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SOME REVERSES OF THE CAUCHY—SCHWARZ INEQUALITY FOR COMPLEX FUNCTIONS OF SELF–ADJOINT OPERATORS IN HILBERT SPACES

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Abstract. We give some ratio and difference reverses of the Cauchy–Schwarz inequality for complex functions of self-adjoint operators in Hilbert spaces, under suitable assumptions for the involved operators. Several examples for particular functions of interest are provided as well.

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on a complex Hilbert space $(\mathscr{H},\langle\cdot,\cdot\rangle)$. An operator $A\in\mathbb{B}(\mathscr{H})$ is said to be *positive* if $\langle Ax,x\rangle\geqslant 0$ holds for all $x\in\mathscr{H}$ and then we write $A\geqslant 0$. For self-adjoint operators $A,B\in\mathbb{B}(\mathscr{H})$, we say that $A\leqslant B$ if $B-A\geqslant 0$. Let $A\in\mathbb{B}(\mathscr{H})$ be self-adjoint. The *continuous functional calculus* $f\mapsto f(A)$ establishes a isometrically $*$-isomorphism Φ between the C^*-algebra $C(\mathrm{sp}(A))$ of all continuous complex-valued functions defined on the spectrum $\mathrm{sp}(A)$ of A and the C^*-algebra $C^*(A)$ generated by A and the identity operator I (see [21]). If f and g are real valued functions on $\mathrm{sp}(A)$, then the following property holds:

$$f(A)\leqslant g(A) \quad\Longleftrightarrow\quad f(t)\leqslant g(t) \quad (t\in \mathrm{sp}(A)).$$

For recent results on various inequalities for functions of self-adjoint operators, see [3, 5, 6] and the references therein.

Let $A\in\mathbb{B}(\mathscr{H})$ be self-adjoint with the spectrum included in the interval $[m,M]$ for some real numbers $m<M$ and $\{E_\lambda\}_\lambda$ be its *spectral family*. For any continuous function $f:[m,M]\to\mathbb{C}$, it is well-known that the following *spectral representation* in terms of a Riemann–Stieltjes integral holds:

$$f(A)=\int_{m-0}^{M} f(\lambda)\,dE_\lambda,$$

which in terms of vectors can be written as

$$\langle f(A)x,y\rangle=\int_{m-0}^{M} f(\lambda)\,d\langle E_\lambda x,y\rangle \tag{1}$$

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for any $x,y \in \mathscr{H}$. The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of bounded variation on $[m,M]$ and $g_{x,y}(m-0) = 0$ and $g_{x,y}(M) = \langle x,y\rangle$ for any $x,y \in \mathscr{H}$. It is also well-known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is monotonic nondecreasing and right continuous on $[m,M]$.

The classical Cauchy–Schwarz inequality asserts that, if x and y are elements of a semi-inner product space, then $|\langle x,y\rangle| \leqslant \|x\|\,\|y\|$. There are mainly two types of the reverse of the Cauchy–Schwarz inequality. In the additive approach (initiated by Ozeki [22], we look for an inequality of the form $\kappa + |\langle x,y\rangle| \geqslant \|x\|\,\|y\|$ for some suitable positive constant κ. In the multiplicative approach (initiated by Polya and Szegö [23]), we seek for an appropriate positive constant κ such that $|\langle x,y\rangle| \geqslant \kappa\|x\|\,\|y\|$. There are many generalizations and applications of the Cauchy–Schwarz inequality and its reverse for integrals, weighted sums and isotone functionals (see the monograph [2]). Moreover, some reverses of the Cauchy–Schwarz inequalities, Cauchy–Schwarz functionals and norm inequalities of Hilbert space operators were presented in [1, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19].

By the Cauchy–Schwarz inequality it holds that $|\langle f(A)x,x\rangle| \leqslant \|f(A)x\|$, where $x \in \mathscr{H}$ with $\|x\| = 1$ and $f : [m,M] \to \mathbb{R}$ is a continuous real-valued function defined on $[m,M]$ containing the spectrum of the self-adjoint operator A. In order to provide upper bounds for the nonnegative quantity $\|f(A)x\|^2 - \langle f(A)x,x\rangle^2$, the first author obtained in [8] the following result:

THEOREM. *Let $A \in \mathbb{B}(\mathscr{H})$ be a self-adjoint operator with $\operatorname{sp}(A) \subseteq [m,M]$ for some scalars $m < M$. If $f : [m,M] \to \mathbb{R}$ is continuous on $[m,M]$ and $\delta := \min_{t\in[m,M]} f(t)$, $\Delta := \max_{t\in[m,M]} f(t)$, then*

$$\begin{aligned}
0 &\leqslant \|f(A)x\|^2 - \langle f(A)x,x\rangle^2 \\
&\leqslant \frac{1}{4}\cdot(\Delta-\delta)^2 - \begin{cases} [\langle \Delta x - f(A)x, f(A)x - \delta x\rangle], \\[2ex] \left|\langle f(A)x,x\rangle - \frac{\Delta+\delta}{2}\right|^2 \end{cases} \\
&\leqslant \frac{1}{4}\cdot(\Delta-\delta)^2
\end{aligned}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$. Moreover, if δ is positive, then

$$0 \leqslant \|f(A)x\|^2 - \langle f(A)x,x\rangle^2 \leqslant \begin{cases} \frac{1}{4}\cdot\frac{(\Delta-\delta)^2}{\Delta\delta}\langle f(A)x,x\rangle^2, \\[2ex] \left(\sqrt{\Delta}-\sqrt{\delta}\right)^2\langle f(A)x,x\rangle \end{cases} \tag{2}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

REMARK 1. We notice that the first inequality in the right hand side of (2) is equivalent to the following:

$$\|f(A)x\| \leqslant \frac{\Delta+\delta}{2\sqrt{\Delta\delta}}\langle f(A)x,x\rangle, \tag{3}$$

while the second inequality is equivalent to

$$0 \leqslant \frac{\|f(A)x\|^2}{\langle f(A)x,x\rangle} - \langle f(A)x,x\rangle \leqslant \left(\sqrt{\Delta}-\sqrt{\delta}\right)^2 \tag{4}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$. We notice that inequality (4) is the operator version of the Klamkin–McLenaghan inequality for sequences of real numbers ([16]).

Motivated by the above results we investigate in the current paper the problem of finding the ratio and difference reverses of $|\langle f(A)x,x\rangle| \leqslant \|f(A)x\|$, where x is a unit vector of a complex Hilbert space $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ for different classes of continuous complex valued functions $f : [m,M] \to \mathbb{C}$ and self-adjoint operators $A \in \mathbb{B}(\mathscr{H})$ with $\operatorname{sp}(A) \subseteq [m,M]$. Some applications are also presented.

2. Main results

We start our work with the following result.

THEOREM 1. *If $f : [m,M] \to \mathbb{C}$ is continuous on $[m,M]$ and $a \in \mathbb{C}$, $r > 0$ with $|a| > r$ and such that*

$$|f(t) - a| \leqslant r \tag{5}$$

for any $t \in [m,M]$, then, for any self-adjoint operator $A \in \mathbb{B}(\mathscr{H})$ with $\operatorname{sp}(A) \subseteq [m,M]$,

$$\|f(A)x\| \leqslant \frac{|a|}{\sqrt{|a|^2 - r^2}} |\langle f(A)x,x\rangle| \tag{6}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

Proof. It follows from (5) that

$$|f(t)|^2 - 2Re[f(t)\bar{a}] + |a|^2 \leqslant r^2 \tag{7}$$

for any $t \in [m,M]$. Let $\{E_t\}_t$ be the spectral family for the operator A. Suppose that $x \in \mathscr{H}$ be a unit vector. The function $g_x(t) := \langle E_t x,x\rangle$ is monotonic nondecreasing. Integrating inequality (7) with the integrator g_x on the interval $[m-\varepsilon, M]$ with $\varepsilon > 0$ and then taking the limit as $\varepsilon \to 0^+$, we get

$$\int_{m-0}^{M} |f(t)|^2 d\langle E_t x,x\rangle - 2Re\left[\bar{a}\int_{m-0}^{M} f(t) d\langle E_t x,x\rangle\right] + |a|^2 \int_{m-0}^{M} d\langle E_t x,x\rangle \\ \leqslant r^2 \int_{m-0}^{M} d\langle E_t x,x\rangle. \tag{8}$$

From (1), we have

$$\int_{m-0}^{M} |f(t)|^2 d\langle E_t x,x\rangle = \|f(A)x\|^2, \quad \int_{m-0}^{M} f(t) d\langle E_t x,x\rangle = \langle f(A)x,x\rangle$$

and $\int_{m-0}^{M} d\langle E_t x,x\rangle = 1$. It follows from (8) that

$$\|f(A)x\|^2 + \left(\sqrt{|a|^2 - r^2}\right)^2 \leqslant 2Re\left(\langle f(A)x,x\rangle \bar{a}\right). \tag{9}$$

In addition,

$$Re\left(\langle f(A)x,x\rangle \bar{a}\right) \leqslant |\langle f(A)x,x\rangle \bar{a}| = |\langle f(A)x,x\rangle|\,|a|. \tag{10}$$

Utilizing the elementary inequality $\alpha^2 + \beta^2 \geqslant 2\alpha\beta$ for any $\alpha,\beta \in \mathbb{R}$, we obtain

$$2\|f(A)x\|\sqrt{|a|^2 - r^2} \leqslant \|f(A)x\|^2 + \left(\sqrt{|a|^2 - r^2}\right)^2. \tag{11}$$

It follows from (9), (10) and (11) we get the desired inequality (6). □

REMARK 2. If $\delta := \min_{t\in[m,M]} f(t)$ and $\Delta := \max_{t\in[m,M]} f(t)$, then the condition (5) holds with $a = \frac{\delta+\Delta}{2}$ and $r = \frac{\Delta-\delta}{2}$.

COROLLARY 1. *Let* $f : [m,M] \to \mathbb{C}$ *be continuous on* $[m,M]$, $a \in \mathbb{C}$, $r > 0$ *with* $|a| > r$ *and*

$$|f(t) - a| \leqslant r$$

for any $t \in [m,M]$, *Then, for any self-adjoint operators* $A_j \in \mathbb{B}(\mathscr{H})$ *with* $\mathrm{sp}(A_j) \subseteq [m,M]$ $(1 \leqslant j \leqslant n)$,

$$\|(f(A_1)x_1,\cdots,f(A_n)x_n)\| \leqslant \frac{|a|}{\sqrt{|a|^2 - r^2}} \left|\left\langle \sum_{j=1}^{n} f(A_j)x_j, x_j \right\rangle\right|$$

for any $x_j \in \mathscr{H}$ $(1 \leqslant j \leqslant n)$ *with* $\sum_{j=1}^{n} \|x_j\|^2 = 1$.

Proof. Considering

$$\widetilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathscr{H}^n, \quad \widetilde{A} = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{pmatrix} \in \mathbb{B}(\mathscr{H}^n)$$

and applying Theorem 1 for $\widetilde{A}$ and $\widetilde{x}$, we deduce the desired result. □

Now, for any $\gamma,\Gamma \in \mathbb{C}$ and an interval of real numbers $[a,b]$, let us define the sets of complex-valued functions as follows:

$$\bar{U}_{[a,b]}(\gamma,\Gamma) := \left\{ f : [a,b] \to \mathbb{C} | Re\left[(\Gamma - f(t))\left(\overline{f(t)} - \overline{\gamma}\right)\right] \geqslant 0, \forall t \in [a,b] \right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma,\Gamma) := \left\{ f : [a,b] \to \mathbb{C} | \left| f(t) - \frac{\gamma+\Gamma}{2} \right| \leqslant \frac{1}{2}|\Gamma - \gamma|, \forall t \in [a,b] \right\}.$$

The following representation result may be stated:

PROPOSITION 2. *Let* $\gamma,\Gamma\in\mathbb{C}$ *with* $\gamma\neq\Gamma$. *Then* $\bar{U}_{[a,b]}(\gamma,\Gamma)$ *and* $\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ *are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\gamma,\Gamma)=\bar{\Delta}_{[a,b]}(\gamma,\Gamma).$$

Proof. For any $z\in\mathbb{C}$, we observe that

$$\left|z-\frac{\gamma+\Gamma}{2}\right|\leqslant\frac{1}{2}|\Gamma-\gamma|$$

if and only if

$$Re\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right]\geqslant 0.$$

This follows from the equality

$$\frac{1}{4}|\Gamma-\gamma|^2-\left|z-\frac{\gamma+\Gamma}{2}\right|^2=Re\left[(\Gamma-z)(\bar{z}-\bar{\gamma})\right]$$

for any $z\in\mathbb{C}$. □

By using an application of the properties of the complex numbers, we can also state the following:

COROLLARY 2. *For any* $\gamma,\Gamma\in\mathbb{C}$ *with* $\gamma\neq\Gamma$, *it holds that*

$$\begin{aligned}\bar{U}_{[a,b]}(\gamma,\Gamma):=\{f:[a,b]\to\mathbb{C}:&\,(Re(\Gamma)-Ref(t))(Ref(t)-Re(\gamma))\\&+(Im(\Gamma)-Imf(t))(Imf(t)-Im(\gamma))\geqslant 0,\ \forall t\in[a,b]\}.\end{aligned}$$

Now, if we assume that $Re(\Gamma)\geqslant Re(\gamma)$ and $Im(\Gamma)\geqslant Im(\gamma)$, then we can define the following set of functions:

$$\begin{aligned}\bar{S}_{[a,b]}(\gamma,\Gamma):=\{f:[a,b]\to\mathbb{C}:&\,Re(\Gamma)\geqslant Ref(t)\geqslant Re(\gamma),\\&Im(\Gamma)\geqslant Imf(t)\geqslant Im(\gamma),\ \forall t\in[a,b]\}.\end{aligned}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma,\Gamma)$ is closed convex and

$$\emptyset\neq\bar{S}_{[a,b]}(\gamma,\Gamma)\subseteq\bar{U}_{[a,b]}(\gamma,\Gamma).$$

Making use of the classes of functions defined above, we can provide a generalization of inequality (3) as follows:

COROLLARY 3. *Let* $A\in\mathbb{B}(\mathscr{H})$ *be a self-adjoint operator with* $\operatorname{sp}(A)\subseteq[m,M]$ *for some scalars* $m<M$. *If* $f:[m,M]\to\mathbb{C}$ *is continuous on* $[m,M]$ *and there exist two numbers* $\gamma,\Gamma\in\mathbb{C}$, $\gamma\neq\Gamma$, *with* $Re(\Gamma\bar{\gamma})>0$ *and such that*

$$f\in\bar{U}_{[m,M]}(\gamma,\Gamma)\left(=\bar{\Delta}_{[m,M]}(\gamma,\Gamma)\right),$$

then

$$\|f(A)x\| \leqslant \frac{|\gamma+\Gamma|}{2\sqrt{Re(\Gamma\bar{\gamma})}} |\langle f(A)x,x\rangle| \tag{12}$$

for any $x \in \mathscr{H}$ *with* $\|x\| = 1$.

Proof. We apply Theorem 1 for $a = \frac{\gamma+\Gamma}{2}$ and $r = \frac{1}{2}|\Gamma-\gamma|$ and observe that

$$|a|^2 - r^2 = \left|\frac{\gamma+\Gamma}{2}\right|^2 - \left|\frac{\Gamma-\gamma}{2}\right|^2 = Re(\Gamma\bar{\gamma}) > 0.$$

Now the desired result (12) is deduced from (6). □

REMARK 3. If $f : [m,M] \to (0,\infty)$ is continuous on $[m,M]$ and $0 < \delta := \min_{t\in[m,M]} f(t)$, $\Delta := \max_{t\in[m,M]} f(t)$, then $f \in \bar{S}_{[a,b]}(\delta,\Delta) \subseteq \bar{U}_{[a,b]}(\delta,\Delta)$ and so, by (12), we get inequality (3). The inequality is also useful for applications (see the last section).

The following result, where the condition $|a| > r$ from Theorem 1 is dropped, may be stated as follows:

THEOREM 3. *If* $f : [m,M] \to \mathbb{C}$ *is continuous on* $[m,M]$, $a \in \mathbb{C}\setminus\{0\}$ *and* $r > 0$ *such that*

$$|f(t) - a| \leqslant r$$

for any $t \in [m,M]$, *then*

$$\|f(A)x\| \leqslant |\langle f(A)x,x\rangle| + \frac{r^2}{2|a|}$$

for any self-adjoint operator $A \in \mathbb{B}(\mathscr{H})$ *with* $\mathrm{sp}(A) \subseteq [m,M]$ *and any* $x \in \mathscr{H}$ *with* $\|x\| = 1$.

Proof. Let $x \in \mathscr{H}$ be a unit vector. As in the proof of Theorem 1, we have

$$2\|f(A)x\|\,|a| \leqslant \|f(A)x\|^2 + |a|^2 \leqslant 2Re(\langle f(A)x,x\rangle\,\bar{a}) + r^2 \leqslant 2|\langle f(A)x,x\rangle\,\bar{a}| + r^2,$$

which immediately gives the desired inequality. □

A computation as used in the proof of Corollary 3 gives rise the following assertion.

COROLLARY 4. *Assume that* $A \in \mathbb{B}(\mathscr{H})$ *is a self-adjoint operator with* $\mathrm{sp}(A) \subseteq [m,M]$ *for some scalars* $m < M$. *If* $f : [m,M] \to \mathbb{C}$ *is continuous on* $[m,M]$ *and there exists two numbers* $\gamma,\Gamma \in \mathbb{C}$ *such that* $\gamma \neq \pm\Gamma$ *such that* $f \in \bar{U}_{[m,M]}(\gamma,\Gamma)\left(= \bar{\Delta}_{[m,M]}(\gamma,\Gamma)\right)$, *then*

$$\|f(A)x\| \leqslant |\langle f(A)x,x\rangle| + \frac{|\Gamma-\gamma|^2}{4|\gamma+\Gamma|} \tag{13}$$

for any $x \in \mathscr{H}$ *with* $\|x\| = 1$.

REMARK 4. If $f:[m,M]\to\mathbb{R}$ is continuous on $[m,M]$ and $\delta := \min_{t\in[m,M]} f(t)$, $\Delta := \max_{t\in[m,M]} f(t)$ with $\Delta \neq -\delta$, then $f \in \bar{S}_{[a,b]}(\delta,\Delta) \subseteq \bar{U}_{[a,b]}(\delta,\Delta)$ and so, by (13), we get

$$\|f(A)x\| \leqslant |\langle f(A)x,x\rangle| + \frac{(\Delta-\delta)^2}{4|\Delta+\delta|}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$. Inequality (13) is also useful for applications (see the next section).

3. Some Examples

For the first example, let $A \in \mathbb{B}(\mathscr{H})$ be a selfadjoint operator with $sp(A) \subseteq [m,M]$ and let $\lambda \in \mathbb{R}$ such that $\lambda \neq 0$ and $\lambda^2 + mM > 0$. Define the complex number

$$a := -\lambda + \frac{m+M}{2} i$$

and the positive number $r := \frac{M-m}{2}$. Then

$$|a|^2 - r^2 = \left(\frac{M+m}{2}\right)^2 + \lambda^2 - \left(\frac{M-m}{2}\right)^2 = \lambda^2 + mM > 0.$$

If we take the complex-valued continuous function

$$f:[m,M]\to\mathbb{C}, f(t) = it - \lambda,$$

then we have

$$\begin{aligned} |f(t)-a| &= \left|it - \lambda - \left(-\lambda + \frac{m+M}{2} i\right)\right| \\ &= \left|t - \frac{m+M}{2}\right| \leqslant \frac{M-m}{2} = r \end{aligned}$$

for any $t \in [m,M]$.

If we apply Theorems 1 and 3, then we reach the following inequalities

$$\|(iA-\lambda I)x\| \leqslant \frac{1}{2}\sqrt{\frac{(M+m)^2 + 4\lambda^2}{mM+\lambda^2}}\,|\langle (iA-\lambda I)x,x\rangle| \tag{14}$$

and

$$\|(iA-\lambda I)x\| \leqslant |\langle (iA-\lambda I)x,x\rangle| + \frac{1}{4}\frac{(M-m)^2}{\sqrt{(M+m)^2+4\lambda^2}} \tag{15}$$

for any $x \in \mathscr{H}$ with $\|x\| = 1$.

We notice that the operator $T_\lambda := iA - \lambda I$ for all $\lambda \in \mathbb{R}\setminus\{0\}$ is not selfadjoint and therefore the usual inequalities for selfadjoint operators cannot be applied to get (14) and (15).

For the second example, consider the class of functions $f_{\alpha,\beta} : [0,M] \subseteq \left[0,\frac{\pi}{2}\right) \to \mathbb{C}$ with $\alpha, \beta > 0$, $\alpha \neq \beta$ and

$$f_{\alpha,\beta}(t) := \alpha \cos t + i\beta \sin t.$$

If we define the complex numbers $\Gamma := \alpha + i\beta \sin M$ and $\gamma := \alpha \cos M$, then we have $f_{\alpha,\beta}(t) \in \bar{U}_{[0,M]}(\gamma, \Gamma)$. Using Corollaries 3 and 4 yield that

$$\left\|f_{\alpha,\beta}(A)x\right\| \leqslant \frac{1}{2} \cdot \sqrt{\frac{\alpha^2(1+\cos M)^2 + \beta^2 \sin^2 M}{\alpha^2 \cos M}} \left|\langle f_{\alpha,\beta}(A)x, x\rangle\right|$$

as well as

$$\left\|f_{\alpha,\beta}(A)x\right\| \leqslant \left|\langle f_{\alpha,\beta}(A)x, x\rangle\right| + \frac{\alpha^2(1-\cos M)^2 + \alpha^2 \sin^2 M}{4\sqrt{\alpha^2(1+\cos M)^2 + \beta^2 \sin^2 M}}$$

for any selfadjoint operator A such that $0 \leqslant A \leqslant MI$ and any $x \in H$ with $\|x\| = 1$.

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