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INTEGRABILITY AND BOUNDEDNESS OF EXTREMAL FUNCTIONS OF A HARDY–SOBOLEV TYPE INEQUALITY

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Abstract. In this paper, we study the properties of positive solutions of an integral equation in R^n

$$u(x)=\int_{R^n}\frac{u^\gamma(y)dy}{|x-y|^{n-\alpha}|y|^{-\sigma}},\quad x\in R^n.$$

Such a nonlinear singular equation is related to the study of the best constant of the Hardy-Sobolev type inequality. According to the Newton potential theory, this integral equation is helpful to understand the Henon type partial differential equation when $\alpha=2$. We use the weighted Hardy-Littlewood-Sobolev inequality to obtain the optimal integrability interval of positive integrable solutions. Namely, if $u\in L^{\frac{2n}{n-\alpha-\sigma}}(R^n)$, then $u\in L^t(R^n)$ for all $t>\frac{n}{n-\alpha-\sigma}$. Based on this result, we prove that those integrable solutions must be bounded.

1. Introduction

In this paper, we study the integrable positive solutions of the following integral equation involving the weighted Riesz potential

$$u(x)=\int_{R^n}\frac{u^\gamma(y)dy}{|x-y|^{n-\alpha}|y|^{-\sigma}},\quad x\in R^n, \tag{1.1}$$

where

$$n\geqslant 3,\quad \sigma\leqslant 0,\quad \alpha+\sigma>0,\quad n-\alpha+\sigma>0,\quad \gamma=\frac{n+\alpha+\sigma}{n-\alpha-\sigma}. \tag{1.2}$$

This type integral equation is associated with the best constant of the Hardy-Sobolev inequality and the more general Caffarelli-Kohn-Nirenberg inequality (cf. [3], [13]). The positive integrable solutions of (1.1) are closely related to the extremal functions of those inequalities. When $\alpha=2$, the equation is the singular Henon type equation [6] as a model to study spherically symmetric clusters of stars. From a mathematical point of view, its quantitative properties are important and interesting in the critical point theory and the nonlinear elliptic equations (cf. [2], [4], [5], [9], [14] and the references therein). The local singularity near the origin was studied in [1].

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Recently, [16] studied the properties of the integrable solutions of the following integral system

$$\begin{cases} u(x) = \displaystyle\int_{R^n} \frac{v^p(y)u^q(y)}{|x-y|^{n-\alpha}|y|^{-\sigma}}dy \\ v(x) = \displaystyle\int_{R^n} \frac{v^q(y)u^p(y)}{|x-y|^{n-\alpha}|y|^{-\sigma}}dy. \end{cases} \tag{1.3}$$

When $u = v$ and $p+q = \gamma$, this system is reduced to (1.1). Thus, we can also see the following results from [16].

PROPOSITION 1.1. *Let $u \in L^s(R^n)$ be a positive solution of (1.1), where $s = \frac{2n}{n-\alpha-\sigma}$. Then u is radially symmetric and monotone decreasing about the origin 0.*

PROPOSITION 1.2. *Under the same assumption of Proposition 1.1, we have $u \in L^t(R^n)$ for all $\frac{1}{t} \in (\frac{-\sigma}{n}, \frac{n-\alpha-\sigma}{n})$.*

PROPOSITION 1.3. *Under the same assumption of Proposition 1.1, if $\alpha + s\sigma \geqslant 0$, then u is bounded.*

In addition, Remark 3.1 in [16] shows that the right end point $\frac{n-\alpha-\sigma}{n}$ of the integrability interval in Proposition 1.2 is optimal. Remark 3.3 in [16] shows that the left end point $\frac{-\sigma}{n}$ is not optimal when $\alpha + s\sigma \geqslant 0$.

In this paper, we remove the restricted condition $\alpha + s\sigma \geqslant 0$, and give a simple proof of the boundedness for the integrable solutions. This simple proof is based on the better integrability interval of those solutions. Namely, $u \in L^t(R^n)$ for all $\frac{1}{t} \in (0, \frac{n-\alpha-\sigma}{n})$. Such an interval was shown by Remark 3.3 in [16] and Proposition 1.2 when $\alpha + s\sigma \geqslant 0$. When $\alpha + s\sigma < 0$, we will extend the left end point from $\frac{-\sigma}{n}$ (which has been given in Proposition 1.2) to 0 in this paper. Thus, we complete thoroughly the integrability interval in all cases.

THEOREM 1.1. *Under the same assumption of Proposition 1.1, we have $u \in L^t(R^n)$ for all $\frac{1}{t} \in (0, \frac{n-\alpha-\sigma}{n})$.*

THEOREM 1.2. *Under the same assumption of Proposition 1.1, u is bounded.*

REMARK. Theorem 1.1 answers the question in Remark 3.2 of [16]. In fact, let $w = u + v$ and $p + q = \gamma$. Then it follows from (1.3) that

$$w(x) \leqslant C \int_{R^n} \frac{w^\gamma(y)dy}{|x-y|^{n-\alpha}|y|^{-\sigma}}.$$

If we set $K(x) = w(x)[\int_{R^n} \frac{w^\gamma(y)dy}{|x-y|^{n-\alpha}|y|^{-\sigma}}]^{-1}$, then $0 < K(x) \leqslant C$ and

$$w(x) = K(x) \int_{R^n} \frac{w^\gamma(y)dy}{|x-y|^{n-\alpha}|y|^{-\sigma}}.$$

Noting the boundedness of $K(x)$, we can also obtain the boundedness of w by the same proof of Theorem 1.1.

To prove Theorem 1.1, we need to use the weighted Hardy-Littlewood-Sobolev (WHLS) inequality (cf. [15])

$$\int_{R^n}\int_{R^n}\frac{f(x)g(y)}{|x|^\alpha|x-y|^\lambda|y|^\beta}dxdy \leqslant C_{\alpha,\beta,s,\lambda,n}\|f\|_r\|g\|_s,$$

where $1<s,r<\infty$, $0<\lambda<n$, $\lambda\leqslant\overline{\lambda}=\lambda+\alpha+\beta\leqslant n$, $\frac{1}{r}+\frac{1}{s}+\frac{\overline{\lambda}}{n}=2$, $\frac{\alpha}{n}<1-\frac{1}{r}<\frac{\lambda+\alpha}{n}$, $\frac{\beta}{n}<1-\frac{1}{s}<\frac{\lambda+\beta}{n}$. In this paper, we will use another equivalent form of the WHLS inequality which can be found in [7] (i.e. (2.8) in [7]): let $Tg(x)=\int_{R^n}\frac{g(y)}{|x|^\alpha|x-y|^\lambda|y|^\beta}dy$, then

$$\|Tg(x)\|_p\leqslant C\|g(x)\|_s \tag{1.4}$$

where $1+\frac{1}{p}=\frac{1}{s}+\frac{\lambda+\alpha+\beta}{n}$, $1<s,p<\infty$, $\alpha+\beta\geqslant 0$, $0<\lambda<n$ and $\frac{1}{p}-\frac{\lambda}{n}<\frac{\alpha}{n}<\frac{1}{p}$.

To find the best constant of the WHLS inequality, Lieb [11] obtained the following Euler-Lagrange system

$$\begin{cases} u(x)=\dfrac{1}{|x|^\alpha}\displaystyle\int_{R^n}\frac{v^q(y)}{|y|^\beta|x-y|^\lambda}dy \\ v(x)=\dfrac{1}{|x|^\beta}\displaystyle\int_{R^n}\frac{u^p(y)}{|y|^\alpha|x-y|^\lambda}dy. \end{cases} \tag{1.5}$$

Such a system has the analogous properties as (1.1). The radial symmetry and the integrability intervals of the integrable solutions of (1.5) were obtained by Jin and Li (cf. [7] and [8]). However, the integrable solutions of (1.5) are not bounded, because those solutions have singularity near the origin 0 (cf. [10]).

2. Proofs of Theorems

Proof of Theorem 1.1. Combining Remark 3.3 in [16] and Proposition 1.2, we know that $u\in L^t(R^n)$ for all $\frac{1}{t}\in(0,\frac{n-\alpha-\sigma}{n})$ when $\alpha+(\gamma+1)\sigma\geqslant 0$. Therefore, we only consider the case of

$$\alpha+(\gamma+1)\sigma<0. \tag{2.1}$$

Step 1. Write

$$a_0=\frac{-\sigma}{n},\quad a_{j+1}=a_j\gamma-\frac{\alpha+\sigma}{n},\quad j=1,2,\cdots. \tag{2.2}$$

By induction we can prove that $\{a_j\}$ is a decreasing sequence.

In fact, $n-\alpha+\sigma>0$ implies

$$-\frac{\sigma}{n}<\frac{n-\alpha-\sigma}{2n},$$

which means $a_0 < \frac{\alpha+\sigma}{n(\gamma-1)}$ or

$$a_0(\gamma-1) - \frac{\alpha+\sigma}{n} < 0. \tag{2.3}$$

This leads to

$$a_1 < a_0.$$

Suppose

$$a_j < a_{j-1} \quad \text{for} \ \ j = 1,2,\cdots,k.$$

By this and (2.3),

$$a_k(\gamma-1) < \frac{\alpha+\sigma}{n}.$$

Therefore,

$$a_{k+1} = a_k\gamma - \frac{\alpha+\sigma}{n} < a_k.$$

Step 2. From (2.1) we have $a_1 > 0$ and

$$\frac{1}{r_1} := \frac{\varepsilon-\sigma}{n}\gamma - \frac{\alpha+\sigma}{n} > 0 \tag{2.4}$$

with $\varepsilon > 0$ sufficiently small.

Applying the weighted Hardy-Littlewood-Sobolev inequality (1.4), from (1.1) we can deduce that

$$\|u\|_{L^{r_1}(R^n)} \leqslant C\|u^\gamma\|_{L^{\frac{nr_1}{n+(\alpha+\sigma)r_1}}(R^n)} = C\|u\|^\gamma_{L^{s_1}(R^n)}, \tag{2.5}$$

where

$$s_1 = \frac{nr_1\gamma}{n+(\alpha+\sigma)r_1}.$$

Noting (2.4), we get

$$\frac{1}{s_1} = \left(\frac{1}{r_1} + \frac{\alpha+\sigma}{n}\right)\frac{1}{\gamma} = \frac{\varepsilon-\sigma}{n}.$$

Taking $\varepsilon > 0$ suitably small, we see

$$\frac{1}{s_1} \in \left(\frac{-\sigma}{n}, \frac{n-\alpha-\sigma}{n}\right).$$

According to Proposition 1.2, we can see $\|u\|_{L^{s_1}(R^n)} < \infty$. Hence, (2.5) leads to $u \in L^{r_1}(R^n)$.

Let $1 < p < q$. If $u \in L^p(R^n) \cap L^q(R^n)$, then $u \in L^\theta(R^n)$ for all $\theta \in (p,q)$ (cf. Ch 2 in [12]).

Thus, combining $u \in L^{r_1}(R^n)$ with Proposition 1.2, we have

$$u \in L^t(R^n), \quad \forall \frac{1}{t} \in \left(\frac{1}{r_1}, \frac{n-\alpha-\sigma}{n}\right).$$

Since a_1 is the limit of the left end point of the **open** interval above, letting $\varepsilon \to 0$, we see easily that

$$u \in L^t(R^n), \quad \forall \frac{1}{t} \in \left(a_1, \frac{n-\alpha-\sigma}{n}\right).$$

By virtue of $0 < a_1 < a_0$, we extend the left end point of the integrability interval from a_0 to a_1.

Step 3. By the same argument of Step 2, from $u \in L^{1/(a_1+\varepsilon)}(R^n)$, we can use the WHLS inequality to improve the integrability interval from $(a_1, \frac{n-\alpha-\sigma}{n})$ to $(a_2, \frac{n-\alpha-\sigma}{n})$ as long as $a_2 > 0$. Similarly, by k steps we can also obtain that the integrability interval is $(a_k, \frac{n-\alpha-\sigma}{n})$ as long as $a_k > 0$.

Suppose $a_j > 0$ for all j. Combining with Step 1, we see that the sequence $\{a_j\}$ is convergent. If we denote $a_j - \frac{\alpha+\sigma}{n(\gamma-1)}$ by b_j, then $\{b_j\}$ is also convergent. On the other hand, (2.2) leads to

$$b_{j+1} = \gamma b_j = \gamma^2 b_{j-1} = \cdots = \gamma^{j+1} b_0.$$

In view of $\gamma > 1$, $\{b_j\}$ is not a convergent sequence. This contradiction shows that there must be l such that $a_l \leqslant 0$.

Step 4. When $a_l = 0$, then Theorem 2.1 is proved. When

$$a_l < 0 \quad \text{and} \quad a_{l-1} > 0,$$

then

$$a_{l-1} < \frac{\alpha+\sigma}{n\gamma}.$$

In addition, $n > \alpha + \sigma$ and $\gamma = \frac{n+\alpha+\sigma}{n-\alpha-\sigma}$ lead to $(\alpha+\sigma)(\gamma+1) < n\gamma$, which implies $\frac{\alpha+\sigma}{n\gamma} < \frac{n-\alpha-\sigma}{n}$. Therefore,

$$\frac{\alpha+\sigma}{n\gamma} \in \left(a_{l-1}, \frac{n-\alpha-\sigma}{n}\right).$$

On the other hand, Step 3 shows that by $l-1$ steps, we can verify $(a_{l-1}, \frac{n-\alpha-\sigma}{n})$ is an integrability interval since $a_{l-1} > 0$. Hence,

$$u \in L^{\frac{n\gamma}{\alpha+\sigma}}(R^n). \tag{2.6}$$

Similar to the argument of Step 2, if we replace $u \in L^{s_1}(R^n)$ by (2.6) and use the WHLS inequality, we also deduce that the integrability interval is $(0, \frac{n-\alpha-\sigma}{n})$. □

Next, we extend the integrability interval from $(0, \frac{n-\alpha-\sigma}{n})$ to $[0, \frac{n-\alpha-\sigma}{n})$.

Proof of Theorem 1.2. According to Proposition 1.1, we know that $u(x)$ is radially symmetric and decreasing about the origin 0. Set

$$U(r) = U(|x|) = u(x).$$

Thus, by Theorem 2.1, for any $\frac{1}{t} \in (0, \frac{n-\alpha-\sigma}{n})$, we have

$$U^t(R)\left(\frac{R}{2}\right)^n \log 2 \leqslant \int_{R/2}^{R} U^t(r) r^{n-1} dr \leqslant C.$$

If we take $\frac{1}{t} = \frac{\varepsilon}{n}$ with ε sufficiently small, then for small $|x|$,

$$u(x) \leqslant C|x|^{-\varepsilon}. \tag{2.7}$$

In view of (1.1), for small $|x|$,

$$u(x) = \int_{B_\delta(0)} \frac{|y|^\sigma u^\gamma(y) dy}{|x-y|^{n-\alpha}} + \int_{R^n \setminus B_\delta(0)} \frac{|y|^\sigma u^\gamma(y) dy}{|x-y|^{n-\alpha}} := I_1 + I_2.$$

By virtue of (2.7), it follows

$$I_1 \leqslant C \int_0^\delta r^{\alpha+\sigma-\gamma\varepsilon} \frac{dr}{r}.$$

Noting ε is sufficiently small, we can see $\alpha + \sigma - \gamma\varepsilon > 0$. Thus $I_1 < \infty$.

When $|x|$ is sufficiently small,

$$I_2 \leqslant C \int_{R^n \setminus B_\delta(0)} \frac{u^\gamma(y) dy}{|y|^{n-\alpha-\sigma}}.$$

By Proposition 2.2, if we take

$$\frac{1}{k} = \frac{\gamma(n-\alpha-\sigma) - \varepsilon}{n},$$

then $u \in L^{k\gamma}(R^n)$. Applying the Hölder inequality, we obtain

$$I_2 \leqslant C\|u\|_{k\gamma}^{\gamma} \left(\int_\delta^\infty r^{n-(n-\alpha-\sigma)k'} \frac{dr}{r} \right)^{1/k'}.$$

Here

$$\frac{1}{k'} = 1 - \frac{1}{k} < \frac{n-\alpha-\sigma}{n}.$$

Therefore, $I_2 < \infty$. The boundedness of u is proved. $\square$

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