Mathematical
Inequalities
& Applications
Volume 17, Number 1 (2014), 49–74 doi:10.7153/mia-17-03

FOURIER MULTIPLIER THEOREMS FOR BESOV AND TRIEBEL–LIZORKIN SPACES WITH VARIABLE EXPONENTS

TAKAHIRO NOI

(*Communicated by L. Pick*)

Abstract. In this paper, we will prove Fourier multiplier theorems on Besov and Triebel–Lizorkin spaces with variable exponents. It was shown by many authors that variable Triebel–Lizorkin spaces coincide with variable Bessel potential spaces, variable Sobolev spaces and variable Lebesgue spaces when appropriate indices are chosen. In consequence of the results, we also have Fourier multiplier theorems on these variable function spaces.

1. Introduction

Recently, variable function spaces have been studied by many authors [3, 8, 5, 6, 9, 12, 13, 14] and in particular the papers about the variable Triebel–Lizorkin and Besov spaces have been published in [1, 7, 10, 17, 18, 19, 20]. More additional references about variable exponent spaces are in the book [6] written by L. Diening, P. Harjulehto, P. Hästö and M. Růžička.

We state Fourier multiplier theorems on Triebel–Lizorkin spaces with variable exponents $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and Besov spaces with variable exponents $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ in section 3 and prove the main theorems in section 5, where the spaces $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ were introduced by L. Diening, P. Hästö and S. Roudenko [7] and A. Almeida and P. Hästö [1], respectively.

J. Xu [18] showed that variable Bessel potential space $L^{s,p(\cdot)}(\mathbb{R}^n)$ coincides with $F^{s}_{p(\cdot),2}(\mathbb{R}^n)$ if $s \geqslant 0$ and $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. L. Diening, P. Hästö and S. Roudenko [7] showed that the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ coincides with $F^{0}_{p(\cdot),2}(\mathbb{R}^n)$ under suitable assumptions on $p(\cdot)$. P. Gurka, P. Harjulehto and A. Nekvinda [9] showed that $L^{k,p(\cdot)}(\mathbb{R}^n)$ coincides with variable Sobolev spaces $W^{k,p(\cdot)}(\mathbb{R}^n)$ if $k \in \mathbb{N}$ and $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. In consequence of these results, we also have Fourier multiplier theorems on these variable function spaces.

Mathematics subject classification (2010): 42B15.
Keywords and phrases: Fourier multiplier theorem, variable exponents Besov spaces, variable exponents Triebel–Lizorkin spaces.

2. Definition of variable function spaces

We first introduce the variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. Let $p(\cdot)$ be a measurable function on $\mathbb{R}^n$ with range in $(0,\infty)$. Let $L^{p(\cdot)}(\mathbb{R}^n)$ denote the set of measurable functions f on $\mathbb{R}^n$ such that, for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x < \infty.$$

The set becomes a quasi Banach function space when it is equipped with the Luxemburg–Nakano norm

$$||f||_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 \ : \ \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} \mathrm{d}x \leqslant 1\right\}.$$

If $p(x) \equiv p$ is a constant function, then the above norm coincides with the usual L^p-norm and so the notation is not confusing. It is remarked that one can define variable Lebesgue spaces on any measurable subset of $\mathbb{R}^n$ ([12]).

Denote by $\mathscr{P}_0(\mathbb{R}^n)$ the set of measurable functions $p(\cdot)$ on $\mathbb{R}^n$ with range in $(0,\infty)$ such that

$$0 < p_- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) = p_+ < \infty.$$

We also denote by $\mathscr{P}(\mathbb{R}^n)$ the set of measurable functions $p(\cdot)$ on $\mathbb{R}^n$ with range in $(1,\infty)$ such that $1 < p_-$ and $p_+ < \infty$. If $f(\cdot)$ is a complex-valued locally Lebesgue-integrable function on $\mathbb{R}^n$, then

$$(\mathscr{M}f)(x) = \sup \frac{1}{|B|}\int_B |f(y)|\,\mathrm{d}y$$

is called Hardy-Littlewood maximal function, where the supremum is taken over all balls B centered at x. There exists some $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ such that the Hardy–Littlewood maximal operator $\mathscr{M}$ is not bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ ([13]), although the operator $\mathscr{M}$ is bounded on $L^p(\mathbb{R}^n)$ for $p > 1$. Let $\mathscr{B}(\mathbb{R}^n)$ be the set of $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ such that the Hardy–Littlewood maximal operator $\mathscr{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$. There are some sufficient conditions on $p(\cdot)$ for the maximal operator $\mathscr{M}$ to be bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ (see [3, 4]). We denote by $C^{\log}(\mathbb{R}^n)$ the set of all real valued functions $p(\cdot) : \mathbb{R}^n \to \mathbb{R}$ satisfying following conditions: There exist constants $C_{\log}(p)$ and $p_\infty \in \mathbb{R}$ such that

$$|p(x) - p(y)| \leqslant \frac{C_{\log}(p)}{\log(e + |x-y|^{-1})} \quad (x,y \in \mathbb{R}^n, x \neq y) \tag{1}$$

and

$$|p(x) - p_\infty| \leqslant \frac{C_{\log}(p)}{\log(e + |x|)} \quad (x \in \mathbb{R}^n). \tag{2}$$

Cruz-Uribe et. al. [2] showed that $C^{\log}(\mathbb{R}^n) \cap \mathscr{P}(\mathbb{R}^n) \subset \mathscr{B}(\mathbb{R}^n)$.

Let $\mathscr{B}_0(\mathbb{R}^n)$ be the set of $p(\cdot)\in\mathscr{P}_0(\mathbb{R}^n)$ for which there exists a positive number $\alpha>0$ such that $\mathscr{M}$ is bounded on $L^{\alpha p(\cdot)}(\mathbb{R}^n)$.

Let $\mathscr{S}(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing and infinitely differentiable functions on $\mathbb{R}^n$. Let $\mathscr{S}'(\mathbb{R}^n)$ be the set of all tempered distributions on $\mathbb{R}^n$. If $\varphi\in\mathscr{S}(\mathbb{R}^n)$, then $\mathscr{F}\varphi$ denotes the Fourier transform of φ, and $\mathscr{F}^{-1}\varphi$ denotes the inverse Fourier transform of φ. We write $\mathscr{F}^{-1}m\mathscr{F}f=\mathscr{F}^{-1}[m\cdot\mathscr{F}f]$ for the sake of simplicity.

Let $s\geqslant 0$ and $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$. The variable exponent Bessel potential space $L^{s,p(\cdot)}(\mathbb{R}^n)$ is the collection of $f\in L^{p(\cdot)}(\mathbb{R}^n)$ such that

$$||f||_{L^{s,p(\cdot)}}=\left|\left|\mathscr{F}^{-1}\left(1+|\cdot|^2\right)^{\frac{s}{2}}\mathscr{F}f(\cdot)\right|\right|_{L^{p(\cdot)}}<\infty.$$

Let $k\in\mathbb{N}$ and $p\in\mathscr{P}(\mathbb{R}^n)$. The variable exponent Sobolev spaces $W^{k,p(\cdot)}(\mathbb{R}^n)$ is the collection of $f\in L^{p(\cdot)}(\mathbb{R}^n)$ such that the derivatives (in the sense of distribution) up to the order k belong to $L^{p(\cdot)}(\mathbb{R}^n)$ and

$$||f||_{W^{k,p(\cdot)}}=\sum_{|\alpha|\leqslant k}||D^\alpha f||_{L^{p(\cdot)}}<\infty,$$

where α is a multi-index and $|\alpha|=\alpha_1+\cdots+\alpha_n$.

The set $\Phi(\mathbb{R}^n)$ is the collection of all systems $\theta=\{\theta_j\}_{j=0}^\infty\subset\mathscr{S}(\mathbb{R}^n)$ such that

$$\begin{cases}\operatorname{supp}\mathscr{F}\theta_0\subset\{x:|x|\leqslant 2\},\\ \operatorname{supp}\mathscr{F}\theta_j\subset\{x:2^{j-1}\leqslant|x|\leqslant 2^{j+1}\}\quad\text{for } j=0,1,2,\cdots,\end{cases}$$

and, for every multi-index α, there exists a positive number c_α such that

$$2^{j|\alpha|}|D^\alpha\mathscr{F}\theta_j(x)|\leqslant c_\alpha$$

for $j=0,1,\cdots$ and $x\in\mathbb{R}^n$ and

$$\sum_{j=0}^\infty\mathscr{F}\theta_j(x)=1$$

for $x\in\mathbb{R}^n$.

To define the variable Besov spaces, we first define the mixed Lebesgue sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$.

Let $p(\cdot)$, $q(\cdot)\in\mathscr{P}_0(\mathbb{R}^n)$. The space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is the collection of all sequences $\{g_j\}_0^\infty$ of measurable functions on $\mathbb{R}^n$ such that

$$||\{g_j\}_{j=0}^\infty||_{\ell^{q(\cdot)}(L^{p(\cdot)})}=\inf\left\{\mu>0:\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\left\{\frac{f_j}{\mu}\right\}_{j=0}^\infty\right)\leqslant 1\right\}<\infty,$$

where

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\{f_j\}_{j=0}^\infty\right)=\sum_{j=0}^\infty\inf\left\{\lambda_j:\int_{\mathbb{R}^n}\left(\frac{|f_j(x)|}{\lambda_j^{\frac{1}{q(x)}}}\right)^{p(x)}\mathrm{d}x\leqslant 1\right\}.$$

Since we assume that $q_+ < \infty$, we have

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\{f_j\}_{j=0}^{\infty}\right) = \sum_{j=0}^{\infty} \left|\left| |f_j|^{q(\cdot)} \right|\right|_{L^{\frac{p(\cdot)}{q(\cdot)}}}. \tag{3}$$

DEFINITION 2.1. Let $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathscr{P}_0(\mathbb{R}^n)$ and $\alpha(\cdot) \in C^{\log}(\mathbb{R}^n)$. Let $\theta = \{\theta_j\}_{j=0}^{\infty} \in \Phi(\mathbb{R}^n)$. The variable Besov spaces $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is the collection of $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$||f||^{(\theta)}_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \left|\left| \left\{ 2^{j\alpha(\cdot)} \theta_j * f \right\}_0^{\infty} \right|\right|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

The variable Triebel–Lizorkin space $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is the collection of $f \in \mathscr{S}'(\mathbb{R}^n)$ such that

$$||f||^{(\theta)}_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = \left|\left| \left\{ 2^{j\alpha(\cdot)} \theta_j * f \right\}_0^{\infty} \right|\right|_{L^{p(\cdot)}(\ell^{q(\cdot)})} < \infty.$$

Here $L^{p(\cdot)}(\ell^{q(\cdot)})$ is the spaces of all sequences $\{g_j\}_0^{\infty}$ of measurable functions on $\mathbb{R}^n$ such that

$$||\{g_j\}||_{L^{p(\cdot)}(\ell^{q(\cdot)})} = ||\ ||\{g_j\}_0^{\infty}||_{\ell^{q(\cdot)}}||_{L^{p(\cdot)}} = \left|\left| \left(\sum_{j=0}^{\infty} |g_j(\cdot)|^{q(\cdot)} \right)^{\frac{1}{q(\cdot)}} \right|\right|_{L^{p(\cdot)}} < \infty.$$

Almeida et. al. [1] showed that the quasi norm $||f||^{(\theta)}_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$ of $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ does not depend on $\theta \in \Phi(\mathbb{R}^n)$. Diening et. al. [7] showed that $||f||^{(\theta)}_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$ and $||f||^{(\rho)}_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$ are equivalent quasi norms on $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, where $\theta,\ \rho \in \Phi(\mathbb{R}^n)$. Hence we write $||f||_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = ||f||^{(\theta)}_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$ and $||f||_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = ||f||^{(\theta)}_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}$.

3. Main results

Let $N \in \mathbb{N}$ and α be a multi-index. Following [16], we write

$$||m||_N = \sup_{|\alpha| \leqslant N} \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^{\frac{|\alpha|}{2}} |D^{\alpha} m(x)|$$

for an infinity differentiable function $m(\cdot)$. Then we have the following Fourier multiplier theorem which is proved in Section 5.

THEOREM 3.1. *Let* $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathscr{P}_0(\mathbb{R}^n)$ *and* $\alpha(\cdot) \in C^{\log}(\mathbb{R}^n)$.

(i) *If* $N > \max\{|\alpha_-|, |\alpha_+|\} + \frac{3n + 9C_{\log}(\alpha)\min\{p_-, q_-\}}{\min\{p_-, q_-\}} + n + 2$, *then there exists a positive number* c *such that*

$$||\mathscr{F}^{-1} m \mathscr{F} f||_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant c ||m||_N ||f||_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \tag{4}$$

for all infinitely differentiable functions $m(\cdot)$ *and all* $f \in F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

(ii) *If* $N > \max\{|\alpha_-|,|\alpha_+|\} + \frac{6n+9C_{\log}(\alpha)\min\{p_-,q_-\}}{\min\{p_-,q_-\}} + n + 2$, *then there exists a positive number* c *such that*

$$||\mathscr{F}^{-1}m\mathscr{F}f||_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant c||m||_N||f||_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \tag{5}$$

for all infinitely differentiable functions $m(\cdot)$ *and all* $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

This theorem corresponds to [16, p. 57, Theorem].

For a real number s,

$$H^s_2(\mathbb{R}^n) = \{f \in \mathscr{S}'(\mathbb{R}^n) \,:\, ||f||_{H^s_2} = ||(1+|\cdot|^2)^{s/2}(\mathscr{F}f)(\cdot)||_{L^2} < \infty\}.$$

Let $\psi(\cdot) \in \mathscr{S}(\mathbb{R}^n)$ and $\varphi(\cdot) \in \mathscr{S}(\mathbb{R}^n)$ such that

$$0 \leqslant \psi \leqslant 1,\ \ \operatorname{supp}\psi \subset \{y : |y| \leqslant 4\},\ \ \psi(x) = 1 \text{ if } |x| \leqslant 2, \tag{6}$$

and

$$0 \leqslant \varphi \leqslant 1,\ \ \operatorname{supp}\varphi \subset \left\{y : \frac{1}{4} \leqslant |y| \leqslant 4\right\},\ \ \varphi(x) = 1 \text{ if } \frac{1}{2} \leqslant |x| \leqslant 2. \tag{7}$$

Following [16], we write

$$||m||_{h^s_2} = ||\psi m||_{H^s_2} + \sup_{k=0,1,2,\cdots} ||\varphi(\cdot)m(2^k\cdot)||_{H^s_2}.$$

Then we obtain the following Fourier multiplier theorem which is proved in Section 5.

THEOREM 3.2. *Let* $p(\cdot),q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathscr{P}_0(\mathbb{R}^n)$ *and* $\alpha(\cdot) \in C^{\log}(\mathbb{R}^n)$.

(i) *If* $v > \frac{n}{2} + \frac{n+3C_{\log}(\alpha)\min\{p_-,q_-\}}{\min\{p_-,q_-\}}$, *then there exists a constant* c *such that*

$$||\mathscr{F}^{-1}m\mathscr{F}f||_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant c||m||_{h^v_2}||f||_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \tag{8}$$

for $m(\cdot) \in L^\infty(\mathbb{R}^n)$ *and* $f \in F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

(ii) *If* $v > \frac{n}{2} + \frac{2n+3C_{\log}(\alpha)\min\{p_-,q_-\}}{\min\{p_-,q_-\}}$, *then there exists a constant* c *such that*

$$||\mathscr{F}^{-1}m\mathscr{F}f||_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \leqslant c||m||_{h^v_2}||f||_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \tag{9}$$

for $m(\cdot) \in L^\infty(\mathbb{R}^n)$ *and* $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

REMARK 3.3. It is shown by Triebel [16] that, in the case where $\alpha(\cdot)$, $p(\cdot)$ and $q(\cdot)$ are constants, Theorem 3.1 and 3.2 holds under the condition $N \geqslant |\alpha| +$

$3n/\min\{p,q\}+n+2$ or $v \geqslant n/2+n/\min\{p,q\}$, respectively. Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. If $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ is not a constant function, Almeida et al. [1] showed the inequality

$$||\{\mathscr{M} f_k\}_0^\infty||_{\ell^{q(\cdot)}(L^{p(\cdot)})} \lesssim ||\{f_k\}_0^\infty||_{\ell^{q(\cdot)}(L^{p(\cdot)})}$$

does not hold for all sequences $\{f_k\}_{k=0}^\infty \subset L^{p(\cdot)}(\mathbb{R}^n)$, and so Theorem 4.10 are applied to prove Theorem 4.13. Hence, main results above do not recover the constant cases in the case of $B^\alpha_{p,q}$. However, if we want to prove directly the theorems where $q(\cdot)$ and $\alpha(\cdot)$ are constants, the proofs are as follows:

Let $p(\cdot) \in \mathscr{B}_0(\mathbb{R}^n)$ and $q \in (0,\infty)$. Then we set

$$r_{p_-,q} = \sup\left\{ r \in \mathbb{R} \, : 0 < r < \min\{p_-,q\} \text{ and } \frac{p(\cdot)}{r} \in \mathscr{B}(\mathbb{R}^n) \right\}$$

Since the definition of $\mathscr{B}_0(\mathbb{R}^n)$, the condition on m in the (ii) of Theorem 4.13 is weakened to $m > n$. Hence, the conditions on N and v in the case of $B^\alpha_{p(\cdot),q}$ are weakened to $N > |\alpha| + 3n/r_{p_-,q} + n + 2$ and $v > n/2 + n/r_{p_-,q}$ respectively.

The next theorem is proved by many authors [18, 7].

THEOREM 3.4. *Let $s \geqslant 0$ and $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then $F^s_{p(\cdot),2}(\mathbb{R}^n)$ coincides with $L^{s,p(\cdot)}(\mathbb{R}^n)$. Let $k \in \mathbb{N}$ and $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$. Then $F^k_{p(\cdot),2}(\mathbb{R}^n)$ coincides with $W^{k,p(\cdot)}(\mathbb{R}^n)$.*

The next corollary is an immediate consequence of Theorem 3.1, 3.2 and 3.4.

COROLLARY 3.5. *Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ and $s \geqslant 0$.*

(i) *If $N \geqslant |s| + 4n + 2$, then there exists a positive number c such that*

$$||\mathscr{F}^{-1} m \mathscr{F} f||_{L^{s,p(\cdot)}} \leqslant c||m||_N ||f||_{L^{s,p(\cdot)}}$$

for all infinitely differentiable functions $m(\cdot)$ and all $f \in L^{s,p(\cdot)}(\mathbb{R}^n)$.

(ii) *If $v \geqslant n/2 + n$, then there exists a constant c such that*

$$||\mathscr{F}^{-1} m \mathscr{F} f||_{L^{s,p(\cdot)}} \leqslant c||m||_{h_2^v} ||f||_{L^{s,p(\cdot)}}$$

for $m(\cdot) \in L^\infty(\mathbb{R}^n)$ and $f \in L^{s,p(\cdot)}(\mathbb{R}^n)$.

We can prove the next corollary by the same arguments in the proof of [16, p. 58, Theorem 2.3.8] with lifting property (Corollary 4.22) and Theorem 3.1 which takes the place of [16, p. 57, Theorem 2.3.7].

COROLLARY 3.6. *Let $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathscr{P}_0(\mathbb{R}^n)$, $\alpha(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $m \in \mathbb{N}$.*

(i) *Let β be a multi index. Then*

$$\sum_{|\beta| \leqslant m} \left|\left| D^\beta f \right|\right|_{B^{\alpha(\cdot)-m}_{p(\cdot),q(\cdot)}} \quad \text{and} \quad ||f||_{B^{\alpha(\cdot)-m}_{p(\cdot),q(\cdot)}} + \sum_{j=1}^n \left|\left| \frac{\partial^m f}{\partial x_j^m} \right|\right|_{B^{\alpha(\cdot)-m}_{p(\cdot),q(\cdot)}}$$

are equivalent quasi-norms on $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

(ii) *Similarly,*

$$\sum_{|\beta|\leqslant m}\left|\left|D^{\beta}f\right|\right|_{f^{\alpha(\cdot)-m}_{p(\cdot),q(\cdot)}} \quad \text{and} \quad ||f||_{F^{\alpha(\cdot)-m}_{p(\cdot),q(\cdot)}}+\sum_{j=1}^{n}\left|\left|\frac{\partial^m f}{\partial x_j^m}\right|\right|_{F^{\alpha(\cdot)-m}_{p(\cdot),q(\cdot)}}$$

are equivalent quasi-norms on $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

4. Preliminaries

We need the following fundamental properties of $L^{p(\cdot)}(\mathbb{R}^n)$.

THEOREM 4.1. ([5, Theorem 8.1]) *Let* $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$. *Then the following conditions are equivalent to each other:*

(a) $p(\cdot)\in\mathscr{B}(\mathbb{R}^n)$.

(b) $p(\cdot)/t\in\mathscr{B}(\mathbb{R}^n)$ *for some* $1<t<p_-$.

(c) $p'(\cdot)\in\mathscr{B}(\mathbb{R}^n)$, *where*

$$p'(x)=\frac{p(x)}{p(x)-1}.$$

In [3, 5], some other equivalent conditions are given.

REMARK 4.2. Let $p(\cdot)\in\mathscr{B}(\mathbb{R}^n)$ and t be the same as Theorem 4.1. Then, for any $w\in(1,t]$, we have $p(\cdot)/w\in\mathscr{B}(\mathbb{R}^n)$. Furthermore, for any $w\in(0,1]$, we have also $p(\cdot)/w\in\mathscr{B}(\mathbb{R}^n)$ by Jensen inequality. The proof and the details are found in [5]. Consequently, if $p(\cdot)\in\mathscr{B}(\mathbb{R}^n)$, then, for any $r\in(0,r_{p_-})$, $p(\cdot)/r\in\mathscr{B}(\mathbb{R}^n)$. For any $r\in(0,r_{p_-,q})$, we have also $p(\cdot)/r\in\mathscr{B}(\mathbb{R}^n)$.

The next theorem gives a generalized Hölder inequality which is shown in [12, 14].

THEOREM 4.3. ([12, 14, Generalized Hölder inequality]) *Let* $p\in\mathscr{P}(\mathbb{R}^n)$. *Then*

$$\int_{\mathbb{R}^n}|f(x)g(x)|\,\mathrm{d}x\leqslant\left(1+\frac{1}{p_-}-\frac{1}{p_+}\right)||f||_{L^{p(\cdot)}}||g||_{L^{p'(\cdot)}}$$

for every $f\in L^{p(\cdot)}(\mathbb{R}^n)$ *and* $g\in L^{p'(\cdot)}(\mathbb{R}^n)$.

Let $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ such that $p(\cdot)$ is not a constant function. Then, according to [12, Example 2.9, Theorem 2.10], for every $L^{p(\cdot)}$, there exists a function $f(\cdot)\in L^{p(\cdot)}(\mathbb{R}^n)$ such that its any translation $f(\beta+\cdot)\notin L^{p(\cdot)}(\mathbb{R}^n)$, where $\beta\in\mathbb{R}^n\setminus\{(0,0,\cdots,0)\}$. The next theorem shows that all elements of $\mathscr{S}(\mathbb{R}^n)$ and any of their corresponding translations belong to $L^{p(\cdot)}(\mathbb{R}^n)$ for every $p(\cdot)\in\mathscr{P}_0(\mathbb{R}^n)$.

THEOREM 4.4. *Let $f(\cdot)$ be a measurable function on $\mathbb{R}^n$ satisfying*

$$\sup_{x\in\mathbb{R}^n}\{(1+|x|^2)^{\frac{n}{p_-}}|f(x)|\}<\infty. \tag{10}$$

Then, for any $p(\cdot)\in\mathscr{P}_0(\mathbb{R}^n)$,

$$||f||_{L^{p(\cdot)}}\leqslant\pi^{\frac{n}{p_-}}\sup_{x\in\mathbb{R}^n}\{(1+|x|^2)^{\frac{n}{p_-}}|f(x)|\}.$$

Furthermore, for any $\alpha\in\mathbb{R}\setminus\{0\}$ and $\beta\in\mathbb{R}^n$,

$$\sup_{\beta\in\mathbb{R}^n}||f(\beta+\alpha\cdot)||_{L^{p(\cdot)}}\leqslant|\alpha|^{-\frac{n}{p_*}}\pi^{\frac{n}{p_-}}\sup_{x\in\mathbb{R}^n}\{(1+|x|^2)^{\frac{n}{p_-}}|f(x)|\},$$

where

$$p_*=\begin{cases}p_- & \text{if } |\alpha|\leqslant 1,\\ p_+ & \text{otherwise.}\end{cases}$$

Proof. We put $f_{\mathrm{c}}=\sup_{x\in\mathbb{R}^n}\{(1+|x|^2)^{n/p_-}|f(x)|\}$. Then

$$|f(x)|^{p(x)}=\left(\frac{(1+|x|^2)^{\frac{n}{p_-}}|f(x)|}{(1+|x|^2)^{\frac{n}{p_-}}}\right)^{p(x)}\leqslant\frac{f_{\mathrm{c}}^{p(x)}}{(1+|x|^2)^n}, \tag{11}$$

since $(1+|x|^2)^{-1}\leqslant 1$ and $p_-<p(x)$. So, we have

$$\begin{aligned}\int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\pi^{\frac{n}{p_-}}f_{\mathrm{c}}}\right)^{p(x)}\mathrm{d}x&\leqslant\int_{\mathbb{R}^n}\left(\frac{f_{\mathrm{c}}}{(1+|x|^2)^{\frac{n}{p_-}}}\frac{1}{\pi^{\frac{n}{p_-}}f_{\mathrm{c}}}\right)^{p(x)}\mathrm{d}x\\&=\int_{\mathbb{R}^n}\left(\frac{1}{(1+|x|^2)^{\frac{n}{p_-}}}\frac{1}{\pi^{\frac{n}{p_-}}}\right)^{p(x)}\mathrm{d}x\\&\leqslant\frac{1}{\pi^n}\left(\int_{-\infty}^{\infty}\frac{1}{1+t^2}\mathrm{d}t\right)^n=1,\end{aligned} \tag{12}$$

where we use (11), $\pi^{-p(x)n/p_-}\leqslant\pi^{-n}$ and $(1+x_1^2)(1+x_2^2)\cdots(1+x_n^2)\leqslant(1+|x|^2)^n$. By (12), we have $||f||_{L^{p(\cdot)}}\leqslant\pi^{n/p_-}\sup_{x\in\mathbb{R}^n}\{(1+|x|^2)^{n/p_-}|f(x)|\}<\infty$. By (10) with x replaced by $\beta+\alpha x$, we have

$$\begin{aligned}|f(\beta+\alpha x)|&\leqslant(1+|\beta+\alpha x|^2)^{\frac{n}{p_-}}|f(\beta+\alpha x)|\\&\leqslant\sup_{x\in\mathbb{R}^n}\{(1+|\beta+\alpha x|^2)^{\frac{n}{p_-}}|f(\beta+\alpha x)|\}<\infty\end{aligned}$$

for any $\alpha\in\mathbb{R}\setminus\{0\}$ and $\beta\in\mathbb{R}^n$. We put $f_{\{\alpha,\beta\}}=\sup_{x\in\mathbb{R}^n}\{(1+|\beta+\alpha x|^2)^{n/p_-}|f(\beta+\alpha x)|\}$. Let

$$p_*=\begin{cases}p_+ & \text{if } |\alpha|\geqslant 1,\\ p_- & \text{if } |\alpha|\leqslant 1.\end{cases}$$

Then we have

$$\int_{\mathbb{R}^n} \left(\frac{|f(\beta+\alpha x)|}{|\alpha|^{-\frac{n}{p_*}} \pi^{\frac{n}{p_-}} f_{\{\alpha,\beta\}}} \right)^{p(x)} \mathrm{d}x \leqslant 1$$

by the same calculation as in (12). Hence we have

$$\begin{aligned} \|f(\beta+\alpha\cdot)\|_{L^{p(\cdot)}} &\leqslant |\alpha|^{\frac{-n}{p_*}} \pi^{\frac{n}{p_-}} \sup_{x\in\mathbb{R}^n} \{(1+|\beta+\alpha x|^2)^{\frac{n}{p_-}} |f(\beta+\alpha x)|\} \\ &\leqslant |\alpha|^{\frac{-n}{p_*}} \pi^{\frac{n}{p_-}} \sup_{z\in\mathbb{R}^n} \{(1+|z|^2)^{\frac{n}{p_-}} |f(z)|\} \end{aligned} \tag{13}$$

by (10). Finally we have

$$\sup_{\beta\in\mathbb{R}^n} \|f(\beta+\alpha\cdot)\|_{L^{p(\cdot)}} \leqslant |\alpha|^{\frac{-n}{p_*}} \pi^{\frac{n}{p_-}} \sup_{x\in\mathbb{R}^n} \{(1+|x|^2)^{\frac{n}{p_-}} |f(x)|\} < \infty$$

by (13). □

COROLLARY 4.5. *Let* $p(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$.

(i) *The inclusion* $\mathscr{S}(\mathbb{R}^n) \subset L^{p(\cdot)}(\mathbb{R}^n)$ *holds. Furthermore, we have*

$$\|f\|_{L^{p(\cdot)}} \leqslant \pi^{\frac{n}{p_-}} \sup_{x\in\mathbb{R}^n} \{(1+|x|^2)^{\frac{n}{p_-}} |f(x)|\}$$

for $f(\cdot) \in \mathscr{S}(\mathbb{R}^n)$.

(ii) *If* $f(\cdot) \in \mathscr{S}(\mathbb{R}^n)$, *then* $f(\beta+\alpha\cdot) \in L^{p(\cdot)}(\mathbb{R}^n)$ *for any* $\alpha \in \mathbb{R}\setminus\{0\}$ *and* $\beta \in \mathbb{R}^n$. *Furthermore, we have*

$$\sup_{\beta\in\mathbb{R}^n} \|f(\beta+\alpha\cdot)\|_{L^{p(\cdot)}} \leqslant |\alpha|^{-\frac{n}{p_*}} \pi^{\frac{n}{p_-}} \sup_{x\in\mathbb{R}^n} \{(1+|x|^2)^{\frac{n}{p_-}} |f(x)|\},$$

where

$$p_* = \begin{cases} p_- & \text{if } |\alpha| \leqslant 1, \\ p_+ & \text{otherwise}. \end{cases}$$

Proof. It is sufficient to prove that (10) holds for $f(\cdot) \in \mathscr{S}(\mathbb{R}^n)$. We recall that the topology in the complete locally convex space $\mathscr{S}(\mathbb{R}^n)$ is generated by semi-norms

$$p_N(f) = \sup_{x\in\mathbb{R}^n} \sum_{|\alpha|+k\leqslant N} (1+|x|^2)^k |D^\alpha f(x)|, \quad N \in \mathbb{N}.$$

Then, for any $x \in \mathbb{R}^n$ and any $N \geqslant [n/p_-]+1$, we have

$$|f(x)| \leqslant \sup_{x\in\mathbb{R}^n} \{(1+|x|^2)^{\frac{n}{p_-}} |f(x)|\} \leqslant p_N(f) < \infty$$

for $f \in \mathscr{S}(\mathbb{R}^n)$. □

DEFINITION 4.6. (i) Let Ω be a compact subset of $\mathbb{R}^n$. Then $\mathscr{S}^{\Omega}(\mathbb{R}^n)$ denotes the space of all elements $\varphi \in \mathscr{S}(\mathbb{R}^n)$ which satisfies $\operatorname{supp}\mathscr{F}\varphi \subset \Omega$.

(ii) Let $p(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathscr{P}_0(\mathbb{R}^n)$. If $\Omega = \{\Omega_k\}_{k=0}^{\infty}$ is a sequence of compact subsets of $\mathbb{R}^n$, then $L^{\Omega}_{p(\cdot)}$ are the spaces of all sequences $\{f_k\}_{k=0}^{\infty}$ of $\mathscr{S}'(\mathbb{R}^n)$ such that

$$\operatorname{supp}\mathscr{F} f_k \subset \Omega_k \tag{14}$$

and $||f_k||_{L^{p(\cdot)}} < \infty$ for $k = 0, 1, 2, \cdots$.

The next theorem gives a relationship between $L^{p(\cdot)}$ and $L^{q(\cdot)}$ norm of $\varphi \in \mathscr{S}^{B}(\mathbb{R}^n)$.

THEOREM 4.7. *Let $B_b = \{x \in \mathbb{R}^n : |x| \leqslant b\}$, $b > 0$ and α be an arbitrary multi-index.*

(i) *Let $p(\cdot)$, $q(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$ satisfy $0 < p(\cdot) \leqslant q(\cdot) < \infty$. If $\varphi \in \mathscr{S}^{B_b}(\mathbb{R}^n)$, then there exists a positive number c such that*

$$||\varphi||_{L^{q(\cdot)}} \leqslant c b^{n\left(\frac{1}{p^*} - \frac{1}{q^{**}}\right)} ||\varphi||_{L^{p(\cdot)}} \quad \text{and} \quad ||\varphi||_{L^\infty} \leqslant c b^{\frac{n}{p^*}} ||\varphi||_{L^{p(\cdot)}},$$

where

$$p^* = \begin{cases} p_- & \text{if } \ b \geqslant 1, \\ p_+ & \text{if } \ b < 1, \end{cases} \qquad \text{and} \quad q^{**} = \begin{cases} q_+ & \text{if } \ b \geqslant 1, \\ q_- & \text{if } \ b < 1, \end{cases}$$

and c is independent of b.

(ii) *Let $p(\cdot)$, $q(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$ satisfy $0 < p(\cdot) \leqslant q(\cdot) < \infty$ and let p^* and q^{**} are the same function as above. If $\varphi \in \mathscr{S}^{B_b}(\mathbb{R}^n)$, then there exists a positive number c such that*

$$||D^\alpha \varphi||_{L^{q(\cdot)}} \leqslant c b^{|\alpha| + n\left(\frac{1}{p^*} - \frac{1}{q^{**}}\right)} ||\varphi||_{L^{p(\cdot)}} \quad \text{and} \quad ||D^\alpha \varphi||_{L^\infty} \leqslant c b^{|\alpha| + \frac{n}{p^*}} ||\varphi||_{L^{p(\cdot)}},$$

where c is independent on b.

Proof. We prove only the $b \geqslant 1$ case because we can prove the $b < 1$ case by the same argument.

Step 1. First we will prove that

$$||\varphi||_{L^\infty} \leqslant c b^{\frac{n}{p_-}} ||\varphi||_{L^{p(\cdot)}} \tag{15}$$

for $\varphi \in \mathscr{S}^{\Omega}(\mathbb{R}^n)$ and $p(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$. Without loss of generality, we can assume $||\varphi||_{L^\infty} = 1$. Let $\psi(y) \in \mathscr{S}(\mathbb{R}^n)$ such that $(\mathscr{F}\psi)(\xi) = 1$ for $\xi \in \{x \in \mathbb{R}^n : |x| \leqslant 1\}$. Then there exists a some positive number c_1 such that

$$\varphi(x) = c_1 \int_{\mathbb{R}^n} \varphi(y) b^n \psi(b(x-y))\, \mathrm{d}y,$$

where c_1 depends only on the definition of Fourier transform $\mathscr{F}$. Then we have

$$\begin{aligned}\frac{|\varphi(x)|}{c_1} &\leqslant \int_{\mathbb{R}^n} b^n|\varphi(y)\psi(b(x-y))|\,\mathrm{d}y\\ &= \int_{\Omega(p>1)} b^n|\varphi(y)\psi(b(x-y))|\,\mathrm{d}y + \int_{\Omega(p\leqslant 1)} b^n|\varphi(y)\psi(b(x-y))|\,\mathrm{d}y\\ &= I_1 + I_2, \end{aligned} \tag{16}$$

where $\Omega(p>1)=\{x\in\mathbb{R}^n:\ p(x)>1\}$ and $\Omega(p\leqslant 1)=\{x\in\mathbb{R}^n:\ p(x)\leqslant 1\}$. Let

$$\tilde{p}(x)=\begin{cases} p(x) & (x\in\Omega(p>1)),\\ 1+\varepsilon & (x\in\Omega(p\leqslant 1)),\end{cases}$$

where $\varepsilon>0$ is an arbitrary fixed number with $1+\varepsilon\leqslant p_+$. By Corollary 4.5, we have

$$\sup_{x\in\mathbb{R}^n} ||b^n\psi(b(x-\cdot))\chi_{\Omega(p>1)}(\cdot)||_{L^{\tilde{p}'(\cdot)}} \leqslant \pi^{\frac{n}{\tilde{p}'_-}} b^{n-\frac{n}{\tilde{p}'_+}} \sup_{y\in\mathbb{R}^n}\{(1+|y|^2)^{\frac{n}{\tilde{p}'_-}}|\psi(y)|\},$$

where $\chi_{\Omega(p>1)}(x)$ is a characteristic function on $\Omega(p>1)$. Hence we have

$$\begin{aligned}|I_1| &\leqslant db^n||\psi(b(x-\cdot))\chi_{\Omega(p>1)}(\cdot)||_{L^{\tilde{p}'(\cdot)}}||\varphi(\cdot)\chi_{\Omega(p>1)}(\cdot)||_{L^{\tilde{p}(\cdot)}}\\ &\leqslant dc_2\pi^{\frac{n}{p'_-}} b^{n\left(1-\frac{1}{\tilde{p}'_+}\right)}||\varphi||_{L^{p(\cdot)}} = dc_2\pi^{\frac{n}{p'_-}} b^{\frac{n}{\tilde{p}_-}}||\varphi||_{L^{p(\cdot)}}\\ &\leqslant dc_2\pi^{\frac{n}{p'_-}} b^{\frac{n}{p_-}}||\varphi||_{L^{p(\cdot)}} \end{aligned} \tag{17}$$

by Hölder inequality, where $d=(1+\frac{1}{\tilde{p}_-}-\frac{1}{\tilde{p}_+})$ and $c_2=\sup_{y\in\mathbb{R}^n}\{(1+|y|^2)^{\frac{n}{\tilde{p}'_-}}|\psi(y)|\}$. By $||\varphi||_{L^\infty}=1$, we have

$$\begin{aligned}|I_2| &\leqslant b^n||\psi||_{L^\infty}\int_{\Omega(p\leqslant 1)}|\varphi(y)|^{p(y)}|\varphi(y)|^{1-p(y)}\,\mathrm{d}y\\ &\leqslant b^n||\psi||_{L^\infty}\int_{\Omega(p\leqslant 1)}|\varphi(y)|^{p(y)}\,\mathrm{d}y\\ &\leqslant b^n||\psi||_{L^\infty}\int_{\mathbb{R}^n}|\varphi(y)|^{p(y)}\,\mathrm{d}y \leqslant b^n||\psi||_{L^\infty}||\varphi||^{p_*}_{L^{p(\cdot)}}, \end{aligned} \tag{18}$$

where

$$p_*=\begin{cases} p_+ & \text{if } ||\varphi||_{L^{p(\cdot)}}\geqslant 1,\\ p_- & \text{if } ||\varphi||_{L^{p(\cdot)}}<1.\end{cases}$$

Hence we have

$$1\leqslant dc_1c_2\pi^{\frac{n}{p'_-}} b^{\frac{n}{p_-}}||\varphi||_{L^{p(\cdot)}} + c_1b^n||\psi||_{L^\infty}||\varphi||^{p_*}_{L^{p(\cdot)}}$$

by $||\varphi||_{L^\infty}=1$ and (16)–(18). It follows that

$$\frac{1}{2}\leqslant dc_1c_2\pi^{\frac{n}{p'_-}} b^{\frac{n}{p_-}}||\varphi||_{L^{p(\cdot)}} \ \text{ or } \ \frac{1}{2}\leqslant c_1b^n||\psi||_{L^\infty}||\varphi||^{p_*}_{L^{p(\cdot)}}.$$

Hence we have

$$||\varphi||_{L^\infty} \leqslant cb^{\frac{n}{p_-}}||\varphi||_{L^{p(\cdot)}},$$

where

$$c = \max\left\{2dc_1c_2\pi^{\frac{n}{p'_-}}, (2c_1||\psi||_{L^\infty})^{\frac{1}{p_-}}, (2c_1||\psi||_{L^\infty})^{\frac{1}{p_+}}\right\}.$$

By $q(x) - p(x) \geqslant 0$ and (15), we have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\frac{|\varphi(x)|}{cb^{n\left(\frac{1}{p_-}-\frac{1}{q_+}\right)}||\varphi||_{L^{p(\cdot)}}}\right)^{q(x)} \mathrm{d}x \\
&= \int_{\mathbb{R}^n}\left(\frac{|\varphi(x)|}{cb^{n\left(\frac{1}{p_-}-\frac{1}{q_+}\right)}||\varphi||_{L^{p(\cdot)}}}\right)^{q(x)-p(x)}\left(\frac{|\varphi(x)|}{cb^{n\left(\frac{1}{p_-}-\frac{1}{q_+}\right)}||\varphi||_{L^{p(\cdot)}}}\right)^{p(x)} \mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^n}\left(\frac{||\varphi||_{L^\infty}}{cb^{n\left(\frac{1}{p_-}-\frac{1}{q_+}\right)}||\varphi||_{L^{p(\cdot)}}}\right)^{q(x)-p(x)}\left(\frac{|\varphi(x)|}{cb^{n\left(\frac{1}{p_-}-\frac{1}{q_+}\right)}||\varphi||_{L^{p(\cdot)}}}\right)^{p(x)} \mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^n} b^{\frac{nq(x)}{q_+}} b^{\frac{-np(x)}{p_-}}\left(\frac{|\varphi(x)|}{c||\varphi||_{L^{p(\cdot)}}}\right)^{p(x)} \mathrm{d}x \leqslant \frac{1}{c} \leqslant 1.
\end{aligned}$$

Hence we have

$$||\varphi||_{L^{q(\cdot)}} \leqslant cb^{n\left(\frac{1}{p^*}-\frac{1}{q^{**}}\right)}||\varphi||_{L^{p(\cdot)}}, \tag{19}$$

which implies (i).

Step 2. We will prove (ii). Let $\psi \in \mathscr{S}(\mathbb{R}^n)$ be the same function as in the Step 1. Then we see that

$$D^\alpha\varphi(x) = c_1\int_{\mathbb{R}^n}\varphi(y)b^{n+|\alpha|}D^\alpha\psi(b(x-y))\,\mathrm{d}y.$$

Hence, we have

$$||D^\alpha\varphi||_{L^\infty} \leqslant cb^{|\alpha|+\frac{n}{p_-}}||\varphi||_{L^{p(\cdot)}}$$

and

$$||D^\alpha\varphi||_{L^{q(\cdot)}} \leqslant cb^{|\alpha|+n\left(\frac{1}{p_-}-\frac{1}{q_+}\right)}||\varphi||_{L^{p(\cdot)}},$$

where

$$c = \max\left\{2dc_1\pi^{\frac{n}{p'_-}}\sup_{y\in\mathbb{R}^n}\{(1+|y|^2)^{\frac{n}{\tilde{p}'_-}}|D^\alpha\psi(y)|\}, (2c_1||D^\alpha\psi||_{L^\infty})^{\frac{1}{p_-}}, (2c_1||D^\alpha\psi||_{L^\infty})^{\frac{1}{p_+}}\right\}.$$

□

This theorem is corresponding to [16, p. 17, Theorem].

D. Cruz-Uribe et al. [3] proves the boundedness of classical operators, for example, singular integral operators and fractional integral operators, on the space $L^{p(\cdot)}(\mathbb{R}^n)$. The next theorem is corresponding to the well-known maximal vector-valued inequality in the classical setting.

THEOREM 4.8. ([3, Corollary 2.1]) *If* $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, *then, for all* $q \in (1,\infty)$, *there exists a constant* c *such that*

$$||\{\mathscr{M} f_k\}_0^\infty||_{L^{p(\cdot)}(\ell^q)} \leqslant c||\{f_k\}_0^\infty||_{L^{p(\cdot)}(\ell^q)} \tag{20}$$

for all sequences $\{f_k\}_{k=0}^\infty \subset L^{p(\cdot)}(\mathbb{R}^n)$.

It is well-known that (20) does not hold if $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ is not a constant function. However, Diening et al. [7] showed the following helpful theorem, which takes the place of Theorem 4.8.

Let

$$\eta_m(x) = (1+|x|)^{-m} \text{ and } \eta_{\nu,m}(x) = 2^{\nu n}\eta_m(2^\nu x)$$

for $\nu \in \mathbb{N}_0$ and a positive real number m.

THEOREM 4.9. ([7, Theorem 3.2]) *Let* $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n)$ *with* $1 < p_- \leqslant p_+ < \infty$ *and* $1 < q_- \leqslant q_+ < \infty$. *Then the inequality*

$$||\{\eta_{k,m} * f_k\}_{k=0}^\infty||_{L^{p(\cdot)}(\ell^{q(\cdot)})} \leqslant c||\{f_k\}_0^\infty||_{L^{p(\cdot)}(\ell^{q(\cdot)})}$$

holds for every sequence $\{f_k\}_{k=0}^\infty$ *of* L^1_{loc}*-functions and* $m > n$.

Almeida et al. [1] showed the following helpful theorem for $\ell^{q(\cdot)}(L^{p(\cdot)})$ quasi norm.

THEOREM 4.10. ([1, Lemma 4.7]) *Let* $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n)$ *with* $1 < p_- \leqslant p_+ < \infty$ *and* $1 < q_- \leqslant q_+ < \infty$. *Then the inequality*

$$||\{\eta_{k,m} * f_k\}_{k=0}^\infty||_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant c||\{f_k\}_0^\infty||_{\ell^{q(\cdot)}(L^{p(\cdot)})}$$

holds for every sequence $\{f_k\}_{k=0}^\infty$ *of* L^1_{loc}*-functions and* $m > 2n$.

We can generalize [7, Lemma A.6], which is called "the r trick", to following lemma.

LEMMA 4.11. *Let* $r > 0$, $\nu \in \mathbb{N}_0$ *and* $m \geqslant n+1$. *Then there exists* $c = c(r,m,n) > 0$ *such that*

$$\frac{|f(x-z)|}{(1+|2^\nu z|)^{\frac{m}{r}}} \leqslant c\,(\eta_{\nu,m} * |f|^r(x))^{\frac{1}{r}}$$

for all $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n$ *and every* $f \in \mathscr{S}'(\mathbb{R}^n)$ *with* $\operatorname{supp}\mathscr{F}f \subset \{\xi : |\xi| \leqslant 2^{\nu+1}\}$.

Proof. We use the same arguments in the proof of [7, Lemma A.6]. Let $k = (k_1,\cdots,k_n) \in \mathbb{Z}^n$ and $l \in \mathbb{Z}^n$. Fix a dyadic cube $\mathscr{Q} = \mathscr{Q}_{\nu,k} = \prod_{i=1}^n [2^\nu k_i, 2^\nu(k_i+1)]$ and $x - z \in \mathscr{Q}$. Then we have

$$|f(x-z)|^r \leqslant \sup_{\omega\in\mathscr{Q}} |f(\omega)|^r \leqslant c_r 2^{\nu n} \sum_{l\in\mathbb{Z}^n} (1+|l|)^{-m} \int_{\mathscr{Q}_{\nu,k+l}} |f(y)|^r \,\mathrm{d}y$$

for $m \geqslant n+1$. Let $y \in \mathscr{Q}_{\nu,k+l}$. Then we have $1+2^\nu|x-z-y| \approx 1+|l|$ by $x-z \in \mathscr{Q}_{\nu,k}$. Since

$$(1+2^\nu|x-y|)^m \leqslant (1+2^\nu|x-z-y|)^m(1+2^\nu|z|)^m,$$

$$\begin{aligned}\sup_{\omega\in\mathscr{Q}} |f(\omega)|^r &\leqslant c_{r,n}2^{\nu n}\sum_{l\in\mathbb{Z}^n}\int_{\mathscr{Q}_{\nu,k+l}}(1+2^\nu|x-z-y|)^{-m}|f(y)|^r\,\mathrm{d}y\\ &\leqslant c_{r,n}2^{\nu n}\int_{\mathbb{R}^n}(1+2^\nu|x-y|)^{-m}(1+2^\nu|z|)^m|f(y)|^r\,\mathrm{d}y\\ &= c_{r,n}(1+2^\nu|z|)^m(\eta_{\nu,m}*|f|^r)(x).\end{aligned}$$

For $x-z\in\mathscr{Q}_{\nu,k}$, we have

$$\frac{|f(x-z)|}{(1+2^\nu|z|)^{\frac{m}{r}}} \leqslant c(r,m,n)\,(\eta_{\nu,m}*|f|^r(x))^{\frac{1}{r}},$$

where $c(r,m,n)$ depends only on r, m and n. For any $x,z\in\mathbb{R}^n$, then, there exists a $k'\in\mathbb{Z}^n$ such that $x-z\in\mathscr{Q}_{\nu,k'}$. Hence we have the desired inequality for all $x\in\mathbb{R}^n$ and $z\in\mathbb{R}^n$. □

We often use the following relation between $\alpha(x)$ and $\alpha(y)$, which is proved in [11].

LEMMA 4.12. ([11, Lemma 19]) *Let $\alpha(\cdot)\in C^{\log}(\mathbb{R}^n)$. Then there exists a positive constant c such that*

$$2^{k\alpha(x)}\eta_{k,m+R}(x-y) \leqslant c2^{k\alpha(y)}\eta_{k,m}(x-y)$$

for all $x,y\in\mathbb{R}^n$ and $R\geqslant C_{\log}(\alpha)$.

THEOREM 4.13. *Let $p(\cdot),q(\cdot)\in C^{\log}(\mathbb{R}^n)\cap\mathscr{P}_0(\mathbb{R}^n)$ and $\alpha(\cdot)\in C^{\log}(\mathbb{R}^n)$. Let $\Omega=\{\Omega_k\}_{k=0}^\infty$ be a sequence of compact subsets of $\mathbb{R}^n$ such that $\Omega_k\subset\{\xi\in\mathbb{R}^n\colon |\xi|\leqslant 2^{k+1}\}$.*

(i) *If $0<r<\min\{p_-,q_-\}$ and $m>n+2C_{\log}(\alpha)\min\{p_-,q_-\}$, then there exists a constant c such that*

$$\left\|\left\{\sup_{z\in\mathbb{R}^n}\frac{2^{k\alpha(\cdot-z)}|f_k(\cdot-z)|}{1+|2^kz|^{\frac{m}{r}}}\right\}_{k=0}^\infty\right\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \leqslant c\|\{2^{k\alpha(\cdot)}f_k\}_0^\infty\|_{L^{p(\cdot)}(\ell^{q(\cdot)})} \tag{21}$$

for all $\{f_k\}_0^\infty\in L^\Omega_{p(\cdot)}$.

(ii) *If $0<r<\min\{p_-,q_-\}$ and $m>2n+2C_{\log}(\alpha)\min\{p_-,q_-\}$, then there exists a constant c such that*

$$\left\|\left\{\sup_{z\in\mathbb{R}^n}\frac{2^{k\alpha(\cdot-z)}|f_k(\cdot-z)|}{1+|2^kz|^{\frac{m}{r}}}\right\}_{k=0}^\infty\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant c\|\{2^{k\alpha(\cdot)}f_k\}_0^\infty\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \tag{22}$$

for all $\{f_k\}_0^\infty\in L^\Omega_{p(\cdot)}$.

Proof. Let $\{f_k\}_0^\infty \in L^\Omega_{p(\cdot)}$ and $R \geqslant C_{\log}(\alpha)$. It is easy to see that

$$\begin{aligned}\frac{2^{k\alpha(x-z)}|f_k(x-z)|}{1+|2^k z|^{\frac{m}{r}}} &\leqslant \max\{2^{\frac{m}{r}-1},1\}\frac{2^{k\alpha(x-z)}|f_k(x-z)|}{(1+|2^k z|)^{\frac{m}{r}}}\\ &\lesssim \max\{2^{\frac{m}{r}-1},1\}\frac{2^{k\alpha(x)}|f_k(x-z)|}{(1+|2^k z|)^{\frac{m-Rr}{r}}}\\ &\lesssim \max\{2^{\frac{m}{r}-1},1\}2^{k\alpha(x)}\left(\eta_{k,m-Rr}*|f_k|^r(x)\right)^{\frac{1}{r}}\\ &\lesssim \max\{2^{\frac{m}{r}-1},1\}\left((\eta_{k,m-2Rr}(y)*2^{kr\alpha(y)}|f_k(y)|^r)(x)\right)^{\frac{1}{r}}\end{aligned}$$

for $k=0,1,\cdots$ by Lemma 4.11 and 4.12. Hence we have

$$\begin{aligned}&\left|\left|\left\{\sup_{z\in\mathbb{R}^n}\frac{2^{k\alpha(\cdot-z)}|f_k(\cdot-z)|}{1+|2^k z|^{\frac{m}{r}}}\right\}_{k=0}^\infty\right|\right|_{L^{p(\cdot)}(\ell^{q(\cdot)})}\\ &\quad\leqslant c||\{(\eta_{k,m-2Rr}(y)*2^{kr\alpha(y)}|f_k(y)|^r)(\cdot)\}_{k=0}^\infty||^{1/r}_{L^{\frac{p(\cdot)}{r}}(\ell^{\frac{q(\cdot)}{r}})}\end{aligned} \tag{23}$$

and

$$\begin{aligned}&\left|\left|\left\{\sup_{z\in\mathbb{R}^n}\frac{2^{k\alpha(\cdot-z)}|f_k(\cdot-z)|}{1+|2^k z|^{\frac{m}{r}}}\right\}_{k=0}^\infty\right|\right|_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})}\\ &\quad\leqslant c||\{(\eta_{k,m-2Rr}(y)*2^{kr\alpha(y)}|f_k(y)|^r)(\cdot)\}_{k=0}^\infty||^{1/r}_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})}.\end{aligned} \tag{24}$$

By $0<r<\min\{p_-,q_-\}$, $p(\cdot)/r$ and $q(\cdot)/r$ satisfy the assumption of Theorem 4.9 and 4.10. If $m>n+2Rr$, then we have

$$\begin{aligned}\left|\left|\left\{\sup_{z\in\mathbb{R}^n}\frac{2^{k\alpha(\cdot-z)}|f_k(\cdot-z)|}{1+|2^k z|^{\frac{m}{r}}}\right\}_{k=0}^\infty\right|\right|_{L^{p(\cdot)}(\ell^{q(\cdot)})} &\leqslant c||\{2^{kr\alpha(\cdot)}|f_k|^r\}_0^\infty||^{1/r}_{L^{\frac{p(\cdot)}{r}}(\ell^{\frac{q(\cdot)}{r}})}\\ &= c||\{2^{k\alpha(\cdot)}f_k\}_{k=0}^\infty||_{L^{p(\cdot)}(\ell^{q(\cdot)})}\end{aligned}$$

by (23) and Theorem 4.9.

If $m>2n+2Rr$, then we have

$$\begin{aligned}\left|\left|\left\{\sup_{z\in\mathbb{R}^n}\frac{2^{k\alpha(\cdot-z)}|f_k(\cdot-z)|}{1+|2^k z|^{\frac{m}{r}}}\right\}_{k=0}^\infty\right|\right|_{\ell^{q(\cdot)}(L^{p(\cdot)})} &\leqslant c||\{2^{kr\alpha(\cdot)}|f_k|^r\}_0^\infty||^{1/r}_{\ell^{\frac{q(\cdot)}{r}}(L^{\frac{p(\cdot)}{r}})}\\ &= c||\{2^{k\alpha(\cdot)}f_k\}_{k=0}^\infty||_{\ell^{q(\cdot)}(L^{p(\cdot)})}\end{aligned}$$

by (24) and Theorem 4.10. □

THEOREM 4.14. *Let Ω be a compact subset of $\mathbb{R}^n$, $p(\cdot) \in \mathscr{B}_0(\mathbb{R}^n)$ and $q(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$ satisfy $0 < p(x) \leqslant q(x) < \infty$. Then there exists a positive constant c such that*

$$||D^\alpha f||_{L^{q(\cdot)}} \leqslant c||f||_{L^{p(\cdot)}} \tag{25}$$

and

$$||D^\alpha f||_{L^\infty} \leqslant c||f||_{L^{p(\cdot)}} \tag{26}$$

for any $f \in L^\Omega_{p(\cdot)}$ and any multi index α.

Proof. We use the same arguments in the proof of [16, p. 22, Theorem]. Let $\varphi \in \mathscr{S}(\mathbb{R}^n)$ with $\varphi(0) = 1$ and $\operatorname{supp}\mathscr{F}\varphi \subset \{y\colon\ |y| \leqslant 1\}$. Let $f_\delta(x) = \varphi(\delta x)f(x)$ with $0 < \delta < 1$ and $f \in L^\Omega_{p(\cdot)}$. By the Paley–Wiener–Schwartz theorems [16, p. 13, Theorem1, Theorem2], we have $f_\delta \in \mathscr{S}^B$, where B is a closed ball, centered at the origin, such that

$$\{y\ :\ \exists x \in \Omega \text{ and } |x-y| \leqslant 1\} \subset B.$$

We apply Theorem 4.7 to $\varphi = f_\delta$ to obtain

$$||D^\alpha f_\delta||_{L^{q(\cdot)}} \leqslant c'||f_\delta||_{L^{p(\cdot)}} \leqslant c||f||_{L^{p(\cdot)}}.$$

Similarly as in the first step of the proof of [16, p. 22, Theorem], $D^\alpha f_\delta(x) \to D^\alpha f(x)$ (pointwise convergence) if $\delta \downarrow 0$ and $D^\alpha f \in L^\infty$. Hence we have (25). This completes the proof of Theorem 4.14. $\square$

Let $f(\cdot) \in L^\Omega_{p(\cdot)}(\mathbb{R}^n)$ and $\mathscr{F}^{-1}M \in L^1(\mathbb{R}^n)$. Then

$$(\mathscr{F}^{-1}M\mathscr{F}f)(x) = c\int_{\mathbb{R}^n}(\mathscr{F}^{-1}M)(x-y)f(y)\,\mathrm{d}y$$

make sense for any $x \in \mathbb{R}^n$ by the classical Hölder inequality and (26).

THEOREM 4.15. *Let $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathscr{P}_0(\mathbb{R}^n)$ and $\alpha(\cdot) \in C^{\log}(\mathbb{R}^n)$. Let $\Omega = \{\Omega_k\}_{k=0}^\infty$ be a sequence of compact subsets of $\mathbb{R}^n$ such that $\Omega_k \subset \{\xi \in \mathbb{R}^n\colon\ |\xi| \leqslant 2^{k+1}\}$.*

(i) *If $\nu > \frac{n}{2} + \frac{n+3C_{\log}(\alpha)\min\{p_-,q_-\}}{\min\{p_-,q_-\}}$, then there exists a number c such that*

$$||\{2^{k\alpha(\cdot)}\mathscr{F}^{-1}M_k\mathscr{F}f_k\}_{k=0}^\infty||_{L^{p(\cdot)}(\ell^{q(\cdot)})} \leqslant c\sup_l||M_l(2^l\cdot)||_{H_2^\nu}||\{2^{k\alpha(\cdot)}f_k\}_0^\infty||_{L^{p(\cdot)}(\ell^{q(\cdot)})}$$

for $\{f_k(x)\}_{k=0}^\infty \in L^\Omega_{p(\cdot)}$ and $\{M_k(x)\}_{k=0}^\infty \in H_2^\nu(\mathbb{R}^n)$.

(ii) *If $\nu > \frac{n}{2} + \frac{2n+3C_{\log}(\alpha)\min\{p_-,q_-\}}{\min\{p_-,q_-\}}$, then there exists a number c such that*

$$||\{2^{k\alpha(\cdot)}\mathscr{F}^{-1}M_k\mathscr{F}f_k\}_{k=0}^\infty||_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant c\sup_l||M_l(2^l\cdot)||_{H_2^\nu}||\{2^{k\alpha(\cdot)}f_k\}_0^\infty||_{\ell^{q(\cdot)}(L^{p(\cdot)})}$$

for $\{f_k(x)\}_{k=0}^\infty \in L^\Omega_{p(\cdot)}$ and $\{M_k(x)\}_{k=0}^\infty \in H_2^\nu(\mathbb{R}^n)$.

Proof. As we mentioned above, $\mathscr{F}^{-1}M_k\mathscr{F}f_k$ is well-defined. By the same arguments in the proof of [16, p. 31, Theorem], using Theorem 4.13, we have

$$\begin{aligned}
&|2^{k\alpha(x)}\mathscr{F}^{-1}M_k\mathscr{F}f_k(x-z)| \\
&\lesssim \int_{\mathbb{R}^n} 2^{k\alpha(x)}\frac{|(\mathscr{F}^{-1}M_k)(x-z-y)|}{(1+2^k|x-y|)^{\frac{m+Rr}{r}}}|f_k(y)|(1+|2^k(x-y)|^{\frac{m+Rr}{r}})\,dy \\
&\lesssim \int_{\mathbb{R}^n} \frac{|(\mathscr{F}^{-1}M_k)(x-z-y)|}{(1+2^k|x-y|)^{\frac{m}{r}}}2^{k\alpha(y)}|f_k(y)|(1+|2^k(x-y)|^{\frac{m+Rr}{r}})\,dy \\
&\lesssim \sup_{u\in\mathbb{R}^n}\frac{2^{k\alpha(u)}|f_k(u)|}{(1+2^k|x-u|)^{\frac{m}{r}}}\int_{\mathbb{R}^n}|(\mathscr{F}^{-1}M_k)(x-z-y)|(1+|2^k(x-y)|^{\frac{m+Rr}{r}})\,dy.
\end{aligned}$$

Since

$$1+|2^k(x-y)|^{\frac{m+Rr}{r}} \leqslant c(1+|2^k(x-y-z)|^{\frac{m+Rr}{r}})(1+|2^kz|^{\frac{m+Rr}{r}}),$$

$$1+|2^kz|^{\frac{m+Rr}{r}} \leqslant 2(1+|2^kz|)^{\frac{m+Rr}{r}},$$

the arguments in the proof of [16, p. 31, Theorem] implies that

$$\sup_{z\in\mathbb{R}^n}\frac{2^{k\alpha(x)}|\mathscr{F}^{-1}M_k\mathscr{F}f_k(x-z)|}{1+|2^kz|^{\frac{m+Rr}{r}}} \leqslant c\sup_{z\in\mathbb{R}^n}\frac{2^{k\alpha(x-z)}|f_k(x-z)|}{1+|2^kz|^{\frac{m}{r}}}||M_k(2^k\cdot)||_{H_2^\nu} \tag{27}$$

for $0<r<\min\{p_-,q_-\}$ and $\nu>\frac{n}{2}+\frac{m+Rr}{r}$. Hence Theorem 4.15 is an immediate consequence of (27), Theorem 4.13 and the estimate

$$2^{k\alpha(x)}|\mathscr{F}^{-1}M_k\mathscr{F}f_k(x)| \leqslant \sup_{z\in\mathbb{R}^n}\frac{2^{k\alpha(x)}|\mathscr{F}^{-1}M_k\mathscr{F}f_k(x-z)|}{1+|2^kz|^{\frac{m+Rr}{r}}}. \qquad \square$$

DEFINITION 4.16. [16, p. 45, Definition1] The set $\Psi(\mathbb{R}^n)$ is the collection of all systems $\varphi=\{\varphi_j\}_{j=0}^\infty\subset\mathscr{S}(\mathbb{R}^n)$ such that

$$\begin{cases} \operatorname{supp}\varphi_0\subset\{x: |x|\leqslant 2\}, \\ \operatorname{supp}\varphi_j\subset\{x: 2^{j-1}\leqslant|x|\leqslant 2^{j+1}\} \quad \text{for } j=1,2,\cdots, \end{cases}$$

for every multi-index α, there exists a positive number c_α such that

$$2^{j|\alpha|}|D^\alpha\varphi_j(x)|\leqslant c_\alpha$$

for $j=0,1,\cdots$ and $x\in\mathbb{R}^n$ and

$$\sum_{j=0}^\infty\varphi_j(x)=1$$

for $x\in\mathbb{R}^n$.

DEFINITION 4.17. ([16]) For a natural number L, Let $\mathscr{A}_L(\mathbb{R}^n)$ denote the collection of all systems $\varphi = \{\varphi_j(x)\}_{j=0}^{\infty} \subset \mathscr{S}(\mathbb{R}^n)$ of functions with compact supports such that

$$C(\varphi) = \sup_{x\in\mathbb{R}^n} |x|^L \sum_{|\alpha|\leqslant L} |D^\alpha \varphi_0(x)| + \sup_{\substack{x\in\mathbb{R}^n\setminus\{0\}\\ j=1,2,\cdots}} (|x|^L + |x|^{-L}) \sum_{|\alpha|\leqslant L} |D^\alpha \varphi_j(2^j x)| < \infty.$$

All elements $\{\varphi_j\} \in \Psi(\mathbb{R}^n)$ belong to $\mathscr{A}_L(\mathbb{R}^n)$ for any natural number L.

DEFINITION 4.18. For a natural number L, $\varphi = \{\varphi_j(x)\}_{j=0}^{\infty} \in \mathscr{A}_L(\mathbb{R}^n)$, $f \in \mathscr{S}'(\mathbb{R}^n)$, and $a > 0$, we set

$$(\varphi_j^* f)(x) = \sup_{y\in\mathbb{R}^n} \frac{|(\mathscr{F}^{-1}\varphi_j \mathscr{F} f)(x-y)|}{1+|2^j y|^a}, \quad x \in \mathbb{R}^n, \tag{28}$$

(Peetre maximal function), for $j = 0,1,2,\cdots$.

The next proposition corresponds to [16, p. 53, Proposition] in the classical setting.

PROPOSITION 4.19. *Let $a > 0$ in (28) be fixed. Let $p(\cdot), q(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$, $\alpha(\cdot) \in C^{\log}(\mathbb{R}^n)$ and L a natural number larger than $\max\{|\alpha_-|,|\alpha_+|\}+3a+n+2$.*

(i) *There exists a positive number c such that*

$$||2^{k\alpha(x)} \sup_{0<\tau<1} (\varphi_k^{\tau *} f)(x)||_{\ell^{q(x)}} \leqslant c \sup_{0<\tau<1} C(\varphi^\tau)||2^{k\alpha(x)}(\varphi_k^* f)(x)||_{\ell^{q(x)}} \tag{29}$$

for $\varphi = \{\varphi_k(x)\}_{k=0}^{\infty} \in \Psi(\mathbb{R}^n)$, $\varphi^\tau = \{\varphi_k^\tau(x)\}_{k=0}^{\infty} \in \mathscr{A}_L(\mathbb{R}^n)$ with $0 < \tau < 1$, $f \in \mathscr{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

(ii) *There exists a positive number c such that*

$$||2^{k\alpha(\cdot)} \sup_{0<\tau<1} (\varphi_k^{\tau *} f)(\cdot)||_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant c \sup_{0<\tau<1} C(\varphi^\tau)||2^{k\alpha(\cdot)}(\varphi_k^* f)(\cdot)||_{\ell^{q(\cdot)}(L^{p(\cdot)})} \tag{30}$$

for $\varphi = \{\varphi_k(x)\}_{k=0}^{\infty} \in \Psi(\mathbb{R}^n)$, $\varphi^\tau = \{\varphi_k^\tau(x)\}_{k=0}^{\infty} \in \mathscr{A}_L(\mathbb{R}^n)$ with $0 < \tau < 1$ and $f \in \mathscr{S}'(\mathbb{R}^n)$.

Proof. (i) is the same as [16, p. 53, Proposition]. In particular, we remark that there exists a positive number $d > 0$ such that

$$2^{k\alpha(x)}(\varphi_k^{\tau *} f)(x) \leqslant dC(\varphi^\tau) \sum_{\ell=0}^{\infty} 2^{(\sigma-L)|\ell-k|} 2^{\ell\alpha(x)} (\varphi_\ell^* f)(x), \tag{31}$$

where $\sigma = \max\{|\alpha_-|,|\alpha_+|\}+3a+n+2$ and $L > \sigma$, which follows from the arguments in [16, p. 53, Proposition]. Hence we have (29) because $q(x) \in (0,\infty)$ for any fixed $x \in \mathbb{R}^n$.

We will prove (ii). Let $p(\cdot), q(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$ and

$$\mu = \left\| \left\{ 2^{j\alpha(\cdot)} (\varphi_j^* f)(\cdot) \right\}_{j=0}^{\infty} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

Let

$$g(x) = \frac{f(x)}{d \sup_{0<\tau'<1} C(\varphi^{\tau'})\mu} \text{ and } h(x) = \frac{f(x)}{\mu}.$$

Then we have

$$2^{k\alpha(x)} \sup_{0<\tau<1} (\varphi_k^{\tau *} g)(x) \leqslant \sum_{\ell=0}^{\infty} 2^{(\sigma-L)|\ell-k|} 2^{\ell\alpha(x)} (\varphi_\ell^* h)(x)$$

by (31).

Hence we have

$$\left\{ 2^{k\alpha(x)} \sup_{0<\tau<1} (\varphi_k^{\tau *} g)(x) \right\}^{q(x)} \leqslant c \sum_{\ell=0}^{\infty} 2^{q_-(\sigma+\varepsilon_1-L)|\ell-k|} \left\{ 2^{\ell\alpha(x)} (\varphi_\ell^* h)(x) \right\}^{q(x)} \tag{32}$$

with $\varepsilon_1 > 0$ and $L > \sigma + \varepsilon_1$, where

$$c = \left(\sum_{k=0}^{\infty} 2^{-\varepsilon_1 k} \right)^{q_+}.$$

Let

$$\lambda = c \sum_{\ell=0}^{\infty} 2^{q_-(\sigma+\varepsilon_2-L)|\ell-k|} \left\| \left\{ 2^{\ell\alpha(\cdot)} (\varphi_\ell^* h) \right\}^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}$$

with $\varepsilon_2 > \varepsilon_1$ and $L > \sigma + \varepsilon_2$.

Then we consider the following integrand

$$I = \int_{\mathbb{R}^n} \left(\frac{\{2^{k\alpha(x)} \sup_{0<\tau<1} (\varphi_k^{\tau *} g)(x)\}^{q(x)}}{\lambda} \right)^{\frac{p(x)}{q(x)}} \mathrm{d}x.$$

Let $\Omega(\frac{p}{q} > 1)$ and $\Omega(\frac{p}{q} \leqslant 1)$ be the same as (16). Then we have

$$I = \int_{\Omega(\frac{p}{q}>1)} \left(\frac{\{2^{k\alpha(x)} \sup_{0<\tau<1} (\varphi_k^{\tau *} g)(x)\}^{q(x)}}{\lambda} \right)^{\frac{p(x)}{q(x)}} \mathrm{d}x + \int_{\Omega(\frac{p}{q}\leqslant 1)} \left(\frac{\{2^{k\alpha(x)} \sup_{0<\tau<1} (\varphi_k^{\tau *} g)(x)\}^{q(x)}}{\lambda} \right)^{\frac{p(x)}{q(x)}} \mathrm{d}x = I_1 + I_2. \tag{33}$$

First we estimate I_1.

Let $\widetilde{\frac{p(\cdot)}{q(\cdot)}}$ be the same as in the proof of Theorem 4.7. Then $L^{\widetilde{\frac{p(\cdot)}{q(\cdot)}}}$-norm has the triangle inequality property because $\widetilde{\frac{p(\cdot)}{q(\cdot)}}$ is a measurable function on $\mathbb{R}^n$ with range in $(1,\infty)$. Hence we have

$$\left\|\left\{2^{k\alpha(\cdot)}\sup_{0<\tau<1}(\varphi_k^{\tau*}g)(\cdot)\chi_{\Omega(\frac{p}{q}>1)}(\cdot)\right\}^{q(\cdot)}\right\|_{L^{\widetilde{\frac{p(\cdot)}{q(\cdot)}}}} \leqslant \lambda',$$

where

$$\lambda' = c\sum_{\ell=0}^{\infty} 2^{q_-(\sigma+\varepsilon_2-L)|\ell-k|}\left\|\left\{2^{\ell\alpha(\cdot)}(\varphi_\ell^* h)(\cdot)\chi_{\Omega(\frac{p}{q}>1)}(\cdot)\right\}^{q(\cdot)}\right\|_{L^{\widetilde{\frac{p(\cdot)}{q(\cdot)}}}}.$$

This implies

$$\int_{\mathbb{R}^n}\left(\frac{\{2^{k\alpha(x)}\sup_{0<\tau<1}(\varphi_k^{\tau*}g)(x)\chi_{\Omega(\frac{p}{q}>1)}(x)\}^{q(x)}}{\lambda'}\right)^{\widetilde{\frac{p(x)}{q(x)}}} \mathrm{d}x \leqslant 1.$$

Hence we have

$$\begin{aligned} I_1 &= \int_{\mathbb{R}^n}\left(\frac{\{2^{k\alpha(x)}\sup_{0<\tau<1}(\varphi_k^{\tau*}g)(x)\chi_{\Omega(\frac{p}{q}>1)}(x)\}^{q(x)}}{\lambda}\right)^{\widetilde{\frac{p(x)}{q(x)}}} \mathrm{d}x \\ &\leqslant \int_{\mathbb{R}^n}\left(\frac{\{2^{k\alpha(x)}\sup_{0<\tau<1}(\varphi_k^{\tau*}g)(x)\chi_{\Omega(\frac{p}{q}>1)}(x)\}^{q(x)}}{\lambda'}\right)^{\widetilde{\frac{p(x)}{q(x)}}} \mathrm{d}x \leqslant 1 \qquad (34)\end{aligned}$$

by $\lambda' \leqslant \lambda$.

Next we estimate the I_2. By (32), we have

$$I_2 \leqslant \int_{\Omega(\frac{p}{q}\leqslant 1)}\left(\frac{c\sum_{\ell=0}^{\infty}2^{q_-(\sigma+\varepsilon_1-L)|\ell-k|}\left\{2^{\ell\alpha(x)}(\varphi_\ell^* h)(x)\right\}^{q(x)}}{\lambda}\right)^{\frac{p(x)}{q(x)}} \mathrm{d}x. \qquad (35)$$

Since $c2^{(\sigma+\varepsilon_2-L)|\ell-k|}\left\|\left\{2^{\ell\alpha(\cdot)}(\varphi_\ell^* h)\right\}^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leqslant \lambda$, it is easy to see that

$$\begin{aligned}&\frac{c}{\lambda}\sum_{\ell=0}^{\infty}2^{q_-(\sigma+\varepsilon_1-L)|\ell-k|}\left\{2^{\ell\alpha(x)}(\varphi_\ell^* h)(x)\right\}^{q(x)} \\ &\qquad\qquad \leqslant \sum_{\ell=0}^{\infty}2^{-q_-\varepsilon_3|\ell-k|}\frac{\left\{2^{\ell\alpha(x)}(\varphi_\ell^* h)(x)\right\}^{q(x)}}{\left\|\left\{2^{\ell\alpha(\cdot)}(\varphi_\ell^* h)(\cdot)\right\}^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}}\end{aligned}$$

for $\ell = 0, 1, \cdots$, where $\varepsilon_3 = \varepsilon_2 - \varepsilon_1 > 0$. By (35) and Jensen inequality, we have

$$
\begin{aligned}
I_2 &\leqslant \int_{\Omega(\frac{p}{q}\leqslant 1)} \left(\sum_{\ell=0}^{\infty} 2^{-q-\varepsilon_3|\ell-k|} \frac{\left\{2^{\ell\alpha(x)}(\varphi_\ell^* h)(x)\right\}^{q(x)}}{\left\|\left\{2^{\ell\alpha(\cdot)}(\varphi_\ell^* h)(\cdot)\right\}^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}} \right)^{\frac{p(x)}{q(x)}} \mathrm{d}x \\
&\leqslant \int_{\Omega(\frac{p}{q}\leqslant 1)} \sum_{\ell=0}^{\infty} 2^{-q-\varepsilon_3|\ell-k|\frac{p(x)}{q(x)}} \left(\frac{\left\{2^{\ell\alpha(x)}(\varphi_\ell^* h)(x)\right\}^{q(x)}}{\left\|\left\{2^{\ell\alpha(\cdot)}(\varphi_\ell^* h)(\cdot)\right\}^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}} \right)^{\frac{p(x)}{q(x)}} \mathrm{d}x \\
&\leqslant \sum_{\ell=0}^{\infty} 2^{-q-\varepsilon_3|\ell-k|(\frac{p}{q})_-} \int_{\mathbb{R}^n} \left(\frac{\left\{2^{\ell\alpha(x)}(\varphi_\ell^* h)(x)\right\}^{q(x)}}{\left\|\left\{2^{\ell\alpha(\cdot)}(\varphi_\ell^* h)(\cdot)\right\}^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}} \right)^{\frac{p(x)}{q(x)}} \mathrm{d}x \\
&\leqslant \sum_{\ell=0}^{\infty} 2^{-q-\varepsilon_3|\ell-k|(\frac{p}{q})_-} < 2\frac{2^{\varepsilon_3 q-(\frac{p}{q})_-}}{2^{\varepsilon_3 q-(\frac{p}{q})_-}-1} < \infty. \qquad (36)
\end{aligned}
$$

By (33), (34) and (36),

$$
I \leqslant I_1 + I_2 \leqslant 1 + 2\frac{2^{\varepsilon_3 q-(\frac{p}{q})_-}}{2^{\varepsilon_3 q-(\frac{p}{q})_-}-1}
$$

holds. This implies

$$
\begin{aligned}
&\|\{2^{k\alpha(\cdot)} \sup_{0<\tau<1} (\varphi_k^{\tau *} g)(\cdot)\}^{q(\cdot)}\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \\
&\qquad\qquad \leqslant c \sum_{\ell=0}^{\infty} 2^{q-(\sigma+\varepsilon_2-L)|\ell-k|} \left\|\left\{2^{\ell\alpha(\cdot)}(\varphi_\ell^* h)\right\}^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}},
\end{aligned}
$$

where

$$
c = \left(\sum_{k=0}^{\infty} 2^{-\varepsilon_1 k}\right)^{q_+} \left(1 + 2\frac{2^{\varepsilon_3 q-(\frac{p}{q})_-}}{2^{\varepsilon_3 q-(\frac{p}{q})_-}-1}\right)^{\frac{1}{(\frac{p}{q})_-}}.
$$

By (3), we have

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \left\|\left\{2^{k\alpha(\cdot)} \sup_{0<\tau<1} (\varphi_k^{\tau *} g)(\cdot)\right\}^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \\
&\quad \leqslant c \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} 2^{q-(\sigma+\varepsilon_2-L)|\ell-k|} \left\|\left\{2^{\ell\alpha(\cdot)}(\varphi_\ell^* h)\right\}^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \\
&\quad = c \sum_{\ell=0}^{\infty} \left\|\left\{\frac{2^{\ell\alpha(\cdot)}(\varphi_\ell^* f)}{\mu}\right\}^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \sum_{k=0}^{\infty} 2^{q-(\sigma+\varepsilon_2-L)|\ell-k|} \\
&\quad \leqslant \delta, \qquad (37)
\end{aligned}
$$

where

$$\delta = \left(\sum_{k=0}^{\infty} 2^{-\varepsilon_1 k}\right)^{q_+} \left(1 + 2\frac{2^{\varepsilon_3 q_- (\frac{p}{q})_-}}{2^{\varepsilon_3 q_- (\frac{p}{q})_-} - 1}\right)^{\frac{1}{(\frac{p}{q})_-}} 2\left(\frac{2^{q_-(L-(\sigma+\varepsilon_2))}}{2^{q_-(L-(\sigma+\varepsilon_2))} - 1}\right).$$

By (37), we have

$$\sum_{k=0}^{\infty} \left\| \left\{ \frac{2^{k\alpha(\cdot)} \sup_{0<\tau<1} (\varphi_k^{\tau *} f)(\cdot)}{\delta^{\frac{1}{q_-}} d \sup_{0<\tau'<1} C(\varphi^{\tau'})\mu} \right\}^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leqslant \frac{1}{\delta} \sum_{k=0}^{\infty} \left\| \left\{ \frac{2^{k\alpha(\cdot)} \sup_{0<\tau<1} (\varphi_k^{\tau *} f)(\cdot)}{d \sup_{0<\tau'<1} C(\varphi^{\tau'})\mu} \right\}^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}} \leqslant 1.$$

Hence we have

$$||\{2^{k\alpha(\cdot)} \sup_{0<\tau<1} (\varphi_k^{\tau *} f)(\cdot)\}_{k=0}^{\infty}||_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant \delta^{\frac{1}{q_-}} d \sup_{0<\tau<1} C(\varphi^{\tau}) ||\{2^{k\alpha(\cdot)} (\varphi_k^{*} f)(\cdot)\}_{k=0}^{\infty}||_{\ell^{q(\cdot)}(L^{p(\cdot)})}$$

by (3). This proves (29) and (30) provided

$$L > \max\{|\alpha_-|, |\alpha_+|\} + 3a + n + 2. \tag{38}$$

□

REMARK 4.20. Let $p(\cdot), q(\cdot) \in \mathscr{P}_0(\mathbb{R}^n)$ and $\alpha(\cdot) \in C^{\log}(\mathbb{R}^n)$. Let $f(\cdot) \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and $\theta_j \in \Phi(\mathbb{R}^n)$ such that $\operatorname{supp} \mathscr{F}\theta_j \subset \Omega_j$. Then, for any $j = 0, 1, \cdots$, $\theta_j * f \in L^{\Omega_j}_{p(\cdot)}(\mathbb{R}^n)$, which is an immediate consequence of definition of $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$-norm. If $f(\cdot) \in F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, then, for any $j = 0, 1, \cdots$,

$$\left\{(2^{j\alpha(x)}|\theta_j * f(x)|)^{q(x)}\right\}^{\frac{1}{q(x)}} \leqslant \left\{\sum_{k=0}^{\infty} (2^{k\alpha(x)}|\theta_k * f(x)|)^{q(x)}\right\}^{\frac{1}{q(x)}}.$$

This implies

$$||\theta_j * f||_{L^{p(\cdot)}} \leqslant 2^{-j(\alpha_-)} ||f||_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} < \infty$$

for any $j = 0, 1, \cdots$. Hence, we have $\theta_j * f \in L^{\Omega_j}_{p(\cdot)}(\mathbb{R}^n)$ for any $f \in A$. Here A be either $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ or $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

THEOREM 4.21. *Let* $p(\cdot), q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathscr{P}_0(\mathbb{R}^n)$ *and* $\alpha(\cdot) \in C^{\log}(\mathbb{R}^n)$.

(i) *If* $a > \frac{n+3C_{\log}(\alpha)\min\{p_-,q_-\}}{\min\{p_-,q_-\}}$ *in* (28) *and the natural number* L *as in* (38) *is larger than* $\max\{|\alpha_-|,|\alpha_+|\} + \frac{3n+9C_{\log}(\alpha)\min\{p_-,q_-\}}{\min\{p_-,q_-\}} + n + 2$*, then there exists a positive number* c *such that*

$$||2^{k\alpha(\cdot)} \sup_{0<\tau<1} (\varphi_k^{\tau *} f)(\cdot)||_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant c \sup_{0<\tau<1} C(\varphi^\tau)||f||_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \tag{39}$$

for $\varphi = \{\varphi_k(x)\}_{k=0}^\infty \in \Psi(\mathbb{R}^n)$, $\varphi^\tau = \{\varphi_k^\tau(x)\}_{k=0}^\infty \in \mathscr{A}_L(\mathbb{R}^n)$ *with* $0 < \tau < 1$ *and* $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

(ii) *Let* $q(\cdot) \in C^{\log}(\mathbb{R}^n) \cap \mathscr{P}_0(\mathbb{R}^n)$. *If* $a > \frac{2n+3C_{\log}(\alpha)\min\{p_-,q_-\}}{\min\{p_-,q_-\}}$ *in* (28) *and* $L > \max\{|\alpha_-|,|\alpha_+|\} + \frac{6n+9C_{\log}(\alpha)\min\{p_-,q_-\}}{\min\{p_-,q_-\}} + n + 2$*, then there exists a positive number* c *such that*

$$||2^{k\alpha(\cdot)} \sup_{0<\tau<1} (\varphi_k^{\tau *} f)(\cdot)||_{L^{p(\cdot)}(\ell^{q(\cdot)})} \leqslant c \sup_{0<\tau<1} C(\varphi^\tau)||f||_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} \tag{40}$$

for $\varphi = \{\varphi_k(x)\}_{k=0}^\infty \in \Psi(\mathbb{R}^n)$, $\varphi^\tau = \{\varphi_k^\tau(x)\}_{k=0}^\infty \in \mathscr{A}_L(\mathbb{R}^n)$ *with* $0 < \tau < 1$ *and* $f \in F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Proof. We use the same arguments in the proof of [16, p. 56, Theorem] with Proposition 4.19. First, we will prove (ii). By (29), it is sufficient to show that

$$||2^{k\alpha(\cdot)}(\varphi_k^* f)(\cdot)||_{L^{p(\cdot)}(\ell^{q(\cdot)})} \leqslant c||2^{k\alpha(\cdot)}(\mathscr{F}^{-1}\varphi_k * f)(\cdot)||_{L^{p(\cdot)}(\ell^{q(\cdot)})} \tag{41}$$

for $f \in F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, where c is independent of $\varphi \in \Psi(\mathbb{R}^n)$. Recall that the right hand side of (41) is $||f||_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$ by $\{\mathscr{F}^{-1}\varphi_k\}_{k=0}^\infty \in \Phi(\mathbb{R}^n)$. Let $R \geqslant C_{\log}(\alpha)$. Then, it is easy to see that

$$2^{k\alpha(x)}(\varphi_k^* f)(x) \leqslant c \sup_{z\in\mathbb{R}^n} 2^{k\alpha(x-z)} \frac{|(\mathscr{F}^{-1}\varphi_k \mathscr{F} f)(x-z)|}{1+|2^k z|^{a-R}},$$

because

$$\begin{aligned} 2^{k\alpha(x)} \frac{|(\mathscr{F}^{-1}\varphi_k \mathscr{F} f)(x-z)|}{1+|2^k z|^a} &\leqslant \max\{2^{a-1},1\} 2^{k\alpha(x)} \frac{|(\mathscr{F}^{-1}\varphi_k \mathscr{F} f)(x-z)|}{(1+|2^k z|)^a} \\ &\lesssim \max\{2^{a-1},1\} 2^{k\alpha(x-z)} \frac{|(\mathscr{F}^{-1}\varphi_k \mathscr{F} f)(x-z)|}{(1+|2^k z|)^{a-R}} \\ &\leqslant \max\{2^a,2\} 2^{k\alpha(x-z)} \frac{|(\mathscr{F}^{-1}\varphi_k \mathscr{F} f)(x-z)|}{1+|2^k z|^{a-R}} \end{aligned}$$

by Lemma 4.12. Hence (41) is an immediate consequence of Remark 4.20 and Theorem 4.13 with $f_k = \mathscr{F}^{-1}\varphi_k \mathscr{F} f = \mathscr{F}^{-1}\varphi_k * f$ and $a > \frac{n+3C_{\log}(\alpha)\min\{p_-,q_-\}}{r}$, where $0 < r < \min\{p_-,q_-\}$.

Finally we will prove (i). By (30), it is sufficient to show that

$$||2^{k\alpha(\cdot)}\varphi_k^* f||_{\ell^{q(\cdot)}(L^{p(\cdot)})} \leqslant c||2^{k\alpha(\cdot)}\mathscr{F}^{-1}\varphi_k\mathscr{F} f||_{\ell^{q(\cdot)}(L^{p(\cdot)})} \tag{42}$$

for $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $a > \frac{2n+3C_{\log}(\alpha)\min\{p_-,q_-\}}{r}$ in (28), where c is independent of $\varphi \in \Psi(\mathbb{R}^n)$ and $k = 0,1,2,\cdots$. The estimate (42) follows from the Theorem 4.13 (ii). $\square$

We can prove the Lifting property by the same argument in [16, p. 58, Theorem 2.3.8] with Theorem 4.21 which takes the place of [16, p. 56, Theorem2.3.6].

COROLLARY 4.22. (Lifting property) *Let $\mathscr{B}^\sigma$ denote the Bessel potential operator $\mathscr{B}^\sigma = \mathscr{F}^{-1}(1+|\xi|^2)^{-\sigma/2}\mathscr{F}$ for $\sigma \in \mathbb{R}$. Let $p(\cdot),q(\cdot) \in C^{\log}(\mathbb{R}^n)\cap\mathscr{P}_0(\mathbb{R}^n)$ and $\alpha(\cdot) \in C^{\log}(\mathbb{R}^n)$. Then the Bessel potential operator $\mathscr{B}^\sigma$ is an isomorphism between $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and $F^{\alpha(\cdot)+\sigma}_{p(\cdot),q(\cdot)}$. The Bessel potential operator $\mathscr{B}^\sigma$ is an isomorphism between $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ and $B^{\alpha(\cdot)+\sigma}_{p(\cdot),q(\cdot)}$.*

We note that Lifting property for $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ is already proved in [7, Lemma 4.4] by the atomic decomposition techniques.

5. Proof of Fourier multiplier theorems

Proof of Theorem 3.1. We use the same arguments in the proof of [16, p. 57, Theorem]. Let $f \in F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $m(\cdot) \in C^\infty(\mathbb{R}^n)$. Let $\{\theta_k\}_{k=0}^\infty \in \Phi(\mathbb{R}^n)$. Then $\{\mathscr{F}^{-1}\theta_k\}_{k=0}^\infty \in \Psi(\mathbb{R}^n)$. So we write $\mathscr{F}^{-1}\theta_k(x) = \varphi_k(x)$. It is obvious that

$$\theta_k * (\mathscr{F}^{-1}m\mathscr{F} f) = \mathscr{F}^{-1}(\mathscr{F}^{-1}\theta_k \cdot m \cdot \mathscr{F} f) = \mathscr{F}^{-1}\varphi_k m\mathscr{F} f \tag{43}$$

and

$$|(\mathscr{F}^{-1}\varphi_k^\tau\mathscr{F} f)(x)| \leqslant (\varphi_k^{\tau*} f)(x) \tag{44}$$

with $\varphi_k^\tau = m\varphi_k$. By (43), (44) and Theorem 4.21 (ii) with $N > \max\{|\alpha_-|,|\alpha_+|\} + \frac{3n+9C_{\log}(\alpha)\min\{p_-,q_-\}}{\min\{p_-,q_-\}} + n + 2$, we have

$$\begin{aligned}
||\mathscr{F}^{-1}m\mathscr{F} f||_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} &= ||2^{k\alpha(\cdot)}(\mathscr{F}^{-1}\varphi_k^\tau\mathscr{F} f)(\cdot)||_{L^{p(\cdot)}(\ell^{q(\cdot)})} \\
&\leqslant ||2^{k\alpha(\cdot)}(\varphi_k^{\tau*} f)(\cdot)||_{L^{p(\cdot)}(\ell^{q(\cdot)})} \\
&\leqslant c \sup_{0<\tau<1} C(\varphi^\tau)||f||_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}},
\end{aligned}$$

where $\varphi^\tau = \{m\varphi_k\}_{k=0}^\infty$. Hence (4) holds. Finally we prove (ii). By Theorem 4.21 (i) with $N > \max\{|\alpha_-|,|\alpha_+|\} + \frac{6n+9C_{\log}(\alpha)\min\{p_-,q_-\}}{\min\{p_-,q_-\}} + n + 2$, we have

$$
\begin{aligned}
||\mathscr{F}^{-1}m\mathscr{F}f||_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} &= ||2^{k\alpha(\cdot)}(\mathscr{F}^{-1}\varphi_k^{\tau}\mathscr{F}f)(\cdot)||_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\
&\leqslant ||2^{k\alpha(\cdot)} \sup_{0<\tau<1} (\varphi_k^{\tau *} f)(\cdot)||_{\ell^{q(\cdot)}(L^{p(\cdot)})} \\
&\leqslant c \sup_{0<\tau<1} C(\varphi^{\tau})||f||_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}}.
\end{aligned}
$$

This implies (5). This completes the proof of Theorem 3.1. □

Proof of Theorem 3.2. We use the same arguments in the proof of [16, p. 74, Proposition]. Let $f \in \mathscr{S}'(\mathbb{R}^n)$ and $\{\theta_k(x)\}_{k=0}^{\infty} \in \Phi(\mathbb{R}^n)$. If ψ and φ are the functions as (6) and (7), then we have

$$\theta_0 * (\mathscr{F}^{-1}m\mathscr{F}f) = \mathscr{F}^{-1}\{m \cdot \varphi_0 \cdot \mathscr{F}(\theta_0 * f)\}$$

and

$$
\begin{aligned}
\theta_j * (\mathscr{F}^{-1}m\mathscr{F}f) &= \mathscr{F}^{-1}(\mathscr{F}\theta_j \cdot m \cdot \mathscr{F}f) = \mathscr{F}^{-1}(\mathscr{F}\theta_j \cdot m \cdot \varphi_j \cdot \mathscr{F}f) \\
&= \mathscr{F}^{-1}\left\{m \cdot \varphi_j \cdot \mathscr{F}(\theta_j * f)\right\}
\end{aligned}
$$

for $j = 1, 2, \cdots$.

Hence we have

$$\theta_j * (\mathscr{F}^{-1}m\mathscr{F}f) = \theta_j * \mathscr{F}^{-1}m * f = \mathscr{F}^{-1}\left\{M_j \cdot \mathscr{F}(\theta_j * f)\right\}, \tag{45}$$

where

$$M_j(x) = \begin{cases} m(x)\psi(x), & \text{if } j = 0, \\ m(x)\varphi(2^{-j}x), & \text{otherwise.} \end{cases}$$

By (45), we have

$$||\mathscr{F}^{-1}m\mathscr{F}f||_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = ||\{2^{k\alpha(\cdot)}\mathscr{F}^{-1}\{M_k \cdot \mathscr{F}(\theta_k * f)\}\}_{k=0}^{\infty}||_{L^{p(\cdot)}(\ell^{q(\cdot)})}$$

for $f \in F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and

$$||\mathscr{F}^{-1}m\mathscr{F}f||_{B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} = ||\{2^{k\alpha(\cdot)}\mathscr{F}^{-1}\{M_k \cdot \mathscr{F}(\theta_k * f)\}\}_{k=0}^{\infty}||_{\ell^{q(\cdot)}(L^{p(\cdot)})}$$

for $f \in B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. Using Theorem 4.15 with $f_k = \theta_k * f$, we have (8) and (9). This completes the proof of Theorem 3.2. □

Acknowledgements. I would like to express my gratitude to Professor Y. Kobayashi for great support in any manner and for his valuable suggestions in many discussions. I also would like to express my gratitude to the anonymous referee for informing me of the recent book [6] by L. Diening, P. Harjulehto, P. Hästö and M. Růžička and of the useful Lemma 19 of Kempka and J. Vybíral's paper [11]. The Lemma 19 improves the main results considerably.

REFERENCES

[1] A. ALMEIDA AND P. HÄSTÖ, *Besov spaces with variable smoothness and integrability*, J. Funct. Anal. **258**, 5 (2010), 1628–1655.

[2] D. CRUZ-URIBE, SFO, A. FIORENZA, J. M. MARTELL AND C. PÉREZ, *The boundedness of classical operators on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **31** (2006), 239–264.

[3] D. CRUZ-URIBE, SFO, A. FIORENZA AND C. J. NEUGEBAUER, *The maximal function on variable L^p spaces*, Ann. Acad. Sci. Fenn. Math. **28** (2003), 223–238.

[4] L. DIENING, *Maximal function in generalized $L^{p(\cdot)}$ spaces*, Math. Inequal. Appl. **7** (2004), 245–253.

[5] L. DIENING, *Maximal function on Musielak–Orilcz spaces and generalized lebesgue spaces*, Bull. Sci. Math. **129** (2005), 657–700.

[6] L. DIENING, P. HARJULEHTO, P. HÄSTÖ AND M. RŮŽIČKA, *Legesue and Sobolev Spaces with Variable Exponents*, Springer, Lecture Notes in Mathematics **2017**, Springer (2011).

[7] L. DIENING, P. HÄSTÖ AND S. ROUDENKO, *Function spaces of variable smoothness and integrability*, J. Funct. Anal. **256**, 6 (2009), 1731–1768.

[8] L. DIENING AND M. RŮŽIČKA, *Calderón–Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluid dynamics*, J. Reine Angew. Math. **563** (2003), 197–220.

[9] P. GURKA, P. HARJULEHTO AND A. NEKVINDA, *Bessel potential spaces with variable exponent*, Math. Inequal. Appl. **10**, 3 (2007), 661–676.

[10] H. KEMPKA, *2-microlocal Besov and Triebel–Lizorkin spaces of variable integrability*, Rev. Mat. Complut. **22**, 1 (2009), 227–251.

[11] H. KEMPKA AND J. VYBÍRAL, *Spaces of variable smoothness and integrability: Characterizations by local means and ball means of differences*, J. Fourier Anal. Appl., **18**, (2012) 852–891.

[12] O. KOVÁČIK AND J. RÁKOSNÍK, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czech. Math. J. **41** (1991), 592–618.

[13] L. PICK AND M. RŮŽIČKA, *An example of a space $L^{p(\cdot)}$ on which the Hardy–Littlewood maximal operator is not bounded*, Expo. Math. **19** (2001), 369–371.

[14] S. G. SAMKO, *Differentiation and Integration of Variable Order and the Spaces $L^{p(x)}$*, Contemp. Math. **212** (1998), 203–219.

[15] E. M. STEIN, *Harmonic analysis*, Princeton University Press, 1993.

[16] H. TRIEBEL, *Theory of Function Spaces*, Birkhäuser, Basel, Boston, 1983.

[17] J. VYBÍRAL, *Sobolev and Jawerth embeddings for spaces with variable smoothness and integrability*, Ann. Acad. Sci. Fenn., Math. **34**, 2 (2009), 529–544.

[18] J. XU, *The relation between variable Bessel potential spaces and Triebel–Lizorkin spases*, Integral Transforms Spec. Funct. **19** (2008), 599–605.

[19] J. XU, *Variable Besov and Triebel–Lizorkin spaces*, Ann. Acad. Sci. Fenn. Math. **33** (2008), 512–522.

[20] J. XU, *An atomic decomposition of variable Besov and Triebel–Lizorkin spaces*, Armen. J. Math. (2009), 1–12.

(Received April 24, 2011)

Takahiro Noi
Department of Mathematics
Graduate School of Science and Engineering
Chuo University
13-27 Kasuga 1-chome
Bunkyoku, Tokyo
Japan
e-mail: s17004@gug.math.chuo-u.ac.jp