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COMMUTATORS FOR THE MAXIMAL FUNCTIONS ON LEBESGUE SPACES WITH VARIABLE EXPONENT

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Abstract. Let M be the Hardy-Littlewood maximal function, the commutator generated by M and a suitable function b is defined by $[M,b]f = M(bf) - bMf$. In this paper, the authors give some characterizations of b for which $[M,b]$ is bounded on the Lebesgue spaces with variable exponent. The similar results are also proved for the commutator of the sharp maximal function.

1. Introduction and main results

Let T be the classical singular integral operator. The commutator $[T,b]$ generated by T and a suitable function b is defined by

$$[T,b]f = T(bf) - bTf.$$

A well known and important result due to Coifman, Rochberg and Weiss [2] states that if $b \in \mathrm{BMO}(\mathbb{R}^n)$, then $[T,b]$ is bounded on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$. They also gave a characterization of BMO in virtue of the L^p-boundedness of the above commutator. In 1990, Milman and Schonbek [14] established a commutator result that applies to the Hardy-Littlewood maximal function as well as a large class of nonlinear operators.

As usual, a cube $Q \subset \mathbb{R}^n$ always means its sides parallel to the coordinate axes. Denote by $|Q|$ the Lebesgue measure of Q and χ_Q the characteristic function of Q. For a function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we write $f_Q = |Q|^{-1} \int_Q f(x)dx$. The Hardy-Littlewood maximal function M is defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy,$$

and the sharp maximal function $M^\sharp f$ is defined by

$$M^\sharp f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|dy,$$

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where the supremum is taken over all cubes $Q\subset\mathbb{R}^n$ containing x.

The commutators of M and $M^\sharp$ with a suitable function b are formally defined by

$$[M,b]f=M(bf)-bMf \quad \text{and} \quad [M^\sharp,b]f=M^\sharp(bf)-bM^\sharp f.$$

In 2000, Bastero, Milman and Ruiz [1] studied the necessary and sufficient conditions for the boundedness of $[M,b]$ and $[M^\sharp,b]$ on L^p spaces. In 2009, the authors [21] considered the same problem for the fractional maximal function.

In this paper, we will extend the results of Bastero, Milman and Ruiz [1] to the variable exponent Lebesgue spaces. To state our results, we first introduce some notations.

Let $p(\cdot):\mathbb{R}^n\to[1,\infty)$ be a measurable function. Consider the convex modular (see Chapter I of [15] for definitions and properties)

$$m(f,p):=\int_{\mathbb{R}^n}|f(x)|^{p(x)}dx.$$

Denote by $L^{p(\cdot)}(\mathbb{R}^n)$ the set of all Lebesgue measurable functions f on $\mathbb{R}^n$ such that $m(f/\lambda,p)<\infty$ for some $\lambda>0$. This set becomes a Banach space with respect to the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}=\inf\left\{\lambda>0:m(f/\lambda,p)=\int_{\mathbb{R}^n}\left(\frac{|f(x)|}{\lambda}\right)^{p(x)}dx\leqslant 1\right\}.$$

The theory of function spaces with variable exponent have been intensely investigated in the past twenty years since some elementary properties were established by Kováčik and Rákosník [11]. One of the main problems on this theory is the boundedness of the Hardy-Littlewood maximal operator. By virtue of the works such as [4, 5, 6, 7, 12, 13, 16, 17, 18, 19], some important conditions on variable exponent have been obtained. For more recent progress and applications on function spaces with variable exponent, we refer the readers to [8].

If a measurable function $p(\cdot):\mathbb{R}^n\to[1,\infty)$ satisfies

$$1<p_-:=\operatorname*{ess\,inf}_{x\in\mathbb{R}^n}p(x),\ \ p_+:=\operatorname*{ess\,sup}_{x\in\mathbb{R}^n}p(x)<\infty, \tag{1.1}$$

then the function $p'(x):=p(x)/(p(x)-1)$ is well defined and satisfies (1.1).

Denote by $\mathscr{P}(\mathbb{R}^n)$ the set of all measurable functions $p(\cdot):\mathbb{R}^n\to[1,\infty)$ such that (1.1) holds. Let $\mathscr{B}(\mathbb{R}^n)$ be the set of all functions $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ such that the Hardy-Littlewood maximal function M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

We say that $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ if $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ and there exists a constant $C>0$ such that for any $x,y\in\mathbb{R}^n$,

$$|p(x)-p(y)|\leqslant\frac{-C}{\log(|x-y|)} \quad \text{if } |x-y|\leqslant 1/2, \tag{1.2}$$

and

$$|p(x)-p(y)|\leqslant\frac{C}{\log(e+|x|)} \quad \text{if } |y|\geqslant|x|. \tag{1.3}$$

Condition (1.2) is usually called the locally log-Hölder continuity or the Dini-Lipschitz condition. (1.3) is the natural analogue of (1.2) at infinity, which is originally defined in this form in [4].

When $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$, Cruz-Uribe, Fiorenza and Neugebauer [4] proved that the Hardy-Littlewood maximal function M is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself (see Theorem 1.5 of [4]), that is, if $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ then $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$.

Let Q_0 be a fixed cube in $\mathbb{R}^n$, the Hardy-Littlewood maximal function relative to Q_0 is given by

$$M_{Q_0}(f)(x) = \sup_{Q_0 \supseteq Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For a function b defined on $\mathbb{R}^n$, we denote by

$$b^-(x) = \begin{cases} 0, & \text{if } b(x) \geqslant 0, \\ |b(x)|, & \text{if } b(x) < 0, \end{cases}$$

and $b^+(x) := |b(x)| - b^-(x)$. Obviously, $b^+(x) - b^-(x) = b(x)$.

Our main result can be stated as follows.

THEOREM 1.1. *Let $q(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$. If $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $b \geqslant 0$, then $[M,b]$ and $[M^\sharp,b]$ are bounded from $L^{q(\cdot)}(\mathbb{R}^n)$ to itself.*

THEOREM 1.2. *Let $b(x)$ be a real valued, locally integrable function in $\mathbb{R}^n$. The following assertions are equivalent:*

(I) $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.

(II) The commutator $[M,b]$ is bounded in $L^{q(\cdot)}(\mathbb{R}^n)$ for all $q(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$.

(III) The commutator $[M,b]$ is bounded in $L^{q(\cdot)}(\mathbb{R}^n)$ for some $q(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$.

(IV) There exists $q(x) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ such that

$$\sup_Q \frac{\left\| \left(b - M_Q(b)\right)\chi_Q \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}} < \infty.$$

(V) For all $q(x) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ we have

$$\sup_Q \frac{\left\| \left(b - M_Q(b)\right)\chi_Q \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}} < \infty.$$

For the commutators of the sharp maximal function, there holds the similar results.

THEOREM 1.3. *Let $b(x)$ be a real valued, locally integrable function in $\mathbb{R}^n$. The following assertions are equivalent:*

(i) $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$.

(ii) The commutator $[M^\sharp,b]$ is bounded in $L^{q(\cdot)}(\mathbb{R}^n)$ for all $q(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$.

(iii) The commutator $[M^\sharp,b]$ is bounded in $L^{q(\cdot)}(\mathbb{R}^n)$ for some $q(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$.

(iv) There exists $q(x) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ *such that*

$$\sup_Q \frac{\left\| \left(b - 2M^\sharp(b\chi_Q)\right)\chi_Q \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}} < \infty.$$

(v) For all $q(x) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ *we have*

$$\sup_Q \frac{\left\| \left(b - 2M^\sharp(b\chi_Q)\right)\chi_Q \right\|_{L^{q(\cdot)}(\mathbb{R}^n)}}{\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}} < \infty.$$

In what follows, the symbol C always means a positive constant independent of the main parameters and may change from one occurrence to another.

2. Some proposition and lemmas

In this section, we recall some know results which will be used in the proof of our theorems. The first lemma is known as the generalized Hölder's inequality on Lebesgue spaces with variable exponent, and the proof can be found in [11].

LEMMA 2.1. (generalized Hölder's inequality) *Suppose that* $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$, *then for any* $f \in L^{p(\cdot)}(\mathbb{R}^n)$ *and any* $g \in L^{p'(\cdot)}(\mathbb{R}^n)$ *we have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leqslant C_p \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \|g\|_{L^{p'(\cdot)}(\mathbb{R}^n)}, \tag{2.1}$$

where $C_p = 1 + 1/p_- - 1/p_+$.

LEMMA 2.2. ([4] Theorem 1.5) *If* $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ *then* $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$.

Noting that $M^\sharp f(x) \leqslant 2Mf(x)$, it follows from Lemma 2.2 that $M^\sharp$ is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself when $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$.

The following lemma is due to Diening (see Theorem 8.1 and Lemma 5.5 in [5]). We remark that Diening has proved general results on Musielak-Orlicz spaces. Here, we describe them for Lebesgue spaces with variable exponent (also see Proposition 2.4 of [9]).

LEMMA 2.3. *(1)* $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ *if and only if there is a constant* $C > 0$ *such that for any family of pairwise disjoint cubes* π *and any* $f \in L^{p(\cdot)}(\mathbb{R}^n)$,

$$\Big\| \sum_{Q \in \pi} (|f|_Q)\chi_Q \Big\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$

(2) $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ *if and only if* $p'(\cdot) \in \mathscr{B}(\mathbb{R}^n)$.

From Lemma 2.1 and the first part of Lemma 2.3, Izuki [9] obtained the following result (see the proof of Lemma 2.9 in [9]).

LEMMA 2.4. *Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that*

$$\frac{1}{|Q|}\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \leqslant C. \tag{2.2}$$

LEMMA 2.5. ([10]) *Let $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$, denote by $\|b\|_*$ the BMO-norm of b. Then there is a constant $C > 0$ such that*

$$C^{-1}\|b\|_* \leqslant \sup_Q \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}}\|(b - b_Q)\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|b\|_*. \tag{2.3}$$

To prove our theroems, we also need the following notation and result due to Xu [20]. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$, we define

$$M_b f(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |b(x) - b(y)||f(y)|dy. \tag{2.4}$$

LEMMA 2.6. ([20]) *Suppose that $p(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ and $b(x) \in \mathrm{BMO}(\mathbb{R}^n)$, then M_b is bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself.*

To end this section, we show that the commutators $[M,b]$ and $[M^\sharp,b]$ are well defined on $L^{p(\cdot)}(\mathbb{R}^n)$ when $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$.

PROPOSITION 2.1. *If $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$, then the commutators $[M,b]$ and $[M^\sharp,b]$ are well-defined on $L^{p(\cdot)}(\mathbb{R}^n)$.*

Proof. For any $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and $p(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, we have

$$\|Mf\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

which implies that $Mf(x)$, and then $b(x)Mf(x)$, is finite almost everywhere in $\mathbb{R}^n$. This shows that $[M,b]f(x) = M(bf)(x) - b(x)Mf(x)$ is well-defined on $L^{p(\cdot)}(\mathbb{R}^n)$.

Since $|b(x)M^\sharp f(x)| \leqslant 2|b(x)|Mf(x) < \infty$ (a.e. $x \in \mathbb{R}^n$), then $[M^\sharp,b]f(x)$ is also well-defined on $L^{p(\cdot)}(\mathbb{R}^n)$. $\square$

3. Proof of the theorems

In this section, we will prove the theorems. Some idea in the proof of Theorems 1.2 and 1.3 comes from [1]. Now, let us prove Theorem 1.1 first.

Proof. (Proof of Theorem 1.1.) Let $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $b \geqslant 0$. For a fixed $x \in \mathbb{R}^n$ such that $Mf(x) < \infty$, noting that $b \geqslant 0$, we have

$$\begin{aligned}
|[M,b]f(x)| &= \left|\sup_{Q \ni x}\frac{1}{|Q|}\int_Q b(y)|f(y)|dy - \sup_{Q \ni x}\frac{1}{|Q|}\int_Q b(x)|f(y)|dy\right| \\
&\leqslant \sup_{Q \ni x}\frac{1}{|Q|}\left|\int_Q b(y)|f(y)|dy - \int_Q b(x)|f(y)|dy\right| \\
&\leqslant \sup_{Q \ni x}\frac{1}{|Q|}\int_Q |b(y) - b(x)||f(y)|dy \\
&= M_b f(x).
\end{aligned} \tag{3.1}$$

Similarly, we have

$$
\begin{aligned}
|[M^\sharp,b]f(x)| &= \left|\sup_{Q\ni x}\frac{1}{|Q|}\int_Q |b(y)f(y)-(bf)_Q|dy - \sup_{Q\ni x}\frac{b(x)}{|Q|}\int_Q |f(y)-f_Q|dy\right| \\
&\leqslant \sup_{Q\ni x}\frac{1}{|Q|}\left|\int_Q |b(y)f(y)-(bf)_Q|dy - \int_Q |b(x)f(y)-b(x)f_Q|dy\right| \\
&\leqslant \sup_{Q\ni x}\frac{1}{|Q|}\int_Q \big|\big(b(y)-b(x)\big)f(y)+b(x)f_Q-(bf)_Q\big|dy \\
&\leqslant \sup_{Q\ni x}\left\{\frac{1}{|Q|}\int_Q |b(y)-b(x)||f(y)|dy + \big|b(x)f_Q-(bf)_Q\big|\right\} \\
&\leqslant M_b f(x) + \sup_{Q\ni x}\left|\frac{b(x)}{|Q|}\int_Q f(z)dz - \frac{1}{|Q|}\int_Q b(z)f(z)dz\right| \\
&\leqslant M_b f(x) + \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |b(x)-b(z)||f(z)|dz \\
&\leqslant 2M_b f(x).
\end{aligned} \tag{3.2}
$$

Since $Mf(x)<\infty$ for a.e $x\in\mathbb{R}^n$ when $f\in L^{q(\cdot)}(\mathbb{R}^n)$ and $q(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$, then (3.1) and (3.2) valid almost everywhere in $\mathbb{R}^n$. It follows from Lemma 2.6 that $[M,b]$ and $[M^\sharp,b]$ are bounded from $L^{p(\cdot)}(\mathbb{R}^n)$ to itself. □

Proof. (Proof of Theorem 1.2) Since the implications (II) $\Longrightarrow$ (III) and (V) $\Longrightarrow$ (IV) follow readily, we only have to prove (I) $\Longrightarrow$ (II), (III) $\Longrightarrow$ (IV) and (IV) $\Longrightarrow$ (I) (the implication (II) $\Longrightarrow$ (V) is similar to (III) $\Longrightarrow$ (IV)).

(I) $\Longrightarrow$ (II). By the definition of $[M,b]$ and noting that $|b|-b=2b^-$ and $M(bf)(x)=M(|b|f)(x)$, we have

$$
\begin{aligned}
&\big|[M,b]f(x)-[M,|b|]f(x)\big| \\
&\leqslant \big|M(bf)(x)-M(|b|f)(x)\big| + \big|\big(|b(x)|-b(x)\big)M(f)(x)\big| \\
&\leqslant 2b^-(x)M(f)(x).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\big|[M,b]f(x)\big| &\leqslant \big|[M,b]f(x)-[M,|b|]f(x)\big| + \big|[M,|b|]f(x)\big| \\
&\leqslant 2b^-(x)M(f)(x) + \big|[M,|b|]f(x)\big|.
\end{aligned} \tag{3.3}
$$

Noting that $|b|\in \mathrm{BMO}(\mathbb{R}^n)$ when $b\in\mathrm{BMO}(\mathbb{R}^n)$, M is bounded in $L^{q(\cdot)}(\mathbb{R}^n)$ and $b^-\in L^\infty(\mathbb{R}^n)$, it follows from (3.3) and Theorem 1.1 that, for all $q(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$

$$
\begin{aligned}
\|[M,b]f\|_{L^{q(\cdot)}(\mathbb{R}^n)} &\leqslant 2\|b^-\|_{L^\infty(\mathbb{R}^n)}\|M(f)\|_{L^{q(\cdot)}(\mathbb{R}^n)} + \|[M,|b|]f\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}
$$

(III) $\Longrightarrow$ (IV). For any $Q\subset\mathbb{R}^n$, noting that we have for all $x\in Q$,

$$M(\chi_Q)(x)=M_Q(\chi_Q)(x)=\chi_Q(x)$$

and

$$M(b\chi_Q)(x) = M_Q(b\chi_Q)(x) = M_Q(b)(x),$$

then, it follows from (III) that there exists $q(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ such that

$$\begin{aligned}\big\|\big(b - M_Q(b)\big)\chi_Q\big\|_{L^{q(\cdot)}(\mathbb{R}^n)} &= \big\|\big(bM(\chi_Q) - M(b\chi_Q)\big)\chi_Q\big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leqslant \big\|bM(\chi_Q) - M(b\chi_Q)\big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &= \big\|[M,b]\chi_Q\big\|_{L^{q(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}.\end{aligned}$$

(IV) $\Longrightarrow$ (I). Let Q be a fixed cube. By Lemma 2.1 (the generalized Hölder's inequality), the hypothesis (IV) and Lemma 2.4, we have

$$\begin{aligned}\frac{1}{|Q|}\int_Q |b(x) - M_Q(b)(x)|dx &= \frac{1}{|Q|}\int_Q \big|\big(b(x) - M_Q(b)(x)\big)\chi_Q(x)\big|dx \\ &\leqslant \frac{C}{|Q|}\big\|\big(b - M_Q(b)\big)\chi_Q\big\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\leqslant \frac{C}{|Q|}\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\ &\leqslant C.\end{aligned} \tag{3.4}$$

Let $E = \{x \in Q : b(x) \leqslant b_Q\}$ and $F = \{x \in Q : b(x) > b_Q\}$. The following equality is trivially true (see [1] p.3331):

$$\int_E |b(x) - b_Q|\mathrm{d}x = \int_F |b(x) - b_Q|dx.$$

Since for any $x \in E$ we have $b(x) \leqslant b_Q \leqslant |b_Q| \leqslant M_Q(b)(x)$, then for $x \in E$ there has

$$|b(x) - b_Q| \leqslant \big|b(x) - M_Q(b)(x)\big|.$$

Applying (3.4), we obtain

$$\begin{aligned}\frac{1}{|Q|}\int_Q |b(x) - b_Q|dx &= \frac{1}{|Q|}\int_{E\cup F} |b(x) - b_Q|dx \\ &= \frac{2}{|Q|}\int_E |b(x) - b_Q|dx \\ &\leqslant \frac{2}{|Q|}\int_E \big|b(x) - M_Q(b)(x)\big|dx \\ &\leqslant \frac{2}{|Q|}\int_Q \big|b(x) - M_Q(b)(x)\big|dx \\ &\leqslant C.\end{aligned}$$

So, using the definition of BMO, we have $b \in \mathrm{BMO}(\mathbb{R}^n)$.

Now, let us show that $b^- \in L^\infty(\mathbb{R}^n)$. Observe that $0 \leqslant b^+(x) \leqslant |b(x)| \leqslant M_Q(b)(x)$ for $x \in Q$, therefore, for any $x \in Q$, there holds

$$0 \leqslant b^-(x) \leqslant M_Q(b)(x) - b^+(x) + b^-(x) = M_Q(b)(x) - b(x).$$

Then for any cube Q, we have

$$\begin{aligned}\frac{1}{|Q|}\int_Q b^-(x)dx &\leqslant \frac{1}{|Q|}\int_Q \big(M_Q(b)(x) - b(x)\big)dx \\ &= \frac{1}{|Q|}\int_Q \big|b(x) - M_Q(b)(x)\big|dx \\ &\leqslant C,\end{aligned}$$

where the last step follows from (3.4).

Thus, by the Lebesgue's differentiation theorem, we have $b^- \in L^\infty(\mathbb{R}^n)$. □

Proof. (Proof of Theorem 1.3) Similar to Theorem 1.2, we only need to prove (i) $\Longrightarrow$ (ii), (iii) $\Longrightarrow$ (iv) and (iv) $\Longrightarrow$ (i).

(i) $\Longrightarrow$ (ii). Noting that $|b| - b = 2b^-$, by the definition of $[M^\sharp, b]$, we have

$$\begin{aligned}&\big|[M^\sharp,b]f(x) - [M^\sharp,|b|]f(x)\big| \\ &\leqslant \big|M^\sharp(bf)(x) - M^\sharp(|b|f)(x)\big| + \big||b(x)|M^\sharp(f)(x) - b(x)M^\sharp f(x)\big| \\ &\leqslant \big|M^\sharp\big((b-|b|)f\big)(x)\big| + 2b^-(x)M^\sharp f(x) \\ &\leqslant M^\sharp(2b^- f)(x) + 2b^-(x)M^\sharp f(x).\end{aligned}$$

Therefore,

$$\begin{aligned}\big|[M^\sharp,b]f(x)\big| &\leqslant \big|[M^\sharp,b]f(x) - [M^\sharp,|b|]f(x)\big| + \big|[M^\sharp,|b|]f(x)\big| \\ &\leqslant M^\sharp(2b^- f)(x) + 2b^-(x)M^\sharp(f)(x) + \big|[M^\sharp,|b|]f(x)\big|.\end{aligned}$$

Since $M^\sharp$ is bounded in $L^{q(\cdot)}(\mathbb{R}^n)$ for all $q(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ and $b^- \in L^\infty(\mathbb{R}^n)$, then, by Theorem 1.1 and the triangle's inequality, we can see that $[M^\sharp, b]$ is bounded in $L^{q(\cdot)}(\mathbb{R}^n)$ for all $q(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$.

(iii) $\Longrightarrow$ (iv). Let Q be a fixed cube and Q_1 be another cube. By the inequality

$4ac \leqslant (a+c)^2$, it is easy to compute that

$$\begin{aligned}
&\frac{1}{|Q_1|}\int_{Q_1}|\chi_Q(x)-(\chi_Q)_{Q_1}|dx\\
&=\frac{1}{|Q_1|}\left\{\int_{Q_1\setminus Q}|\chi_Q(x)-(\chi_Q)_{Q_1}|dx+\int_{Q_1\cap Q}|\chi_Q(x)-(\chi_Q)_{Q_1}|dx\right\}\\
&=\frac{1}{|Q_1|}\left\{\int_{Q_1\setminus Q}|(\chi_Q)_{Q_1}|dx+\int_{Q_1\cap Q}|1-(\chi_Q)_{Q_1}|dx\right\}\\
&=\frac{1}{|Q_1|}\left\{\int_{Q_1\setminus Q}\left|\frac{1}{|Q_1|}\int_{Q_1\cap Q}\chi_Q(y)dy\right|dx\right.\\
&\qquad\left.+\int_{Q_1\cap Q}\left|\frac{1}{|Q_1|}\int_{Q_1}\chi_{Q_1}(y)dy-\frac{1}{|Q_1|}\int_{Q_1}\chi_Q(y)\cdot\chi_{Q_1}(y)dy\right|dx\right\}\\
&=\frac{1}{|Q_1|}\left\{\frac{|Q_1\cap Q||Q_1\setminus Q|}{|Q_1|}+\frac{1}{|Q_1|}\int_{Q_1\cap Q}\left|\int_{Q_1}\chi_{Q_1}(y)\big(1-\chi_Q(y)\big)dy\right|dx\right\}\\
&=\frac{1}{|Q_1|^2}\Big\{|Q_1\cap Q||Q_1\setminus Q|+|Q_1\cap Q||Q_1\setminus Q|\Big\}\\
&=\frac{2|Q_1\cap Q||Q_1\setminus Q|}{(|Q_1\cap Q|+|Q_1\setminus Q|)^2}\leqslant\frac{1}{2}.
\end{aligned}\tag{3.5}$$

On the other hand, for any $x\in Q$, there always exists a cube $Q_0\supset Q$ such that $|Q_0|=2|Q|$. Then, it follows from (3.5) and $|Q_0\setminus Q|=|Q_0\cap Q|=|Q|$ that

$$\frac{1}{|Q_0|}\int_{Q_0}|\chi_Q(x)-(\chi_Q)_{Q_0}|dx=\frac{2|Q_0\cap Q||Q_0\setminus Q|}{(|Q_0\cap Q|+|Q_0\setminus Q|)^2}=\frac{1}{2}.$$

This shows that for all $x\in\mathbb{R}^n$,

$$\big(M^\sharp(\chi_Q)\chi_Q\big)(x)=\sup_{Q_1\ni x}\frac{1}{|Q_1|}\int_{Q_1}|\chi_Q(y)-(\chi_Q)_{Q_1}|dy=\frac{1}{2}=\frac{1}{2}\chi_Q(x).$$

Note that $\chi_Q(x)\in L^{q(\cdot)}(\mathbb{R}^n)$. Then, using the hypothesis (iii), we have

$$\begin{aligned}
\big\|\big(b-2M^\sharp(b\chi_Q)\big)\chi_Q\big\|_{L^{q(\cdot)}(\mathbb{R}^n)}&=\Big\|2\Big(\frac{1}{2}b\chi_Q-M^\sharp(b\chi_Q)\Big)\chi_Q\Big\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&=\Big\|2\big(bM^\sharp(\chi_Q)\chi_Q-M^\sharp(b\chi_Q)\big)\chi_Q\Big\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&=\Big\|2\big(bM^\sharp(\chi_Q)-M^\sharp(b\chi_Q)\big)\chi_Q\Big\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leqslant\big\|2[M^\sharp,b](\chi_Q)\big\|_{L^{q(\cdot)}(\mathbb{R}^n)}\\
&\leqslant C\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}.
\end{aligned}$$

Hence the conclusion is proved.

(iv) $\Longrightarrow$ (i). For a cube $Q\subset\mathbb{R}^n$, Bastero, Milman and Ruiz obtained the following inequality (see (2) in [1] p.3333):

$$|b_Q|\leqslant 2M^\sharp(b\chi_Q)(x),\quad\text{for } x\in Q.\tag{3.6}$$

Now, we can achieve that $b \in \mathrm{BMO}(\mathbb{R}^n)$. Indeed, let $E = \{x \in Q : b(x) \leqslant b_Q\}$ and $F = \{x \in Q : b(x) > b_Q\}$, then

$$\int_E |b(x) - b_Q| dx = \int_F |b(x) - b_Q| dx.$$

Since for any $x \in E$, we have $b(x) \leqslant b_Q \leqslant |b_Q| \leqslant 2M^\sharp(b\chi_Q)(x)$, then

$$|b(x) - b_Q| \leqslant \big|b(x) - 2M^\sharp(b\chi_Q)(x)\big|, \quad \text{for } x \in E.$$

By Lemma 2.1, the hypothesis (iv) and Lemma 2.4, we obtain

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |b(x) - b_Q| dx &= \frac{1}{|Q|}\int_{E\cup F} \big|b(x) - b_Q\big| dx \\
&= \frac{2}{|Q|}\int_E |b(x) - b_Q| dx \\
&\leqslant \frac{2}{|Q|}\int_E \big|b(x) - 2M^\sharp(b\chi_Q)(x)\big| dx \\
&\leqslant \frac{2}{|Q|}\int_Q \big|b(x) - 2M^\sharp(b\chi_Q)(x)\big| dx \\
&\leqslant \frac{C}{|Q|}\big\|\big(b - 2M^\sharp(b\chi_Q)\big)\chi_Q\big\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\leqslant \frac{C}{|Q|}\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C,
\end{aligned}$$

which implies $b \in \mathrm{BMO}(\mathbb{R}^n)$.

Now, let us show that $b^- \in L^\infty(\mathbb{R}^n)$. By (3.6) we have

$$2M^\sharp(b\chi_Q)(x) - b(x) \geqslant |b_Q| - b(x) = |b_Q| - b^+(x) + b^-(x), \quad x \in Q.$$

Then

$$\begin{aligned}
\frac{1}{|Q|}\int_Q \big|2M^\sharp(b\chi_Q)(x) - b(x)\big| dx &\geqslant \frac{1}{|Q|}\int_Q \big(2M^\sharp(b\chi_Q)(x) - b(x)\big) dx \\
&\geqslant \frac{1}{|Q|}\int_Q \big(|b_Q| - b^+(x) + b^-(x)\big) dx \qquad (3.7)\\
&= |b_Q| - \frac{1}{|Q|}\int_Q b^+(x) dx + \frac{1}{|Q|}\int_Q b^-(x) dx.
\end{aligned}$$

On the other hand, applying Lemma 2.1, the hypothesis (iv) and Lemma 2.4, similar to (3.4), we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q \big|2M^\sharp(b\chi_Q)(x) - b(x)\big| dx \\
&\leqslant \frac{C}{|Q|}\big\|\big(2M^\sharp(b\chi_Q)(x) - b(x)\big)\chi_Q\big\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C|Q|^{-1}\|\chi_Q\|_{L^{q(\cdot)}(\mathbb{R}^n)}\|\chi_Q\|_{L^{q'(\cdot)}(\mathbb{R}^n)} \\
&\leqslant C.
\end{aligned}$$

This, together with (3.7), gives

$$|b_Q| - \frac{1}{|Q|}\int_Q b^+(x)dx + \frac{1}{|Q|}\int_Q b^-(x)dx \leqslant C.$$

Let $|Q| \to 0$ with $x \in Q$, Lebesgue's differentiation theorem assures that

$$C \geqslant |b(x)| - b^+(x) + b^-(x) = 2b^-(x) = 2|b^-(x)|$$

and the desired result follows.

Therefore, the proof is completed. □

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