 1 - c, \quad \lim_{x\to+\infty} g(x) = 0.$$

If $c > \dfrac{2}{3}$, then there exists $x_1 > 0$ such that $\dfrac{2}{3}\mathscr{I}_{\frac{3}{2}}(x_1) = c$. g is strictly decreasing on $]-\infty, -x_1]$ and $[x_1, +\infty[$, and strictly increasing on $[-x_1, x_1]$,

$$g(-x_1) = \theta_1(-x_1) < g(0) = 1 - c < g(x_1) = \theta_1(x_1) < \frac{1}{3}, \quad g(x_1) > 0,$$

$$\lim_{x\to-\infty} g(x) = +\infty, \quad \lim_{x\to+\infty} g(x) = 0,$$

where θ_1 is given by (19). $\square$

Proof of Proposition 4.16. Let $x \in \mathbb{R}\setminus\{0\}$, $c \in \mathbb{R}$ and

$$h(x) := \frac{\psi_{-i}^{-\frac{1}{2}}(x) - c}{\psi_{-i}^{\frac{1}{2}}(x)} = \frac{2x^2(e^x - c)}{(2x-1)e^x + e^{-x}}.$$

We have $h'(x) = \dfrac{2e^x \mathscr{I}_{\frac{3}{2}}(x)}{3\left(\psi_{-i}^{\frac{1}{2}}(x)\right)^2}(c h_1(x)+1)$, where

$$h_1(x) := \frac{3\theta_1(x)}{2\mathscr{I}_{\frac{3}{2}}(x)} = \frac{-(x+2)e^{-2x}+2x^2-3x+2}{2[(x-1)e^x+(x+1)e^{-x}]},$$

where θ_1 is given by (19), $h_1'(x) = -\dfrac{9\psi_{-i}^{\frac{1}{2}}(x)d_1(x)}{4x^4\left(\mathscr{I}_{\frac{3}{2}}(x)\right)^2}$, where d_1 is given by (18). h_1 is strictly increasing on $]-\infty,0]$ and strictly decreasing on $[0,+\infty[$,

$$\lim_{x\to-\infty} h_1(x) = -\infty, \quad h_1(a_0) = 0, \quad h_1(0) = \frac{1}{2}, \quad \lim_{x\to+\infty} h_1(x) = 0,$$

where a_0 is given by (10).

If $-2 \leqslant c \leqslant 0$, then h is strictly increasing on $\mathbb{R}$,

$$\lim_{x\to-\infty} h(x) = 0 < h(a_0) = \frac{e^{a_0}-c}{b_0} < h(0) = 1-c, \quad \lim_{x\to+\infty} h(x) = +\infty,$$

where b_0 is given by (11).

If $c > 0$, then there exists $x_1 < \min\{a_0, \ln c\}$ such that $-\dfrac{1}{h_1(x_1)} = c$. h is strictly decreasing on $]-\infty,x_1]$ and strictly increasing on $[x_1,+\infty[$,

$$\lim_{x\to-\infty} h(x) = 0, \quad h(x_1) = \frac{1}{\theta_1(x_1)} < 0 = h(\ln c), \quad \lim_{x\to+\infty} h(x) = +\infty.$$

If $c < -2$, then there exist $x_2, x_3 \in \mathbb{R}$ such that $a_0 < x_2 < 0 < x_3$, $-\dfrac{1}{h_1(x_2)} = -\dfrac{1}{h_1(x_3)} = c$. h is strictly increasing on $]-\infty,x_2]$ and $[x_3,+\infty[$, and strictly decreasing on $[x_2,x_3]$,

$$\lim_{x\to-\infty} h(x) = 0 < h(x_3) = \frac{1}{\theta_1(x_3)} < h(0) = 1-c < h(x_2) = \frac{1}{\theta_1(x_2)},$$
$$\lim_{x\to+\infty} h(x) = +\infty. \quad \square$$

Proof of Theorem 3.2. Let $\alpha > -\dfrac{1}{2}$, l_1, $r_1 \in \left]-\dfrac{1}{2\alpha+1}, +\infty\right[$, l_2, r_2, a, $b \in \mathbb{R}$; $a \leqslant 0 \leqslant b$, $a < b$ and $x \in [a,b]$. If $l_1\mathscr{I}_{-\frac{1}{2}}(x) + l_2 \leqslant \mathscr{I}_{\frac{1}{2}}(x) \leqslant r_1\mathscr{I}_{-\frac{1}{2}}(x) + r_2$, then by using Proposition 4.17 and $B\left(\alpha+\dfrac{1}{2}, \dfrac{3}{2}\right) = \dfrac{B\left(\alpha+\frac{1}{2},\frac{1}{2}\right)}{2(\alpha+1)} = \dfrac{B\left(\alpha+\frac{3}{2},\frac{1}{2}\right)}{2\alpha+1}$, we deduce

$$\frac{2(\alpha+1)l_1\mathscr{I}_\alpha(x)+l_2}{(2\alpha+1)l_1+1} \leqslant \mathscr{I}_{\alpha+1}(x) \leqslant \frac{2(\alpha+1)r_1\mathscr{I}_\alpha(x)+r_2}{(2\alpha+1)r_1+1}.$$

Theorem 3.1 finishes the proof. $\square$

Proof of Theorem 3.3. We get the result from Propositions 4.14, 4.15, 4.16 and Remarks 5.2, 5.4, 5.5. $\square$

Proof of Theorem 3.4. Let $\alpha > -\frac{1}{2}$, l_1, $r_1 \in \left]-\frac{1}{2\alpha+1}, +\infty\right[$, l_2, r_2, a, $b \in \mathbb{R}$; $a \leqslant 0 \leqslant b$, $a < b$ and $x \in [a,b]$. If $l_1 \psi_{-i}^{-\frac{1}{2}}(x) + l_2 \leqslant \psi_{-i}^{\frac{1}{2}}(x) \leqslant r_1 \psi_{-i}^{-\frac{1}{2}}(x) + r_2$, then by using Proposition 4.17 and $B\left(\alpha+\frac{1}{2}, \frac{3}{2}\right) = \frac{B\left(\alpha+\frac{1}{2},\frac{1}{2}\right)}{2(\alpha+1)} = \frac{B\left(\alpha+\frac{3}{2},\frac{1}{2}\right)}{2\alpha+1}$, we deduce

$$\frac{2(\alpha+1)l_1\psi_{-i}^{\alpha}(x)+l_2}{(2\alpha+1)l_1+1} \leqslant \psi_{-i}^{\alpha+1}(x) \leqslant \frac{2(\alpha+1)r_1\psi_{-i}^{\alpha}(x)+r_2}{(2\alpha+1)r_1+1}.$$

Theorem 3.3 finishes the proof. $\square$

Proof of Theorem 3.7. Let $\alpha > -\frac{1}{2}$, l_1, l_2, r_1, r_2, a, $b \in \mathbb{R}$; $a \leqslant 0 \leqslant b$, $a < b$ and $x \in [a,b]$.

If $l_1 \mathscr{I}_{-\frac{1}{2}}(x) + l_2 \leqslant \psi_{-i}^{-\frac{1}{2}}(x) \leqslant r_1 \mathscr{I}_{-\frac{1}{2}}(x) + r_2$, then by using Proposition 4.17 and Corollary 4.7, we deduce

$$l_1 \mathscr{I}_{\alpha}(x) + l_2 \leqslant \psi_{-i}^{\alpha}(x) \leqslant r_1 \mathscr{I}_{\alpha}(x) + r_2. \quad \square$$

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