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ESTIMATES FOR LOWER BOUNDS OF EIGENVALUES OF THE KLEIN–GORDON OPERATOR

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Abstract. In this paper, we establish some inequalities for eigenvalues of the Klein-Gordon operator on a bounded domain in an n-dimensional Euclidean space. These inequalities give some sharper estimates for lower bounds of the sums of its first k eigenvalues, which improve the recent results of Yildirim Yolcu (Proc. Amer. Math. Soc. **138** (2010), 4059–4066).

1. Introduction

Let Ω be a bounded domain in an n-dimensional Euclidean space $\mathbb{R}^n$. Consider the quadratic form

$$Q(\varphi)=\int_\Omega \overline{\varphi}\sqrt{-\Delta}\varphi,$$

where $\varphi\in C_c^\infty(\Omega)$. The Klein-Gordon operator $H_{0,\Omega}=\sqrt{-\Delta}$ is defined as the Friedrichs extension (cf. [1]) of the quadratic form Q on $L^2(\Omega)$. As a special case of the Klein-Gordon Hamiltonian, it can be used to model relativistic particles in quantum mechanics. Moreover, it is the generator of the Cauchy stochastic process with a killing condition on $\partial\Omega$ (see [3, 4]). It is also called the fractional Laplacian with power $\frac{1}{2}$. Denote by β_j the j-th eigenvalue of $H_{0,\Omega}$. Then its eigenvalues satisfy

$$0<\beta_1<\beta_2\leqslant\beta_3\leqslant\cdots\leqslant\beta_j\leqslant\cdots\to\infty,$$

where each eigenvalue is repeated according to its multiplicity. The purpose of this paper is to give some estimates for the lower bound of $\sum_{j=1}^k\beta_j$.

To begin with, we give a brief review about some related results in this direction. Denote by λ_j the j-th eigenvalues of the Dirichlet Laplacian on Ω. The asymptotic behavior of its k-th eigenvalue λ_k relates to geometric properties of Ω when $k\to\infty$. In fact, the following Weyl's asymptotic formula (see [19]) holds

$$\lambda_k\sim\frac{(2\pi)^2}{(\omega_nV(\Omega))^{\frac{2}{n}}}k^{\frac{2}{n}},\quad \text{as } k\to\infty, \tag{1.1}$$

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where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$ and $V(\Omega)$ denotes the volume of Ω. In 1961, Pólya [15] proved that

$$\lambda_k \geqslant \frac{(2\pi)^2}{(\omega_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}} \tag{1.2}$$

holds on tiling domains in $\mathbb{R}^2$. His proof also works on tiling domains in $\mathbb{R}^n$. Moreover, he conjectured that (1.2) holds for any bounded domain in $\mathbb{R}^n$. Berezin [5] and Lieb [13] made some contributions to the partial solution of this conjecture. In 1983, Li and Yau [12] proved the following so-called Li-Yau inequality

$$\frac{1}{k}\sum_{j=1}^{k} \lambda_j \geqslant \frac{n}{n+2} \frac{(2\pi)^2}{\left(\omega_n V(\Omega)\right)^{\frac{2}{n}}} k^{\frac{2}{n}}. \tag{1.3}$$

Observe that (1.3) is very sharp in the sense of average. In 2000, Laptev and Weidl [11] pointed out that (1.3) can be derived by the Legendre transform of a result derived by Berezin [5]. Hence, (1.3) is also called the Berezin-Li-Yau inequality. In 2003, adding an additional positive term to the right-hand side of (1.3), Melas [14] improved (1.3) to

$$\frac{1}{k}\sum_{j=1}^{k} \lambda_j \geqslant \frac{n}{n+2} \frac{(2\pi)^2}{(\omega_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}} + \frac{1}{24(n+2)} \frac{V(\Omega)}{I(\Omega)}, \tag{1.4}$$

where $I(\Omega) = \min\limits_{a\in\mathbb{R}^n} \int_\Omega |x-a|^2 dx$ is the moment of inertia of Ω. In 2009, when $n=2$ and assuming some geometric properties of the boundary of Ω, Kovařík, Vugalter and Weidl [9] improved (1.4) by adding a positive correction term to its right-hand side (cf. [18]). Recently, Ilyin [10] derived the following asymptotic lower bound for the Dirichlet Laplacian:

$$\frac{1}{k}\sum_{j=1}^{k} \lambda_j \geqslant \frac{n}{n+2} \frac{(2\pi)^2}{(\omega_n V(\Omega))^{\frac{2}{n}}} k^{\frac{2}{n}} + \frac{n}{48} \frac{V(\Omega)}{I(\Omega)} \left(1-\varepsilon_n(k)\right), \tag{1.5}$$

where $0 \leqslant \varepsilon_n(k) = O(k^{-\frac{2}{n}})$ is a infinitesimal of $k^{-\frac{2}{n}}$. Moreover, he obtained some explicit inequalities for some particular cases of n.

For the Klein-Gordon operator, Blumenthal and Getoor [6] established the following analogous formula of the Weyl asymption formula:

$$\beta_k \sim \frac{2\pi}{\left(\omega_n V(\Omega)\right)^{\frac{1}{n}}} k^{\frac{1}{n}}. \tag{1.6}$$

In 2009, Harrell II and Yildirim Yolcu [8] gave a new proof of (1.6). At the same time, they obtained

$$\frac{1}{k}\sum_{j=1}^{k} \beta_j \geqslant \frac{n}{n+1} \frac{2\pi}{(\omega_n V(\Omega))^{\frac{1}{n}}} k^{\frac{1}{n}}. \tag{1.7}$$

In 2010, adding a positive term to the right side of (1.7), Yildirim Yolcu [20] further strengthened (1.7) to

$$\frac{1}{k}\sum_{j=1}^{k}\beta_j \geqslant \frac{n}{n+1}\frac{2\pi}{\left(\omega_n V(\Omega)\right)^{\frac{1}{n}}}k^{\frac{1}{n}} + \frac{nC}{4(n^2-1)}\frac{\left(\omega_n V(\Omega)\right)^{\frac{1}{n}}}{2\pi}\frac{V(\Omega)}{I(\Omega)}k^{-\frac{1}{n}}, \tag{1.8}$$

where

$$C = \min\left\{\frac{1}{6}, \frac{4(n-1)(2\pi)^{2-n}I(\Omega)}{(2n+1)\left(\omega_n V(\Omega)\right)^{\frac{2}{n}}}k^{\frac{2}{n}}\right\}.$$

For more results in this direction, we refer the reader to [7, 20] and the references therein.

In this paper, we obtain the following results for the Klein-Gordon operator:

THEOREM 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Then the eigenvalues of the Klein-Gordon operator $H_{0,\Omega}$ satisfy*

$$\begin{aligned}\frac{1}{k}\sum_{j=1}^{k}\beta_j \geqslant & \frac{n}{n+1}\frac{2\pi}{\left(\omega_n V(\Omega)\right)^{\frac{1}{n}}}k^{\frac{1}{n}} \\ & + \frac{n}{96}\frac{\left(\omega_n V(\Omega)\right)^{\frac{1}{n}}}{2\pi}\frac{V(\Omega)}{I(\Omega)}k^{-\frac{1}{n}}\left(1-\varepsilon_n(k)\right),\end{aligned} \tag{1.9}$$

where $0 \leqslant \varepsilon_n(k) = O(k^{-\frac{2}{n}})$ is an infinitesimal of $k^{-\frac{2}{n}}$.

REMARK 1.1. (1.9) is sharp in view of the asymptotic formula (1.6). Moreover, observe that the first term of (1.9) is the same as that of (1.8) and the second term of (1.9) is $\frac{n^2-1}{4}$ larger than (1.8) when $n \geqslant 3$ (see Remark 2.1). Hence, we get a sharper estimate for the lower bound of $\frac{1}{k}\sum_{j=1}^{k}\beta_j$ when $n \geqslant 3$.

Furthermore, for the special case of $n = 3$, we can give the following explicit lower bounds of $\frac{1}{k}\sum_{j=1}^{k}\beta_j$.

THEOREM 1.2. *Let Ω be a bounded domain in $\mathbb{R}^3$. Then the eigenvalues of the Klein-Gordon operator $H_{0,\Omega}$ satisfy*

$$\begin{aligned}\frac{1}{k}\sum_{j=1}^{k}\beta_j \geqslant & \frac{3}{4}\frac{2\pi}{(\omega_3 V(\Omega))^{\frac{1}{3}}}k^{\frac{1}{3}} + \frac{1}{32}\frac{\left(\omega_3 V(\Omega)\right)^{\frac{1}{3}}}{2\pi}\frac{V(\Omega)}{I(\Omega)}k^{-\frac{1}{3}} \\ & - \frac{7}{5120}\frac{\omega_3 V(\Omega)}{(2\pi)^3}\left(\frac{V(\Omega)}{I(\Omega)}\right)^2 k^{-1}.\end{aligned} \tag{1.10}$$

The first two terms of (1.10) coincide with those of (1.9). The negative contribution from the the third term in (1.10) can be compensated by a $(1-\alpha)$-part of the positive second term where $0 < \alpha < 1$ and α is sufficiently close to 1. Namely we have the following corollary:

COROLLARY 1.1. *Under the same assumptions as Theorem 1.2, we have*

$$\frac{1}{k}\sum_{j=1}^{k}\beta_j \geqslant \frac{3}{4}\frac{2\pi}{(\omega_3 V(\Omega))^{\frac{1}{3}}}k^{\frac{1}{3}}+\frac{\alpha}{32}\frac{\left(\omega_3 V(\Omega)\right)^{\frac{1}{3}}}{2\pi}\frac{V(\Omega)}{I(\Omega)}k^{-\frac{1}{3}}, \tag{1.11}$$

where $\alpha = 0.987$.

2. Proofs of the main results

Let u_j be an orthonormal eigenfuction corresponding to the j-th eigenvalue β_j of the Klein-Gordon operator $H_{0,\Omega}$. Then we define a function $f_j(x)$ by

$$f_j(x)=\begin{cases} u_j(x), & x\in\Omega,\\ 0, & \mathbb{R}^n\backslash\Omega.\end{cases}$$

Denote by $\widehat{f_j}(\xi)$ the Fourier transform of $f_j(x)$, which is given by

$$\widehat{f_j}(\xi)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n} f_j(x)e^{-ix\cdot\xi}dx=(2\pi)^{-\frac{n}{2}}\int_{\Omega}u_j(x)e^{-ix\cdot\xi}dx, \tag{2.1}$$

where $\xi\in\mathbb{R}^n$. Plancherel's theorem implies

$$\int_{\Omega}\widehat{f_j}(\xi)\widehat{f_l}(\xi)d\xi=\delta_{jl}. \tag{2.2}$$

Set $h(\xi)=\sum_{j=1}^{k}|\widehat{f_j}(\xi)|^2$. Using (2.2) and Bessel inequality, one can get

$$0\leqslant h(\xi)\leqslant(2\pi)^{-n}\int_{\Omega}|e^{-ix\cdot\xi}|^2dx=(2\pi)^{-n}V(\Omega). \tag{2.3}$$

Moreover, according to Parseval's identity, we have

$$\begin{aligned}\int_{\mathbb{R}^n}h(\xi)d\xi&=\sum_{j=1}^{k}\int_{\mathbb{R}^n}|\widehat{f_j}(\xi)|^2d\xi=\sum_{j=1}^{k}\int_{\mathbb{R}^n}|f_j(x)|^2dx\\&=\sum_{j=1}^{k}\int_{\Omega}u_j^2(x)dx=k.\end{aligned} \tag{2.4}$$

Since

$$\nabla\widehat{f_j}(\xi)=-(2\pi)^{-\frac{n}{2}}\int_{\Omega}ixu_j(x)e^{-ix\cdot\xi}dx,$$

we have

$$\sum_{j=1}^{k}|\nabla\widehat{f_j}(\xi)|^2\leqslant(2\pi)^{-n}\int_{\Omega}|ixe^{-ix\cdot\xi}|^2dx=(2\pi)^{-n}I(\Omega). \tag{2.5}$$

By translating the domain Ω, we may assume that

$$I(\Omega)=\int_{\Omega}|x|^2dx.$$

Using (2.3) and (2.5), we get

$$|\nabla h(\xi)| \leqslant 2\Big(\sum_{j=1}^{k}|\widehat{f}_j(\xi)|^2\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^{k}|\nabla\widehat{f}_j(\xi)|^2\Big)^{\frac{1}{2}} \leqslant 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)}, \tag{2.6}$$

for every $\xi \in \mathbb{R}^n$. Since

$$\beta_j = \langle u_j, H_{0,\Omega}u_j\rangle = \int_{\mathbb{R}^n}|\xi||\widehat{u}_j(\xi)|^2 d\xi,$$

we have

$$\sum_{j=1}^{k}\beta_j = \int_{\mathbb{R}^n}|\xi|h(\xi)d\xi. \tag{2.7}$$

Therefore, according to (2.3), (2.4), (2.6) and (2.7), our purpose is to find a better solution $M_{A,B}(k)$ of the following minimization problem:

$$\int_{\mathbb{R}^n}|\xi|h(\xi)d\xi \geqslant M_{A,B}(k), \tag{2.8}$$

under the conditions:

$$0 \leqslant h(\xi) \leqslant A, \quad \int_{\mathbb{R}^n}h(\xi)d\xi = k \quad \text{and} \quad |\nabla h(\xi)| \leqslant B,$$

where $A = (2\pi)^{-n}V(\Omega)$ and $B = 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)}$. Denote by $h^*(\xi) = \psi(|\xi|)$ the symmetric decreasing rearrangement of h. By approximating h, we may assume that the decreasing function $\psi : [0,+\infty) \to [0,(2\pi)^{-n}V(\Omega)]$ is absolutely continuous. It follows from (2.4) that

$$k = \int_{\mathbb{R}^n}h(\xi)d\xi = \int_{\mathbb{R}^n}h^*(\xi)d\xi = n\omega_n\int_0^{+\infty}r^{n-1}\psi(r)dr. \tag{2.9}$$

It yields

$$\int_0^{+\infty}r^{n-1}\psi(r)dr = \frac{k}{n\omega_n}. \tag{2.10}$$

Using the well-known properties of the symmetric decreasing rearrangement (see [2, 16]), one can get

$$\sum_{j=1}^{k}\beta_j = \int_{\mathbb{R}^n}|\xi|h(\xi)d\xi \geqslant \int_{\mathbb{R}^n}|\xi|h^*(\xi)d\xi = n\omega_n\int_0^{+\infty}r^n\psi(r)dr. \tag{2.11}$$

Moreover, it is well known (cf. [17]) that

$$0 \leqslant h^*(\xi) \leqslant A, \quad \int_0^{+\infty}h^*(\xi)d\xi = k \quad \text{and} \quad |\nabla h^*(\xi)| \leqslant \text{ess sup}|\nabla h(\xi)|,$$

where ess sup$f$ is the essential supremum of the function f. Hence, according to (2.11), problem (2.8) is equivalent to find a sharper lower bound $M_{A,B}(k)$ of $n\omega_n\int_0^{+\infty}r^n\psi(r)dr$ under the conditions:

$$0 \leqslant \psi(r) \leqslant A, \quad \int_0^{+\infty}r^{n-1}\psi(r)dr = \frac{k}{n\omega_n} \quad \text{and} \quad 0 \leqslant -\psi'(r) \leqslant B.$$

For this goal, we use the following lemma derived by Ilyin [10] to find a sharper lower bound of $\int_0^{+\infty} r^n \psi(r)dr$.

LEMMA 2.1. *Let*

$$\Psi_s(r) = \begin{cases} A, & \text{for} \quad 0 \leqslant r \leqslant s; \\ A - B(r-s), & \text{for} \quad s \leqslant r \leqslant s + \dfrac{A}{B}; \\ 0, & \text{for} \quad r \geqslant s + \dfrac{A}{B}. \end{cases}$$

Assume that $m = \int_0^{+\infty} r^b \Psi_s(r)dr$ *and* $d \geqslant b$. *Then for any decreasing and absolutely continuous function* F *satisfying the conditions*

$$0 \leqslant F \leqslant A, \quad \int_0^{+\infty} r^b F(r)dr = m, \quad 0 \leqslant -F' \leqslant B,$$

the following inequality holds:

$$\int_0^{+\infty} r^d F(r)dr \geqslant \int_0^{+\infty} r^d \Psi_s(r)dr. \tag{2.12}$$

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. According to (2.11), we need to estimate $\int_0^{+\infty} r^n \psi(r)dr$ in order to give an estimate for the lower bound of $\sum_{j=1}^k \beta_j$. Taking $F = \psi$, $b = n-1$, $d = n$ in Lemma 2.1, we have

$$\int_0^{+\infty} r^{n-1} \psi(r)dr = \int_0^{+\infty} r^{n-1} \Psi_s(r)dr. \tag{2.13}$$

and

$$\int_0^{+\infty} r^n \psi(r)dr \geqslant \int_0^{+\infty} r^n \Psi_s(r)dr. \tag{2.14}$$

Then it follows from (2.11) and (2.14) that

$$\sum_{j=1}^k \beta_j \geqslant M_{A,B}(k) = n\omega_n \int_0^{+\infty} r^n \Psi_s(r)dr \tag{2.15}$$

with $A = (2\pi)^{-n} V(\Omega)$, $B = 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)}$ and $m = k(n\omega_n)^{-1}$.

Now we estimate the lower bound of $\int_0^{+\infty} r^n \Psi_s(r)dr$. By a straightforward calculation, one can get

$$\int_0^{+\infty} r^l \Psi_s(r)dr = \frac{A^{l+2}}{(l+1)(l+2)B^{l+1}} \left[(t+1)^{l+2} - t^{l+2} \right], \tag{2.16}$$

where $t = A^{-1}Bs$. Using (2.10), (2.13) and (2.16), we obtain

$$\begin{aligned} \frac{k}{n\omega_n} &= \int_0^{+\infty} r^{n-1} \psi(r)dr = \int_0^{+\infty} r^{n-1} \Psi_s(r)dr \\ &= \frac{A^{n+1}}{n(n+1)B^n} \left[(t+1)^{n+1} - t^{n+1} \right]. \end{aligned}$$

It yields

$$(t+1)^{n+1}-t^{n+1}=k_*, \tag{2.17}$$

where

$$k_*=\frac{(n+1)B^n}{\omega_n A^{n+1}}k.$$

Since $k^*\geqslant 1$ for $k\geqslant 1$ (see (3.11) in [10]), the equation (2.17) has a unique positive solution $t=t(k^*)$. Set

$$\eta=t+\frac{1}{2}. \tag{2.18}$$

Then (2.17) becomes

$$\Big(\eta+\frac{1}{2}\Big)^{n+1}-\Big(\eta-\frac{1}{2}\Big)^{n+1}=k_*. \tag{2.19}$$

The asymptotic expansion for the unique positive root of (2.19) is

$$\eta(k_*)=\zeta-\frac{n-1}{24}\zeta^{-1}+\frac{(n-1)(n-3)(2n+1)}{5760}\zeta^{-3}+\cdots, \tag{2.20}$$

where $\zeta=\big(\frac{k_*}{n+1}\big)^{\frac{1}{n}}$. Making use of (2.15) and (2.16), we know that the lower bound of $\sum_{j=1}^{k}\beta_j$ is

$$n\omega_n\int_0^{+\infty}r^n\Psi_s(r)dr=\frac{n\omega_n A^{n+2}}{(n+1)(n+2)B^{n+1}}\Big[\big(t(k^*)+1\big)^{n+2}-t(k^*)^{n+2}\Big]. \tag{2.21}$$

Using (2.18) and (2.20), we have

$$\begin{aligned}
&\big(t(k^*)+1\big)^{n+2}-t(k^*)^{n+2}\\
=&\Big[\frac{1}{2}+\zeta-\frac{n-1}{24}\zeta^{-1}+\frac{(n-1)(n-3)(2n+1)}{5760}\zeta^{-3}+\cdots\Big]^{n+2}\\
&-\Big[-\frac{1}{2}+\zeta-\frac{n-1}{24}\zeta^{-1}+\frac{(n-1)(n-3)(2n+1)}{5760}\zeta^{-3}+\cdots\Big]^{n+2}\\
=&\binom{n+2}{1}\zeta^{n+1}+2\Big[\frac{1}{2^3}\binom{n+2}{3}-\frac{n-1}{48}\binom{n+2}{2}\binom{2}{1}\Big]\zeta^{n-1}\\
&+2\Big[\frac{1}{2^5}\binom{n+2}{5}-\frac{n-1}{192}\binom{n+2}{4}\binom{4}{1}+\frac{(n-1)^2}{1152}\binom{n+2}{3}\binom{3}{1}\\
&+\frac{(n-1)(n-3)(2n+1)}{11520}\binom{n+2}{2}\binom{2}{1}\Big]\zeta^{n-3}+\cdots\\
=&(n+2)\Big[\zeta^{n+1}+\frac{n+1}{24}\zeta^{n-1}-\frac{(n+1)(n-1)(2n+1)}{1920}\zeta^{n-3}+\cdots\Big],
\end{aligned} \tag{2.22}$$

where the binomial coefficient is $\binom{p}{q}=\frac{p!}{q!(p-q)!}$. Inserting (2.22) into (2.21), we obtain

$$\begin{aligned}
n\omega_n\int_0^{+\infty}r^n\Psi_s(r)dr=&\frac{n\omega_n A^{n+2}}{(n+1)B^{n+1}}\Big[\Big(\frac{k^*}{n+1}\Big)^{1+\frac{1}{n}}+\frac{n+1}{24}\Big(\frac{k^*}{n+1}\Big)^{1-\frac{1}{n}}\\
&-\frac{(n+1)(n-1)(2n+1)}{1920}\Big(\frac{k^*}{n+1}\Big)^{1-\frac{3}{n}}+\cdots\Big].
\end{aligned} \tag{2.23}$$

Substituting $k_* = \frac{(n+1)B^n}{\omega_n A^{n+1}} k$, $A = (2\pi)^{-n} V(\Omega)$ and $B = 2(2\pi)^{-n}\sqrt{V(\Omega)I(\Omega)}$ into (2.23), we deduce

$$\begin{aligned} & n\omega_n \int_0^{+\infty} r^n \Psi_s(r)dr \\ =& \frac{n}{n+1}(A\omega_n)^{-\frac{1}{n}} k^{1+\frac{1}{n}} + \frac{n}{24}(A\omega_n)^{\frac{1}{n}}\left(\frac{A}{B}\right)^2 k^{1-\frac{1}{n}} \\ & - \frac{n(n-1)(2n+1)}{1920}(A\omega_n)^{\frac{3}{n}}\left(\frac{A}{B}\right)^4 k^{1-\frac{3}{n}} + O(k^{1-\frac{5}{n}}) \\ =& \frac{n}{n+1}\frac{2\pi}{(\omega_n V(\Omega))^{\frac{1}{n}}} k^{1+\frac{1}{n}} + \frac{n}{96}\frac{(\omega_n V(\Omega))^{\frac{1}{n}}}{2\pi}\frac{V(\Omega)}{I(\Omega)} k^{1-\frac{1}{n}} \\ & - \frac{n(n-1)(2n+1)}{30720}\frac{(\omega_n V(\Omega))^{\frac{3}{n}}}{(2\pi)^3}\left(\frac{V(\Omega)}{I(\Omega)}\right)^2 k^{1-\frac{3}{n}} + O(k^{1-\frac{5}{n}}). \end{aligned} \tag{2.24}$$

Combining (2.15) and (2.24), we know that (1.9) is true. This completes the proof of Theorem 1.1. □

REMARK 2.1. In fact, (1.8) of Yildirim Yolcu depends on (2.3) of Lemma 2.2 in [20]. Using the same notations as our paper, we observe that it is (see (2.18) of [20]) actually

$$n\omega_n \int_0^{+\infty} r^n \psi(r)dr \geqslant \frac{n}{n+1}(A\omega_n)^{-\frac{1}{n}} k^{1+\frac{1}{n}} + \frac{n}{6(n^2-1)}(A\omega_n)^{\frac{1}{n}}\left(\frac{A}{B}\right)^2 k^{1-\frac{1}{n}}. \tag{2.25}$$

In the proof of Theorem 1.1, we give the first three terms of the asymptotic expansion of the solution $M_{A,B}(k)$. They are the descending powers of k: $k^{1+\frac{1}{n}}$, $k^{1-\frac{1}{n}}$, $k^{1-\frac{3}{n}}$, $\cdots$. Namely we have

$$n\omega_n \int_0^{+\infty} r^n \psi(r)dr \geqslant M_{A,B}(k) = M_0(k) + O(k^{1-\frac{5}{n}}),$$

where $O(k^{1-\frac{5}{n}})$ is an infinitesimal of $k^{1-\frac{5}{n}}$ and

$$\begin{aligned} M_0(k) =& \frac{n}{n+1}(A\omega_n)^{-\frac{1}{n}} k^{1+\frac{1}{n}} + \frac{n}{24}(A\omega_n)^{\frac{1}{n}}\left(\frac{A}{B}\right)^2 k^{1-\frac{1}{n}} \\ & - \frac{n(n-1)(2n+1)}{1920}(A\omega_n)^{\frac{3}{n}}\left(\frac{A}{B}\right)^4 k^{1-\frac{3}{n}}. \end{aligned} \tag{2.26}$$

It is not difficult to find that the first term of (2.26) is the same as that of (2.25) and the second term of (2.26) is $\frac{n^2-1}{4}$ larger than (2.25) when $n \geqslant 3$.

Proof of Theorem 1.2. When $n = 3$, it follows from (2.15) that

$$\sum_{j=1}^{k} \beta_j \geqslant 3\omega_3 \int_0^{\infty} r^3 \Psi_s(r)dr. \tag{2.27}$$

When $n=3$, the equation (2.17) becomes

$$(t+1)^4 - t^4 = k_*. \tag{2.28}$$

The positive root $t(k_*)$ of (2.28) is

$$t(k_*) = \frac{1}{2}\big(\theta(k_*) - \vartheta(k_*)\big) - \frac{1}{2}, \tag{2.29}$$

where

$$\theta(k_*) = \Big(k_* + \sqrt{k_*^2 + \frac{1}{27}}\Big)^{\frac{1}{3}} \quad \text{and} \quad \vartheta(k_*) = \Big(-k_* + \sqrt{k_*^2 + \frac{1}{27}}\Big)^{\frac{1}{3}}.$$

According to (2.21), we have

$$3\omega_3 \int_0^{+\infty} r^3 \Psi_s(r)dr = \frac{3A^5\omega_3}{20B^4}\Big[\big(t(k^*)+1\big)^5 - t(k^*)^5\Big] \tag{2.30}$$

when $n=3$. Hence we need to estimate $\big(t(k_*)+1\big)^5 - t(k_*)^5$. Set $\kappa(k_*) = \frac{1}{2}\big(\theta(k_*) - \vartheta(k_*)\big)$. Using (2.29), we can deduce

$$\begin{aligned}
\big(t(k_*)+1\big)^5 - t(k_*)^5 =& \Big(\kappa(k_*) + \frac{1}{2}\Big)^2 - \Big(\kappa(k_*) - \frac{1}{2}\Big)^2 \\
=& 5\kappa(k_*)^4 + \frac{5}{2}\kappa(k_*)^2 + \frac{1}{16} \\
=& \frac{5}{16}\Big(k_* + \sqrt{k_*^2 + \frac{1}{27}}\Big)^{\frac{4}{3}} + \frac{5}{16}\Big(-k_* + \sqrt{k_*^2 + \frac{1}{27}}\Big)^{\frac{4}{3}} \\
&+ \frac{5}{24}\Big(k_* + \sqrt{k_*^2 + \frac{1}{27}}\Big)^{\frac{2}{3}} + \frac{5}{24}\Big(-k_* + \sqrt{k_*^2 + \frac{1}{27}}\Big)^{\frac{2}{3}} - \frac{7}{48} \\
\geqslant& \frac{5\cdot 2^{\frac{1}{3}}}{8}k_*^{\frac{4}{3}} + \frac{5\cdot 2^{\frac{2}{3}}}{24}k_*^{\frac{2}{3}} - \frac{7}{48}.
\end{aligned} \tag{2.31}$$

Substituting(2.31) into (2.30), we have

$$3\omega_3 \int_0^{+\infty} r^3 \Psi_s(r)dr \geqslant \frac{3\cdot 2^{\frac{1}{3}}A^5\omega_3}{32B^4}k_*^{\frac{4}{3}} + \frac{2^{\frac{2}{3}}A^5\omega_3}{32B^4}k_*^{\frac{2}{3}} - \frac{7A^5\omega_3}{320B^4}. \tag{2.32}$$

Inserting $k_* = \frac{4B^3}{\omega_3 A^4}k$, $A = (2\pi)^{-3}V(\Omega)$ and $B = 2(2\pi)^{-3}\sqrt{V(\Omega)I(\Omega)}$ into (2.32), we derive

$$\begin{aligned}
3\omega_3 \int_0^{+\infty} r^3 \Psi_s(r)dr \geqslant& \frac{3}{4}\frac{2\pi}{(\omega_3 V(\Omega))^{\frac{1}{3}}}k^{\frac{4}{3}} + \frac{1}{32}\frac{(\omega_3 V(\Omega))^{\frac{1}{3}}}{2\pi}\frac{V(\Omega)}{I(\Omega)}k^{\frac{2}{3}} \\
&- \frac{7}{5120}\frac{\omega_3 V(\Omega)}{(2\pi)^3}\Big(\frac{V(\Omega)}{I(\Omega)}\Big)^2.
\end{aligned} \tag{2.33}$$

Combining (2.27) with (2.33), we can get (1.10). This concludes the proof of Theorem 1.2. □

Proof of Corollary 1.1. Noticing that

$$k_* \geqslant \frac{(n+1)(4\pi)^n}{\omega_n^2}\left(\frac{n}{n+2}\right)^{\frac{n}{2}},$$

it is not difficult to observe that

$$k_* = \frac{4kB^3}{\omega_3 A^4} \geqslant \tau := \frac{432\sqrt{15}\pi}{25} \approx 210.25$$

when $n=3$. Hence, when $\sigma \geqslant \frac{7}{48}\tau^{-\frac{2}{3}}$, the inequality $\sigma k_*^{\frac{2}{3}} \geqslant \frac{7}{48}$ holds for $k_* \in [\tau, +\infty)$. Since

$$1 - \frac{12}{5}\cdot 2^{\frac{1}{3}}\sigma \leqslant 1 - \frac{7}{20}\cdot 2^{\frac{1}{3}}\tau^{-\frac{2}{3}} \approx 0.9875,$$

we can conclude that

$$\big(t(k_*)+1\big)^5 - t(k_*)^5 \geqslant \frac{5\cdot 2^{\frac{1}{3}}}{8}k_*^{\frac{4}{3}} + \frac{5\cdot 2^{\frac{2}{3}}}{24}\alpha k_*^{\frac{2}{3}}, \tag{2.34}$$

where $\alpha = 0.987$. Inserting (2.34) into (2.30), we obtain

$$3\omega_3\int_0^{+\infty} r^3\Psi_s(r)dr \geqslant \frac{3\cdot 2^{\frac{1}{3}}A^5\omega_3}{32B^4}k_*^{\frac{4}{3}} + \alpha\frac{2^{\frac{2}{3}}A^5\omega_3}{32B^4}k_*^{\frac{2}{3}}. \tag{2.35}$$

Then it follows from (2.27) and (2.35) that

$$\sum_{j=1}^{k}\beta_j \geqslant \frac{3}{4}\frac{2\pi}{(\omega_3 V(\Omega))^{\frac{1}{3}}}k^{\frac{4}{3}} + \frac{\alpha}{32}\frac{(\omega_3 V(\Omega))^{\frac{1}{3}}}{2\pi}\frac{V(\Omega)}{I(\Omega)}k^{\frac{2}{3}}. \tag{2.36}$$

Thus, (1.11) holds. This completes the proof of Corollary 1.1. □

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