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ON NECESSARY AND SUFFICIENT CONDITIONS FOR VARIABLE EXPONENT HARDY INEQUALITY

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Abstract. In this paper we derive close necessary and sufficient conditions on the regularity of the exponent functions p,β such that the variable exponent Hardy inequality holds

$$\left\| x^{\beta(x)-1} \int_0^x f(t)dt \right\|_{L^{p(\cdot)}(0,l)} \leqslant C \left\| x^{\beta(x)} f \right\|_{L^{p(\cdot)}(0,l)}$$

1. Introduction

There are several works devoted to the study of sufficient conditions for the variable exponent Hardy inequality

$$\left\| |x|^{\beta(\cdot)-1} Hf \right\|_{L^{p(\cdot)}(0,l)} \leqslant C \left\| |x|^{\beta(\cdot)} f \right\|_{L^{p(\cdot)}(0,l)}, \quad f \geqslant 0 \tag{1}$$

(see, [3], [4], [6], [7], [1], [10], [11], [12], [13], [14], [15], [16]). According to the mentioned works [3], [7], [10] the sufficient conditions for β,p are $\beta(0) < 1 - \frac{1}{p(0)}$, $p^- > 1$ and the regularity condition

$$p,\beta \in \Lambda.$$

Here Λ is a class of measurable functions $g:(0,l) \to (-\infty,\infty)$ satisfying the condition

$$\limsup_{x\to 0} |g(x)-g(0)| \ln \frac{1}{|x|} < \infty; \tag{2}$$

$Hf(x) = \int_0^x f(t)dt$ is Hardy's operator, $p^- := \inf\{p(x) : x \in (0,l)\}$.

In this paper, we study mainly the regularity conditions. In this way, we derive a new proof of sufficiency, which distinguishes with its simplicity. In different from sufficiency, the necessity part was not studied yet enough. The main objective of our work is to fill this gap. We show that a condition weaker then (2) is necessary for the

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inequality (1) to hold. Also we derive close necessary and sufficient conditions for the inequality (1) to hold when the exponents p,β are monotone functions.

Note, if the powers are not monotone, the condition (2) is sharp in some sense since there exists an example of exponents β,p with $\beta(0)<1-\frac{1}{p(0)}$ and a sequence $\{f_k\}$, which violates for large k the inequality (1) and the condition (2) simultaneously. See, [3], [7] on referred example and the necessity of conditions $p^->1$ and $\beta(0)<1-\frac{1}{p(0)}$.

For problems of boundedness classical integral operators in variable exponent Lebesgue spaces and regularity results for nonlinear elliptic and parabolic equations with nonstandard growth condition see, [2] and references therein.

2. Main Results

As to the basic properties of spaces $L^{p(\cdot)}$, we refer to [8] (see, also [5], [2], [17]).

By $C,C_1,C_2,\ldots$ we denote the positive constants which may change their value at every appearance and such that the value not essential for main purpose of the paper.

We use the notation $u\sim v$ that means there exist a constant C such that

$$C^{-1}u(x)\leqslant v(x)\leqslant Cu(x).$$

Throughout this paper it is assumed that $p(x)$ is a measurable function in $(0,l)$ taking its values from the interval $[1,\infty)$ with $p^+:=\sup\{p(x):x\in(0,l)\}<\infty$.

The space of functions $L^{p(\cdot)}(0,l)$ is introduced as the class of measurable functions $f(x)$ in $(0,l)$, which have a finite $I_{p;(0,l)}(f):=\int_0^l|f(x)|^{p(x)}dx$-modular (we use also notation $I_p(f)$). A norm in $L^{p(\cdot)}(0,l)$ is given in the form

$$\|f\|_{L^{p(\cdot)}(0,l)}=\inf\left\{\lambda>0:I_p\left(\frac{f}{\lambda}\right)\leqslant 1\right\}.$$

In our proofs we shall use following Lemma several times.

LEMMA 1. *Let $s:(0,\delta)\to\mathbb{R}$ be a measurable function satisfying the condition (2). Then the condition (2) for the function s implies the estimate*

$$C^{-1}x^{s(0)}\leqslant x^{s(x)}\leqslant Cx^{s(0)}.$$

To prove this Lemma, note that the inequality is equivalently to

$$-C\leqslant[s(x)-s(0)]\ln\frac{1}{x}\leqslant C,$$

which is the condition (2).

THEOREM 2. *Let $\beta,\ p:(0,l)\to\mathbb{R}$ be measurable functions such that $-\infty<\beta^-\leqslant\beta(x)\leqslant\beta^+<\infty$, $1<p^-\leqslant p(x)\leqslant p^+<\infty$ for $x\in(0,l)$. Then for the inequality (1) to hold it is sufficient that $p,\ \beta\in\Lambda$ and $\beta(0)<1-\frac{1}{p(0)}$.*

In the following two theorems we show that if functions p, β are monotone then a logarithmic condition weaker then (2) is necessary for the functions p, β for the inequality (1) to hold.

THEOREM 3. *Let $\beta \in \mathbb{R}$ a function $p:(0,l) \longrightarrow [1,\infty)$ be increasing on $(0,\varepsilon)$ and such that $p(0) = \lim_{x\to 0} p(x)$ exists, $\beta < 1 - \frac{1}{p(0)}$, $p^- > 1$. Then for the inequality (1) to hold it is necessary that:*

$$|p(2x) - p(x)| \ln \frac{1}{x} \leqslant C; \quad 0 < x < l \tag{3}$$

THEOREM 4. *Let $p \in \mathbb{R}$, $\beta : (0,l) \to [-\infty,\infty)$ be a function decreasing on $(0.\varepsilon)$ such that $\beta(0) = \lim_{x\to 0} \beta(x)$ exists and the conditions $\beta(0) < 1 - \frac{1}{p}$, $p^- > 1$ be satisfied. Then for the inequality (1) to hold it is necessary that*

$$|\beta(x) - \beta(2x)| \ln \frac{1}{x} \leqslant C; \quad 0 < x < l. \tag{4}$$

3. Proof of main Results

Proof of Theorem 2. Let $f(t) \geqslant 0$ be such that $\left\| |x|^{\beta(.)} f \right\|_{L^{p(.)}(0,l)} \leqslant 1$. Then

$$I_{p(.)}\left(x^{\beta(.)} f\right) \leqslant 1. \tag{5}$$

We have to show that

$$I_{p(.);(0,l)}\left(x^{\beta(.)-1} Hf\right) \leqslant C. \tag{6}$$

By Minkowski inequality, for $L^{p(.)}$ norms, we get

$$\left\| x^{\beta-1} Hf \right\|_{L^{p(.)}(0,l)} \leqslant \left\| x^{\beta-1} Hf \right\|_{L^{p(.)}(0,\delta)} + \left\| x^{\beta-1} Hf \right\|_{L^{p(.)}(\delta,l)}$$

$$:= i_1 + i_2 \tag{7}$$

where δ is such that the condition is satisfied in $(0,\delta)$:

$$\beta(0) < \frac{1}{\left(p^-_{(0,\delta)}\right)'}. \tag{8}$$

To achieve simplicity of notation, we denote by p^- the $p^-_{(0,\delta)}$ in this paragraph.

We have

$$I_{p(.);(0,\delta)}\left(x^{\beta(.)-1} Hf\right) = \int_0^\delta x^{(\beta(.)-1)p(.)} \left(\int_0^x f(t)dt\right)^{p(x)} dx.$$

According to (5), (8), condition $\beta\in\Lambda,$ Lemma 1 and Holder's inequality for $x\in(0,\delta)$ we have the estimates

$$\int_0^x f(t)dt=\int_0^x f(t)\chi_{\{t^\beta f\geqslant 1\}}dt+\int_0^x f(t)\chi_{\{t^\beta f\leqslant 1\}}dt$$

$$\leqslant\int_0^x\left(t^{\beta(.)}f(t)\right)^{\frac{p(t)}{p^-}}t^{-\beta(.)}dt+\int_0^x t^{-\beta(.)}dt$$

$$\leqslant\left(\int_0^x\left(t^{\beta(.)}f(t)\right)^{p(t)}dt\right)^{\frac{1}{p^-}}\left(\int_0^x t^{-\beta(.)(p^-)'}dt\right)^{\frac{1}{(p^-)'}}+C\int_0^x t^{-\beta(0)}dt$$

$$\leqslant\left(C_1\int_0^x t^{-\beta(0)(p^-)'}dt\right)^{\frac{1}{(p^-)'}}+\frac{C\delta^{1-\beta(0)}}{1-\beta(0)}$$

$$\leqslant\left(\frac{C_1\delta^{1-\beta(0)(p^-)'}}{1-\beta(0)(p^-)'}\right)^{\frac{1}{(p^-)'}}+\frac{C\delta^{1-\beta(0)}}{1-\beta(0)}=C_2 \tag{9}$$

It follows from the estimate (9) and the inequality (5) that

$$i_1=\int_0^\delta x^{(\beta(.)-1)p(.)}\left(\int_0^x f(t)dt\right)^{p(x)}dx$$

$$\leqslant\int_0^\delta x^{(\beta(.)-1)p(.)}\left(\frac{1}{C_2}\int_0^x f(t)dt\right)^{p(x)}C_2{}^{p(x)}dx$$

$$\leqslant C_1^{p^+-p^-}\int_0^\delta x^{(\beta(.)-1)p(.)}\left(\int_0^x f(t)dt\right)^{p^-}dx \tag{10}$$

Now we need on classical Hardy's inequality with weights (see, e.g. in [9]):

$$\int_a^b v(x)\left(Hf(x)\right)^R dx\leqslant C\int_a^b \omega(x)f(x)^R dx,\quad f\geqslant 0 \tag{11}$$

where $-\infty\leqslant a<b\leqslant\infty,\ R>1;\ v(x),\omega(x):(a,b)\to(0,\infty)$ are measurable functions. For the inequality (11) to hold it is necessary and sufficient that (see, f.e. in [9])

$$\sup_{a<x<b}\left(\int_x^b v(t)dt\right)\left(\int_a^x \omega^{-\frac{1}{R-1}}(t)dt\right)^{R-1}<\infty. \tag{12}$$

Let us verify the validity of the condition (12) with

$$v(t)=x^{(\beta(.)-1)p(.)},\quad \omega(t)=t^{\beta(t)p^-},\quad R=p^-,\quad a=0,\quad b=\delta.$$

Then it follows from Lemma 1 that

$$x^{(\beta(.)-1)p(.)}\sim x^{(\beta(0)-1)p(0)}\quad\text{and}\quad t^{\beta(t)}\sim t^{\beta(0)}.$$

Therefore, and using (5), we infer the finiteness of supremum

$$\sup_{0<x<\delta}\left(\int_x^\delta x^{(\beta(.)-1)p(.)}dt\right)\left(\int_0^x t^{-\frac{\beta(t)p^-}{p^--1}}dt\right)^{p^--1}$$

$$=\sup_{0<x<\delta}\left(\int_x^\delta x^{(\beta(0)-1)p(0)}dt\right)\left(\int_0^x t^{-\frac{\beta(0)p^-}{p^--1}}dt\right)^{p^--1}$$

$$=C(p^-,p(0),\beta(0))<\infty. \tag{13}$$

Hence the condition (12) is satisfied.

Now, applying the inequality (11) from (13) and (10) we infer

$$\int_0^\delta x^{(\beta(.)-1)p(.)}\left(\int_0^x f(t)dt\right)^{p(x)}dx \leqslant C\int_0^\delta \left(t^{\beta(t)}f(t)\right)^{p^-}dt$$

$$\leqslant \int_0^\delta\left(\left(t^{\beta(t)}f(t)\right)^{p(t)}+1\right)dt = C(\delta+1).$$

Hence $I_{p(.);(0,\delta)}\left(x^{\beta(.)-1}Hf\right)\leqslant C(\delta+1)$ and this imply

$$i_1\leqslant C_1=C_1(\delta,p^-,p(0),\beta(0)). \tag{14}$$

To estimate i_2 note that

$$\int_0^l f dx\leqslant \|x^\beta f\|_{p(x)}\|x^{-\beta}\|_{p'(x)}\leqslant \|x^{-\beta}\|_{p'(x)}.$$

Then from the assumption it follows that $\|x^{-\beta}\|_{p'(x)}\leqslant C$ on $(0,l)$, so f is uniformly bounded in $L^1(0,l)$. Now one can argue as following

$$i_2=\left\|x^{\beta-1}Hf\right\|_{L^{p(.)}(\delta,l)}\leqslant\left(\int_0^l f(t)dt\right)\left\|x^{\beta-1}\right\|_{L^{p(.)}(\delta,l)} \tag{15}$$

This estimate and (14) together with (7) complete the proof of Theorem 2. □

Proof of Theorem 3. Put $\delta_k=\varepsilon 4^{-k}$; $k\in\mathbb{N}$, and $f_k(x)=x^{-\frac{1}{p(x)}-\beta}\chi_{(\delta_k,2\delta_k)}(x)$; $x\in(0,l)$. Then for sufficiently large k,

$$I_{p(.)}\left(x^{\beta(x)}f_k(x)\right)=\int_{\delta_k}^{2\delta_k}\left(t^\beta t^{-\frac{1}{p(t)}-\beta}\right)^{p(t)}dt$$

$$=\int_{\delta_k}^{2\delta_k}t^{-1}dt=\ln 2$$

Also

$$I_{p(.)}\left(x^{\beta(x)-1}H\left(f_k(x)\right)\right) \geqslant \int\limits_{3\delta_k}^{4\delta_k}\left(x^{(\beta-1)}\int\limits_{\delta_k}^{2\delta_k} t^{-\frac{1}{p(t)}-\beta}dt\right)^{p(x)} dx$$
$$\geqslant C\int\limits_{3\delta_k}^{4\delta_k}\left(x^{(\beta-1)}\delta_k^{\left(1-\frac{1}{p(2\delta_k)}-\beta\right)}\right)^{p(x)} dx$$
$$\geqslant C\delta_k^{1-\frac{p(3\delta_k)}{p(2\delta_k)}} = Ce^{\frac{1}{p^+}[p(3\delta_k)-p(2\delta_k)]\ln\frac{1}{2\delta_k}}$$

Applying the inequality (1), we have

$$|p(3\delta_k)-p(2\delta_k)|\ln\frac{1}{2\delta_k}\leqslant C, \quad k\in\mathbb{N}$$

which by using of monotony of p implies (3).

This completes the proof of Theorem 3 □

Proof of Theorem 4. Put $\delta_k = \varepsilon 4^{-k}$; $k\in\mathbb{N}$ and $f_k(x) = x^{-\frac{1}{p}-\beta(x)}\chi_{(\delta_k,2\delta_k)}(x)$. Then

$$I_{p(.)}\left(x^{\beta(x)}f_k(x)\right) = \int\limits_{\delta_k}^{2\delta_k}\left(t^{\beta(t)}t^{-\frac{1}{p}-\beta(t)}\right)^{p(t)}dt$$
$$= \int\limits_{\delta_k}^{2\delta_k} t^{-1}dt = \ln 2$$

Also

$$I_{p(.)}\left(x^{\beta(x)-1}H\left(f_k(x)\right)\right) \geqslant \int\limits_{3\delta_k}^{4\delta_k}\left(x^{\beta(x)-1}\int\limits_{\delta_k}^{2\delta_k} t^{-\frac{1}{p}-\beta(t)}dt\right)^{p(x)} dx$$
$$\geqslant C\delta_k^{[\beta(3\delta_k)-\beta(2\delta_k)]p} \geqslant Ce^{p[\beta(3\delta_k)-\beta(2\delta_k)]\ln\frac{1}{\delta_k}}$$

Applying the inequality (1), we have

$$|\beta(2\delta_k)-\beta(3\delta_k)|\ln\frac{1}{\delta_k}\leqslant C, \quad k\in\mathbb{N}$$

which by using of monotony of β implies (4).

This complete the proof of Theorem 4. □

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