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ON MEAN VALUES OF FOURIER TRANSFORMS

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Abstract. We show that there exists a sequence $\{n_k, k \geqslant 1\}$ growing at least geometrically such that for any finite non-negative measure ν such that $\widehat{\nu} \geqslant 0$, any $T > 0$,

$$\int_{-2^{n_k}T}^{2^{n_k}T} \widehat{\nu}(x)\mathrm{d}x \ll_\varepsilon T\, 2^{2^{(1+\varepsilon)n_k}} \int_{\mathbb{R}} \Big|\frac{\sin xT}{xT}\Big|^{n_k^2} \nu(\mathrm{d}x).$$

1. Introduction

Let ν be a finite non-negative measure on $\mathbb{R}$, $\widehat{\nu}(t) = \int_{\mathbb{R}} e^{itx}\nu(\mathrm{d}x)$, then

$$\frac{1}{T}\int_{-T}^{T} \widehat{\nu}(t)\mathrm{d}t = \int_{\mathbb{R}} \frac{\sin Tu}{Tu}\,\nu(\mathrm{d}u).$$

Assume $\widehat{\nu} \geqslant 0$, then

$$\Big|\int_{\mathbb{R}} \Big(\frac{\sin Tu/2}{Tu/2}\Big)^2 \nu(\mathrm{d}u)\Big| \leqslant \frac{1}{T}\int_{-T}^{T} \widehat{\nu}(x)\mathrm{d}x \leqslant 3\int_{\mathbb{R}} \Big(\frac{\sin Tu/2}{Tu/2}\Big)^2 \nu(\mathrm{d}u). \tag{1.1}$$

The first inequality is in turn true at any order: for any positive integer κ,

$$\Big|\int_{\mathbb{R}} \Big(\frac{\sin Tu/2}{Tu/2}\Big)^{2\kappa} \nu(\mathrm{d}u)\Big| \leqslant \frac{1}{T}\int_{-\kappa T}^{\kappa T} |\widehat{\nu}(x)|\mathrm{d}x. \tag{1.2}$$

The question whether the second inequality admits a similar extension arises naturally. We show the existence of a general form of that inequality in which appear constants growing fastly with κ.

THEOREM 1.1. *There exists a sequence $\{n_k, k \geqslant 1\}$ growing at least geometrically such that for any finite non-negative measure ν such that $\widehat{\nu} \geqslant 0$, any $T > 0$, we have*

$$\int_{-2^{n_k}T}^{2^{n_k}T} \widehat{\nu}(x)\mathrm{d}x \ll_\varepsilon T\, 2^{2^{(1+\varepsilon)n_k}} \int_{\mathbb{R}} \Big|\frac{\sin xT}{xT}\Big|^{n_k^2} \nu(\mathrm{d}x).$$

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We don't know whether the constant $2^{2^{(1+\varepsilon)n_k}}$ can be significantly weakened. We believe that this is an important question because of the natural interpretation of mean value problem for Dirichlet polynomials and for the Riemann Zeta function as Fourier inversion formula, see the survey [2], Section 3 and the recent work [3] related to Dirichlet polynomials. The proof of Theorem 1.1 is rather delicate. In order to prepare it, and also to provide the necessary hints concerning inequalities (1.1), (1.2), we begin with introducing some auxiliary functions and displaying some related properties as well.

Let $K(t)=(1-|t|)^+$, $T>0$ and define $K_T(t)=K(t/T)=(1-|t|/T)\chi_{\{|t|\leqslant T\}}$. Then

$$\widehat{K}(u)=\Big(\frac{\sin u/2}{u/2}\Big)^2, \qquad \widehat{K_T}(u)=\frac{1}{T}\Big(\frac{\sin Tu/2}{u/2}\Big)^2.$$

It is easy to check that $K_T(t)+K_T(t+T)+K_T(t-T)=1$, if $|t|\leqslant T$. It follows that

$$\chi_{\{|t-H|\leqslant T\}}\leqslant K_T(t-H)+K_T(t-H+T)+K_T(t-H-T). \tag{1.3}$$

This can be used to prove (1.1). Since $\int_{\mathbb{R}}K_T(t-S)\widehat{\nu}(t)\mathrm{d}t=\int_{\mathbb{R}}e^{iSx}\widehat{K_T}(x)\nu(\mathrm{d}x)$, we deduce

$$\begin{aligned}\int_{H-T}^{H+T}\widehat{\nu}(t)\mathrm{d}t &\leqslant \int_{\mathbb{R}}\Big[K_T(t-H)+K_T(t-H+T)+K_T(t-H-T)\Big]\widehat{\nu}(t)\mathrm{d}t\\ &\leqslant \int_{\mathbb{R}}\widehat{K_T}(u)\Big[e^{iHu}+e^{i(H-T)u}+e^{i(H+T)u}\Big]\nu(\mathrm{d}u)\\ &=\frac{1}{T}\int_{\mathbb{R}}\Big(\frac{\sin Tu/2}{u/2}\Big)^2 e^{iHu}[1+2\cos Tu]\nu(\mathrm{d}u).\end{aligned}$$

This immediately implies the second inequality in (1.1). Notice also by using Fubini's theorem, that for any reals S, γ, T, $T>0$ and any integer $\kappa>0$,

$$\int_{\mathbb{R}}\Big(\frac{\sin T(u-\gamma)/2}{T(u-\gamma)/2}\Big)^{2\kappa}e^{iSu}\nu(\mathrm{d}u)=\frac{1}{T}\int_{\mathbb{R}}e^{-i\gamma(y-S)}K^{*\kappa}\Big(\frac{y-S}{T}\Big)\widehat{\nu}(y)\mathrm{d}y. \tag{1.4}$$

Letting $\kappa=1$, $\gamma=S=0$, gives

$$\frac{1}{T}\int_{\mathbb{R}}\Big(\frac{\sin Tu/2}{u/2}\Big)^2\nu(\mathrm{d}u)=\int_{\mathbb{R}}K_T(y)\widehat{\nu}(y)\mathrm{d}y.$$

As $0\leqslant K_T(y)=\big(1-|y|/T\big)\chi_{\{|y|\leqslant T\}}\leqslant\chi_{\{|y|\leqslant T\}}$, we deduce

$$\Big|\frac{1}{T}\int_{\mathbb{R}}\Big(\frac{\sin Tu/2}{u/2}\Big)^2\nu(\mathrm{d}u)\Big|\leqslant\int_{\mathbb{R}}K_T(y)|\widehat{\nu}(y)|\mathrm{d}y\leqslant\int_{-T}^{T}|\widehat{\nu}(y)|\mathrm{d}y,$$

which yields the first inequality in (1.1).

As to (1.2), some properties of basic convolutions products are needed. Consider for $A>0$ the elementary measures μ_A with density $g_{\mu_A}(x)=\chi_{\{[-A,A]\}}(x)$. Let $0<A\leqslant B$. Plainly

$$\begin{aligned}g_{\mu_A}*g_{\mu_B}(x)&=\int_{\mathbb{R}}g_{\mu_A}(x-y)g_{\mu_B}(y)\mathrm{d}y=\int_{-B}^{B}\chi_{\{[x-A,x+A]\}}(y)\mathrm{d}y\\ &=\lambda\big([-B,B]\cap[x-A,x+A]\big)\\ &=\big[B\wedge(x+A)-(x-A)\vee(-B)\big]\chi_{\{[-A-B,A+B]\}}(x).\end{aligned} \tag{1.5}$$

In particular, introducing the function $\mathbf{g}(x)=\chi_{[-\frac{1}{2},\frac{1}{2}]}(x)$, we have $K(t)=\mathbf{g}*\mathbf{g}(t)$.

More generally, using the notation $g_{\mu_A*\mu_B*\ldots}(x)=g_{\mu_A}*g_{\mu_B}*\ldots(x)$,

LEMMA 1.2. *Let* $0<A_1\leqslant A_2\leqslant\ldots\leqslant A_J$ *and* $\mu=\mu_{A_1}*\mu_{A_2}*\ldots*\mu_{A_J}$. *Then* μ *has density* g *satisfying*

$$0\leqslant g(x)\leqslant G_J\cdot\chi_{\{[-(A_1+A_2+\ldots+A_J),A_1+A_2+\ldots+A_J]\}}(x),$$

where

$$G_J=2^JA_1\cdot\big((A_1+A_2)\wedge A_3\big)\cdot\big((A_1+A_2+A_3)\wedge A_4\big)\ldots\big((A_1+\ldots+A_{J-1})\wedge A_J\big).$$

Proof. We prove it by induction. By (1.5), for every real x

$$0\leqslant g_{\mu_{A_1}*\mu_{A_2}}(x)\leqslant 2A_1\cdot\chi_{\{[-A_1-A_2,A_1+A_2]\}}(x)=2A_1\cdot g_{\mu_{A_1+A_2}}(x).$$

The case $J=2$ is proved. Now for $J=3$, by what preceeeds

$$\begin{aligned}g_{\mu_{A_1}*\mu_{A_2}*\mu_{A_3}}(x)&=\int_{-A_3}^{A_3}g_{\mu_{A_1}*\mu_{A_2}}(x-y)dy\leqslant 2A_1\int_{-A_3}^{A_3}g_{\mu_{A_1+A_2}}(x-y)dy\\&=2A_1\,g_{\mu_{A_1+A_2}*\mu_{A_3}}\\&\leqslant 2A_1\,2(A_3\wedge A_1+A_2)\cdot\chi_{\{[-A_1-A_2-A_3,A_1+A_2+A_3]\}}(x).\end{aligned}$$

The general case follows by iterating the same argument.

In particular, for any positive J,

$$0\leqslant K^{*J}(x)\leqslant\chi_{\{[-J,J]\}}(x). \tag{1.6}$$

Indeed, apply Lemma 1.2 with $A_j\equiv 1/2$. We get

$$0\leqslant K^{*J}(x)=\mathbf{g}^{*2J}(x)\leqslant G_{2J}\cdot\chi_{\{[-J,J]\}}(x),$$

and $G_{2J}=2^{2J}\cdot 2^{-2J}=1$. Inequality (1.2) is yet a direct consequence of (1.4) and (1.6).

Now, we pass to the preparation of the proof of Theorem 1.1, and begin to explain how we shall proceed. By using (1.3) with $H=0$, $T=1/2$, we get

$$\mathbf{g}(x)\leqslant\mathbf{g}^{*2}(2x)+\mathbf{g}^{*2}(2x+1)+\mathbf{g}^{*2}(2x-1). \tag{1.7}$$

An important intermediate step towards the proof of Theorem 1.1 will consist to generalizing that inequality. Our approach can be described as follows. As $\mathbf{g}^{*2}(2v)=\int_{\mathbb{R}}\mathbf{g}(2v-y)\mathbf{g}(y)\mathrm{d}y$, (1.7) can be used to bound the integration term $\mathbf{g}(y)$. And by next reporting this into (1.7), it follows that $\mathbf{g}(x)$ can also be bounded by a sum of terms of type $\mathbf{g}^{*3}$. Call $\mathscr{E}$ this operation. By iterating $\mathscr{E}$, we similarly obtain variant forms of (1.3), involving higher convolution powers of $\mathbf{g}$. The study of the iterated action of $\mathscr{E}$, as well as the order of the constants generated is made in the next section. The action of $\mathscr{E}$ will be first described as the combination of two elementary transforms acting alternatively.

2. Stacks and shifts

We first introduce some operators and related auxiliary results, as well as the necessary notation. Given $f:\mathbb{R}\to\mathbb{R}$ and $a>0$, let $T_af(x)=f(\frac{x}{a})$ be the dilation of f by a^{-1}. Plainly $T_aT_b=T_{ab}$. Notice also that

$$T_a(h*f)=\frac{1}{a}T_ah*T_af, \qquad h\in L^1(\mathbb{R}), \quad f\in L^\infty(\mathbb{R}). \tag{2.1}$$

Indeed

$$\begin{aligned}T_a(h*f)(u)&=\int_{\mathbb{R}}h\Big(\frac{u}{a}-x\Big)f(x)\mathrm{d}x=\int_{\mathbb{R}}T_ah(u-ax)T_af(ax)\mathrm{d}x\\&=\frac{1}{a}\int_{\mathbb{R}}T_ah(u-v)T_af(v)\mathrm{d}v=\frac{1}{a}T_ah*T_af(u).\end{aligned}$$

More generally

$$T_a(h_1*\ldots*h_n)=\frac{1}{a^{n-1}}T_a(h_1)*\ldots*T_a(h_n), \qquad h_1,\ldots,h_n\in L^\infty(\mathbb{R}). \tag{2.2}$$

Introduce also the $\mathbf{g}$-dilations

$$\mathbf{g}_k=T_{2^{-k}}\mathbf{g}, \qquad k=1,2,\ldots.$$

Now let I be a finite subset of $\mathbb{R}$. It will be convenient to denote

$$\Sigma[f(x):I]=\sum_{\rho\in I}f(x+\rho). \tag{2.3}$$

We have

$$\Sigma[f(bx):I]=\Sigma\left[T_{\frac{1}{b}}f(x):\frac{1}{b}I\right]. \tag{2.4}$$

And

$$\widehat{\Sigma}[f:I](t)=\widehat{f}(t)\Big(\sum_{\rho\in I}e^{-it\rho}\Big). \tag{2.5}$$

The linear operator $f\mapsto\Sigma[f:I]$ on $L^1(\mathbb{R})$ commutes with the convolution operation:

$$f*\Sigma[h:I]=\Sigma[f*h:I]. \tag{2.6}$$

Further $\Sigma[f:I]\geqslant 0$ if $f\geqslant 0$. We use the standard arithmetical set notation: $\lambda I=\{\lambda\rho:\rho\in I\}$ and if I, J are two finite subsets, $I+J=\{\rho+\eta:\rho\in I,\eta\in J\}$, repetitions are counted. This is relevant since

$$\Sigma\big[\Sigma[f(x):I]:J\big]=\Sigma[f(x):I+J]. \tag{2.7}$$

Let $j_0<j_1\ldots<j_k$ be a finite set of positive integers, which we denote J. Let $C=\{c_j, j\in J\}$ be some other set of positive integers, not necessarily distinct. We identify (J,C) with $\mathbf{U}:=\{(j,c_j),\ j\in J\}$, and put

$$J^*=\begin{cases}J & \text{if } c_{j_0}>1\\ J\backslash\{j_0\} & \text{if } c_{j_0}=1.\end{cases} \tag{2.8}$$

Define the transform $J \to J_1$ as follows

$$J_1 = \mathscr{D}(J) := J^* \cup (j_0 + J). \tag{2.9}$$

Next define $C \to C_1$ by putting

$$C_1 = \mathscr{T}(C) := \{c_j^1, j \in J_1\}, \tag{2.10}$$

where

$$c_j^1 = \begin{cases} c_j + c_{j-j_0} & \text{if } j \in J^* \cap (j_0 + J) \\ c_{j-j_0} & \text{if } j \in (J^*)^c \cap (j_0 + J) \\ c_j & \text{if } j \in J^* \cap (j_0 + J)^c \text{ and } j > j_0 \\ c_{j_0 - 1} & \text{if } j_0 \in J^*. \end{cases} \tag{2.11}$$

Similarly we identify (J_1, C_1) with $\mathbf{U}_1 := \{(j, c_j^1),\ j \in J_1\}$ The successive transforms $(J,C) \to (J_1,C_1) \to (J_2,C_2) \to \ldots$ turn up to describe the iterated of $\mathscr{E}$, and may be compared to the action of superposing shifted functions. We start with $J = \{1\}$, $C = \{2\}$ corresponding to the basic set

$$\mathbf{U} = \{(1,2)\}.$$

It is easy to check that the iterated transforms of $\mathbf{U}$ progressively generate the sequence of sets

$$\begin{array}{l}
(1,1),(2,2) \\
\qquad (2,3),(3,2) \\
\qquad (2,2),(3,2),(4,3),(5,2) \\
\qquad (2,1),(3,2),(4,5),(5,4),(6,3),(7,2) \\
\qquad\qquad (3,2),(4,6),(5,6),(6,8),\ (7,6),\ (8,3),\ (9,2) \\
\qquad\qquad (3,1),(4,6),(5,6),(6,10),(7,12),(8,9),(9,10),(10,6),(11,3),(12,2) \\
\qquad\qquad\quad \ldots
\end{array}$$

At the m-th step, the set J_m is an interval of integers $\{a_m, \ldots, b_m\}$ with $a_m \to \infty$ slowly, whereas $b_m \to \infty$ very rapidly. More precisely, let for $k = 1,2,\ldots$

$$r_k = \max_{m \geqslant 1} c_k^m.$$

Then $r_1 = 2, r_2 = 3, r_3 = 2, r_4 = 6, \ldots$ etc. And define

$$R_k = r_1 + \ldots + r_k, \qquad \zeta_k = 1 + r_1 + 2r_2 + \ldots + kr_k. \tag{2.12}$$

Let $R_{k-1} < m \leqslant R_k$. At step m, J_m is realized by first shifting J_{m-1} on the right from a length k, next taking union with J_{m-1}^* and in turn

$$J_m = \big\{k, k+1, \ldots, \zeta_{k-1} + (m - R_{k-1})k\big\},$$

if $m < R_k$, whereas $J_{R_k} = \big\{k+1, \ldots, \zeta_k\big\}$.

Write $m = R_{k-1} + h$, $1 \leqslant h \leqslant r_k$. Then we have the relations

$$c_j^m = \begin{cases} c_{j-k}^{R_{k-1}} + [r_k + \ldots + (r_k - h + 1)] & 2k \leqslant j \leqslant \zeta_{k-1} + (m - R_{k-1})k, \\ c_j^{R_{k-1}} & k \leqslant j < 2k. \end{cases} \tag{2.13}$$

After the steps $R_{k-1}+1, R_{k-1}+2, \ldots, R_k$, the function $h \mapsto c_n^{R_{k-1}+h}$ will have increased from

$$r_k + (r_k - 1) + \ldots + 2 + 1 = \frac{r_k(r_k+1)}{2}$$

for all $n \in \{2k, \ldots, \zeta_{k-1}\}$. It follows that

$$\min_{n \in \{2k, \ldots, \zeta_{k-1}\}} r_n \geqslant \frac{r_k^2}{2}. \tag{2.14}$$

Therefore $r_{2k} \geqslant r_k^2/2$. This being true for all k, yields by iteration

$$r_{2^j} \geqslant \frac{1}{2}(r_{2^{j-1}})^2 \geqslant \frac{1}{2}\frac{1}{2^2}(r_{2^{j-2}})^{2^2} \geqslant \ldots \geqslant \frac{1}{2^{1+2+\ldots+2^{H-1}}}(r_{2^{j-H}})^{2^H} = \big(\frac{r_{2^{j-H}}}{2}\big)^{2^H}.$$

We have $r_2 = 3$. Thus

$$r_{2^j} \geqslant \left(\frac{3}{2}\right)^{2^{j-1}}, \qquad j = 1, 2, \ldots. \tag{2.15}$$

We shall deduce from this and (2.14) that r_k grows at least geometrically. Let n and let j be such that $2^{j+1} \leqslant n < 2^{j+2}$. Apply (2.14) with $k = 2^j$. As $n \geqslant 2k$, we have $r_n \geqslant r_{2^j}^2/2$ once $2^{j+2} \leqslant \zeta_{2^j-1}$. But

$$\zeta_{2^j-1} \geqslant \zeta_{2^{j-1}} = 1 + r_1 + 2r_2 + \ldots + 2^{j-1} r_{2^{j-1}} \geqslant 2^{j-1}\left(\frac{3}{2}\right)^{2^{j-2}} \gg 2^{j+2}.$$

Thereby, for j large

$$r_n \geqslant \frac{1}{2} r_{2^j}^2 \geqslant \frac{1}{2}\left(\frac{3}{2}\right)^{2^{j-1}} = \frac{1}{2}\left(\frac{3}{2}\right)^{\frac{2^{j+2}}{8}} \geqslant \frac{1}{2}\left(\frac{3}{2}\right)^{\frac{n}{8}} = \frac{1}{2} e^{(\frac{1}{8}\log\frac{3}{2})n}.$$

Consequently, there is a numerical constant $\rho > 1$, such that for all $n \geqslant 1$, we have

$$r_n \geqslant \rho^n. \tag{2.16}$$

Let $j_m := \#\{J_m\}$. Since $j_m = \zeta_{k-1} + (m - R_{k-1} - 1)k$ if $R_{k-1} < m \leqslant R_k$, we have

$$\sum_{R_{k-1} < m \leqslant R_k} j_m = \sum_{R_{k-1} < m \leqslant R_k} (\zeta_{k-1} + (m - R_{k-1} - 1)k) = r_k \zeta_{k-1} + k \sum_{u=1}^{r_k - 1} u.$$

We thus notice for later use that

$$\sum_{R_{k-1} < m \leqslant R_k} j_m = r_k \zeta_{k-1} + k \frac{r_k(r_k+1)}{2}. \tag{2.17}$$

Let $\prod_j^* f_j$ denotes the convolution product of f_j's. Finally we put

$$\mathbf{I} = \left\{\frac{-1}{2}, 0, \frac{1}{2}\right\}.$$

The key estimate for proving Theorem 1.1 is provided in our next result, which extends inequality (1.7) to arbitrary convolution powers of $\mathbf{g}$.

PROPOSITION 2.1. *Let $k \geqslant 1$ and $R_{k-1} < m \leqslant R_k$. Then*

$$\mathbf{g}(x) \leqslant C_m\, \Sigma\Big[\prod_{(j,c_j)\in \mathbf{U}_m}^{*} \mathbf{g}_j^{*c_j}(x) : \mathbf{I}_m\Big],$$

where $\mathbf{I}_m, C_m$ are defined by the recurrence relations: $\mathbf{I}_0 = \{\frac{-1}{2}, 0, \frac{1}{2}\}$, $C_0 = 2$ and

$$\mathbf{I}_m = \mathbf{I}_{m-1} + r_k\, \mathbf{I}_{m-1}, \qquad\qquad C_m = 2^{k(j_{m-1}-1)} C_{m-1}^2.$$

Proof. We use repeatively the relation (see (2.1))

$$T_{\frac{1}{2}}(h * f) = 2\, T_{\frac{1}{2}} h * T_{\frac{1}{2}} f,$$

$f \in L^\infty(\mathbb{R})$, $h \in L^1(\mathbb{R})$. By (1.7),

$$\begin{aligned}\mathbf{g}(x) &\leqslant 2\left\{\mathbf{g}_1^{*2}(x) + \mathbf{g}_1^{*2}\left(x + \frac{1}{2}\right) + \mathbf{g}_1^{*2}\left(x - \frac{1}{2}\right)\right\} = C_0 \Sigma\big[\mathbf{g}_1^{*2}(x) : \mathbf{I}\big] \\ &= \Sigma\Big[\prod_{(j,c_j)\in \mathbf{U}}^{*} \mathbf{g}_j^{*c_j}(x) : \mathbf{I}\Big]. \end{aligned} \tag{2.18}$$

Now we apply $\mathscr{E}$. We begin with the "stack" of 1's of height $r_1 = 2$. At first

$$\begin{aligned}\mathbf{g}_1^{*2}(x) &= \int_{\mathbb{R}} \mathbf{g}_1(x-y)\mathbf{g}_1(y)\mathrm{d}y = \int_{\mathbb{R}} \mathbf{g}_1(x-y)\mathbf{g}(2y)\mathrm{d}y \\ &\leqslant C_0 \int_{\mathbb{R}} \mathbf{g}_1(x-y)\Sigma\big[\mathbf{g}_1^{*2}(2y) : \mathbf{I}\big]\mathrm{d}y.\end{aligned}$$

But by (2.4), next (2.1)

$$\Sigma\big[\mathbf{g}_1^{*2}(2y) : \mathbf{I}\big] = \Sigma\Big[T_{\frac{1}{2}}(\mathbf{g}_1^{*2})(y) : \frac{1}{2}\mathbf{I}\Big] = \Sigma\Big[2(T_{\frac{1}{2}}\mathbf{g}_1)^{*2}(y) : \frac{1}{2}\mathbf{I}\Big] = 2\Sigma\Big[\mathbf{g}_2^{*2}(y) : \frac{1}{2}\mathbf{I}\Big].$$

Therefore

$$\mathbf{g}_1^{*2}(x) \leqslant 2C_0 \int_{\mathbb{R}} \mathbf{g}_1(x-y)\Sigma\Big[\mathbf{g}_2^{*2}(y) : \frac{1}{2}\mathbf{I}\Big]\mathrm{d}y \;= 2C_0\Sigma\Big[\mathbf{g}_1 * \mathbf{g}_2^{*2}(x) : \frac{1}{2}\mathbf{I}\Big].$$

By reporting in (2.18), we obtain

$$\begin{aligned}\mathbf{g}(x) &\leqslant C_0(2C_0)\Sigma\Big[\Sigma\Big[\mathbf{g}_1 * \mathbf{g}_2^{*2}(x) : \frac{1}{2}\mathbf{I}\Big] : \mathbf{I}\Big] = C_1\Sigma\big[\mathbf{g}_1 * \mathbf{g}_2^{*2}(x) : \mathbf{I}_1\big] \\ &= C_1\, \Sigma\Big[\prod_{(j,c_j)\in \mathbf{U}_1}^{*} \mathbf{g}_j^{*c_j}(x) : \mathbf{I}_1\Big].\end{aligned} \tag{2.19}$$

And $C_1 = 8$.

We now apply $\mathscr{E}$ once again, and bound the generic product $\mathbf{g}_1 * \mathbf{g}_2^{*2}(x)$ by applying (2.19) to $\mathbf{g}_1$. Concretely

$$\begin{aligned}\int_{\mathbb{R}} \mathbf{g}_2^{*2}(x-y)\mathbf{g}_1(y)\mathrm{d}y &= \int_{\mathbb{R}} \mathbf{g}_2^{*2}(x-y)\mathbf{g}(2y)\mathrm{d}y \\ &\leqslant C_1 \int_{\mathbb{R}} \mathbf{g}_2^{*2}(x-y)\Sigma\big[\mathbf{g}_1 * \mathbf{g}_2^{*2}(2y) : \mathbf{I}_1\big]\mathrm{d}y \\ &= 2^{3-1}C_1 \int_{\mathbb{R}} \mathbf{g}_2^{*2}(x-y)\Sigma\Big[\mathbf{g}_2 * \mathbf{g}_3^{*2}(y) : \frac{1}{2}\mathbf{I}_1\Big]\mathrm{d}y \\ &= 2^2 C_1 \Sigma\Big[\mathbf{g}_2^{*3} * \mathbf{g}_3^{*2}(x) : \frac{1}{2}\mathbf{I}_1\Big].\end{aligned}$$

By reporting in (2.19), we obtain

$$\begin{aligned}\mathbf{g}(x) &\leqslant C_1(2^2C_1)\Sigma\Big[\Sigma\Big[\mathbf{g}_2^{*3} * \mathbf{g}_3^{*2}(x) : \frac{1}{2}\mathbf{I}_1\Big] : \mathbf{I}_1\Big] \\ &= C_2\Sigma\big[\mathbf{g}_2^{*3} * \mathbf{g}_3^{*2}(x) : \mathbf{I}_2\big] = C_2\ \Sigma\Big[\prod^{*}_{(j,c_j)\in \mathbf{U}_2} \mathbf{g}_j^{*c_j}(x) : \mathbf{I}_2\Big].\end{aligned} \tag{2.20}$$

And $C_2 = 256$. For the next $\mathscr{E}$-iteration, as we have exhausted the stack of 1's, we now use the stack of 2's of height $r_2 = 3$. We bound the new the generic product $\mathbf{g}_2^{*3} * \mathbf{g}_3^{*2}(x)$ by applying (2.20) to $\mathbf{g}_2(x)$ as follows:

$$\begin{aligned}&\int_{\mathbb{R}} \mathbf{g}_2^{*2} * \mathbf{g}_3^{*2}(x-y)\mathbf{g}_2(y)\mathrm{d}y = \int_{\mathbb{R}} \mathbf{g}_2^{*2} * \mathbf{g}_3^{*2}(x-y)\mathbf{g}(2^2y)\mathrm{d}y \\ &\leqslant C_2 \int_{\mathbb{R}} \mathbf{g}_2^{*2} * \mathbf{g}_3^{*2}(x-y)\Sigma\Big[\mathbf{g}_2^{*3} * \mathbf{g}_3^{*2}(4y) : \mathbf{I}_3\Big]\mathrm{d}y \\ &= 2^{2(3+2-1)}C_2 \int_{\mathbb{R}} \mathbf{g}_2^{*2} * \mathbf{g}_3^{*2}(x-y)\Sigma\Big[\mathbf{g}_4^{*3} * \mathbf{g}_5^{*2}(y) : \frac{1}{4}\mathbf{I}_2\Big]\mathrm{d}y \\ &= 2^8 C_2\ \Sigma\Big[\mathbf{g}_2^{*2} * \mathbf{g}_3^{*2} * \mathbf{g}_4^{*3} * \mathbf{g}_5^{*2}(x) : \frac{1}{4}\mathbf{I}_2\Big].\end{aligned}$$

By reporting in (2.20), we obtain

$$\begin{aligned}\mathbf{g}(x) &\leqslant 2^8 C_2^2\ \Sigma\Big[\Sigma\Big[\mathbf{g}_2^{*2} * \mathbf{g}_3^{*2} * \mathbf{g}_4^{*3} * \mathbf{g}_5^{*2}(x) : \frac{1}{4}\mathbf{I}_2\Big] : \mathbf{I}_2\Big] \\ &= C_3\Sigma\big[\mathbf{g}_2^{*2} * \mathbf{g}_3^{*2} * \mathbf{g}_4^{*3} * \mathbf{g}_5^{*2}(x) : \mathbf{I}_3\big],\end{aligned} \tag{2.21}$$

with $C_3 = 16777216$. And so on.

To simplify, let $k \geqslant 1$ and $R_{k-1} < m \leqslant R_k$. At step m, we play with the stack of k's of height r_k and apply the bound previously obtained to the least dilation of $\mathbf{g}$ in the generic product $G = \prod^*_{(j,c_j)\in \mathbf{U}_{m-1}} \mathbf{g}_j^{*c_j}(x)$ from the previous step. The dilation factor being 2^k, the bound of $\mathbf{g}_k(x)$ thereby produces the new terms $T_{2^{-k}}(G)(x) = \prod^*_{(j,c_j)\in \mathbf{U}_{m-1}} \mathbf{g}_{j+k}^{*c_j}(x)$. Hence by (2.2), after integration, a constant factor $2^{k(j_{m-1}-1)}C_{m-1}$. Next we report the bound obtained for the generic products in the inequality from the preceding step. This is exactly what describes transform $\mathscr{D}$. This generates a new constant factor C_{m-1}. Together with the preceding constant factor, this gives the constant

$2^{k(j_{m-1}-1)}C_{m-1}^2 = C_m$. The rule concerning constants C_m being the same at each step inside the block $]R_{k-1}, R_k]$, we have the recurrence relation

$$C_m = 2^{k(j_{m-1}-1)}C_{m-1}^2. \tag{2.22}$$

And the transform $c_j^{m-1} \mapsto c_j^m$ is described by $\mathscr{T}$. □

Let $k \geqslant 1$. Put

$$\gamma_k = \sum_{j \in J_{R_k}} c_j^{R_k}, \qquad d_k = \sum_{j \in J_{R_k}} j c_j^{R_k}. \tag{2.23}$$

We shall now deduce the following estimate.

PROPOSITION 2.2. *Let ν be a finite measure such that $\widehat{\nu} \geqslant 0$. Then for any $W > 0$*

$$\frac{1}{2W}\int_{-W}^{W} \widehat{\nu}(t)\mathrm{d}t \leqslant C_{R_k} 2^{-d_k+1} \int_{\mathbb{R}} \prod_{(j,c_j)\in \mathbf{U}_{R_k}} \left[\frac{\sin(\frac{2Wx}{2^j})}{\frac{2Wx}{2^j}}\right]^{c_j} \Big| \sum_{\rho \in 2W\mathbf{I}_{R_k}} e^{-i\rho x}\Big| \nu(\mathrm{d}x).$$

Proof. Recall that $J_{R_k} = \{k+1, \ldots, \zeta_k\}$. Further, by (2.12)

$$\gamma_k \geqslant \frac{r_k^2}{2}(\zeta_{k-1} - 2k) = \frac{r_k^2}{2}\zeta_{k-1}\Big(1 - \frac{2k}{\zeta_{k-1}}\Big) \geqslant \Big(\frac{1-\varepsilon}{2}\Big) r_k^2 \zeta_{k-1}, \tag{2.24}$$

once $k \geqslant k_\varepsilon$. Similarly

$$d_k \geqslant \frac{r_k^2}{2} \sum_{j \in \{2k, \ldots, \zeta_{k-1}\}} j \geqslant \frac{r_k^2}{4}(\zeta_{k-1}^2 - 4k^2) \geqslant \Big(\frac{1-\varepsilon}{4}\Big) r_k^2 \zeta_{k-1}^2 \tag{2.25}$$

for k large enough.

By Proposition 2.1, with $m = R_k$

$$\mathbf{g}(t) \leqslant C_{R_k} \Sigma\Big[\prod^{*}_{(j,c_j)\in \mathbf{U}_{R_k}} \mathbf{g}_j^{*c_j}(t) : \mathbf{I}_{R_k}\Big].$$

Let $W > 0$. Then by (2.4), next (2.2)

$$\begin{aligned}
\chi_{[-W,W]}(t) = g\Big(\frac{t}{2W}\Big) &\leqslant C_{R_k} \Sigma\Big[\prod^{*}_{(j,c_j)\in \mathbf{U}_{R_k}} \mathbf{g}_j^{*c_j}\Big(\frac{t}{2W}\Big) : \mathbf{I}_{R_k}\Big] \\
&= C_{R_k} \Sigma\Big[T_{2W}\Big(\prod^{*}_{(j,c_j)\in \mathbf{U}_{R_k}} \mathbf{g}_j^{*c_j}\Big)(t) : 2W\mathbf{I}_{R_k}\Big] \\
&= \frac{C_{R_k}}{(2W)^{\gamma_k - 1}} \Sigma\Big[\prod^{*}_{(j,c_j)\in \mathbf{U}_{R_k}} (T_{2W}\mathbf{g}_j)^{*c_j}(t) : 2W\mathbf{I}_{R_k}\Big].
\end{aligned} \tag{2.26}$$

By (2.5)

$$\widehat{\Sigma}\Big[\prod^{*}_{(j,c_j)\in\mathbf{U}_{R_k}}(T_{2W}\mathbf{g}_j)^{*c_j}:2W\mathbf{I}_{R_k}\Big](x)=\prod_{(j,c_j)\in\mathbf{U}_{R_k}}\widehat{T_{2W}\mathbf{g}_j}(x)^{c_j}\,\Big\{\sum_{\rho\in 2W\mathbf{I}_{R_k}}e^{-i\rho x}\Big\}. \qquad (2.27)$$

But

$$\widehat{T_{2W}\mathbf{g}_j}(x)=\int_{\mathbb{R}}e^{ixu}\mathbf{g}\Big(\frac{2^j u}{2W}\Big)\mathrm{d}u=\frac{2W}{2^j}\int_{\mathbb{R}}e^{\frac{i2W}{2^j}xv}\mathbf{g}(v)\mathrm{d}v=\frac{2W}{2^j}\widehat{\mathbf{g}}\Big(\frac{2Wx}{2^j}\Big).$$

Hence

$$\begin{aligned}&\widehat{\Sigma}\Big[\prod^{*}_{(j,c_j)\in\mathbf{U}_{R_k}}(T_{2W}\mathbf{g}_j)^{*c_j}:2W\mathbf{I}_{R_k}\Big](x)\\&=(2W)^{\gamma_k}2^{-d_k}\prod_{(j,c_j)\in\mathbf{U}_{R_k}}\widehat{\mathbf{g}}\Big(\frac{2Wx}{2^j}\Big)^{c_j}\,\Big\{\sum_{\rho\in 2W\mathbf{I}_{R_k}}e^{-i\rho x}\Big\}.\end{aligned} \qquad (2.28)$$

And by the Parseval relation

$$\begin{aligned}\frac{1}{2W}\int_{-W}^{W}\widehat{\nu}(t)\mathrm{d}t&\leqslant\frac{C_{R_k}}{(2W)^{\gamma_k}}\int_{\mathbb{R}}\Sigma\Big[\prod^{*}_{(j,c_j)\in\mathbf{U}_{R_k}}(T_{2W}\mathbf{g}_j)^{*c_j}(t):2W\mathbf{I}_{R_k}\Big]\widehat{\nu}(t)\mathrm{d}t\\&=\frac{C_{R_k}}{(2W)^{\gamma_k}}(2W)^{\gamma_k}2^{-d_k}\int_{\mathbb{R}}\prod_{(j,c_j)\in\mathbf{U}_{R_k}}\widehat{\mathbf{g}}\Big(\frac{2Wx}{2^j}\Big)^{c_j}\Big\{\sum_{\rho\in 2W\mathbf{I}_{R_k}}e^{-i\rho x}\Big\}\nu(\mathrm{d}x)\\&=C_{R_k}2^{-d_k+1}\int_{\mathbb{R}}\prod_{(j,c_j)\in\mathbf{U}_{R_k}}\Bigg(\frac{\sin(\frac{2Wx}{2^j})}{\frac{2Wx}{2^j}}\Bigg)^{c_j}\Big\{\sum_{\rho\in 2W\mathbf{I}_{R_k}}e^{-i\rho x}\Big\}\nu(\mathrm{d}x).\quad\square\end{aligned} \qquad (2.29)$$

3. Proof of Theorem 1.1

By assumption $\nu\geqslant 0$. Choose $W=2^{\zeta_k}T$. Then

$$\frac{\sin(\frac{2Wx}{2^j})}{\frac{2Wx}{2^j}}=\frac{\sin(2^{\zeta_k+1-j}xT)}{2^{\zeta_k+1-j}xT}$$

But we have that

$$\Big|\sin\sum_{k=1}^{n}x_k\Big|\leqslant\sum_{k=1}^{n}\sin x_k, \qquad (3.1)$$

if $0<x_k<\pi$ and $n>1$, see [1] p. 236. From this easily follows that $|\sin nx|\leqslant n|\sin x|$ for any real x and any integer n. Indeed, write $x=x'+k\pi$ with $0<x'<\pi$. Then $|\sin nx|=|\sin(nx'+nk\varphi)|=|\sin nx'|\leqslant n|\sin nx'|=n|\sin nx|$.

Consequently

$$\Big|\frac{\sin 2^{\zeta_k+1-j}xT}{2^{\zeta_k+1-j}xT}\Big|\leqslant\Big|\frac{\sin xT}{xT}\Big|. \qquad (3.2)$$

By reporting and since $\#\{\mathbf{I}_{R_k}\} = 3^{R_k}$ we get

$$\frac{1}{2.2^{\zeta_k}T}\int_{-2^{\zeta_k}T}^{2^{\zeta_k}T}\widehat{\nu}(\mathrm{d}t) \leqslant C_{R_k}3^{R_k}2^{-d_k}\int_{\mathbb{R}}\left|\frac{\sin xT}{xT}\right|^{\gamma_k}\nu(\mathrm{d}x). \tag{3.3}$$

And by using estimates (2.24), (2.25)

$$\frac{1}{2.2^{\zeta_k}T}\int_{-2^{\zeta_k}T}^{2^{\zeta_k}T}\widehat{\nu}(\mathrm{d}t) \leqslant C_{R_k}3^{R_k}2^{-\frac{1}{5}r_k^2\zeta_{k-1}^2}\int_{\mathbb{R}}\left|\frac{\sin xT}{xT}\right|^{\frac{1}{3}r_k^2\zeta_{k-1}}\nu(\mathrm{d}x). \tag{3.4}$$

We now estimate C_{R_k}. By iterating inside the block of integers $]R_{k-1}, R_k]$ the recurrence relation $C_m = 2^{k(j_{m-1}-1)}C_{m-1}^2$ obtained in Proposition 2.1, we obtain

$$C_{R_k} = 2^{k\{(j_{R_{k-1}}-1)+\ldots+(j_{R_k}-1)\}}C_{R_{k-1}}^{2^{r_k}}.$$

According to (2.17), we have

$$\begin{aligned}k\{(j_{R_{k-1}}-1)+\ldots+(j_{R_k}-1)\} &= k\sum_{R_{k-1}<m\leqslant R_k}(j_m-1)\\ &= kr_k(\zeta_{k-1}-1)+k^2\frac{r_k(r_k+1)}{2}.\end{aligned}$$

As $\zeta_k = 1+r_1+2r_2+\ldots+kr_k$, it follows that

$$kr_k(\zeta_{k-1}-1)+k^2\frac{r_k(r_k+1)}{2} \leqslant kr_k\zeta_{k-1}+k^2r_k^2 \leqslant \zeta_k\zeta_{k-1}+\zeta_k^2 \leqslant 2\zeta_k^2.$$

Thus

$$C_{R_k} \leqslant 2^{2\zeta_k^2}\, C_{R_{k-1}}^{2^{r_k}}. \tag{3.5}$$

By successively iterating this, and since $C_{R_1} = 2$, we get

$$\begin{aligned}C_{R_k} &\leqslant 2^{2\{\zeta_k^2+\zeta_{k-1}^2 2^{r_k}+\zeta_{k-2}^2(2^{r_k}+2^{r_{k-1}})+\ldots+\zeta_2^2(2^{r_k}+\ldots+2^{r_3})\}}2^{2^{r_k}+2^{r_{k-1}}+\ldots+2^{r_2}}\\ &\leqslant 2^{2.2^{r_k}k\zeta_k^2}.\end{aligned}$$

But $r_k \geqslant \rho^k$ by (2.16), so that

$$R_k \leqslant \zeta_k = 1+r_1+2r_2+\ldots+kr_k \ll_\varepsilon 2^{\varepsilon r_k}.$$

Hence also

$$C_{R_k} \leqslant 2^{2^{(1+\varepsilon)r_k}}. \tag{3.6}$$

Finally,

$$\begin{aligned}\frac{1}{T}\int_{-2^{\zeta_k}T}^{2^{\zeta_k}T}\widehat{\nu}(\mathrm{d}t) &\ll_\varepsilon 2^{2^{(1+\varepsilon)r_k}}2^{\zeta_k-\frac{1}{5}r_k^2\zeta_{k-1}^2}\int_{\mathbb{R}}\left|\frac{\sin xT}{xT}\right|^{\frac{1}{3}r_k^2\zeta_{k-1}}\nu(\mathrm{d}x)\\ &\ll_\varepsilon 2^{2^{(1+\varepsilon)r_k}}\int_{\mathbb{R}}\left|\frac{\sin xT}{xT}\right|^{r_k^2}\nu(\mathrm{d}x).\end{aligned} \tag{3.7}$$

Thereby, since $\zeta_k \geqslant r_k$

$$\int_{-2^{r_k}T}^{2^{r_k}T}\widehat{\nu}(\mathrm{d}t) \ll_\varepsilon T\,2^{2^{(1+\varepsilon)r_k}}\int_{\mathbb{R}}\left|\frac{\sin xT}{xT}\right|^{r_k^2}\nu(\mathrm{d}x). \tag{3.8}$$

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