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LIOUVILLE TYPE THEOREM FOR HIGHER ORDER HARDY–HÉNON SYSTEM OF INEQUALITIES

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Abstract. In this paper, we prove some new Liouville type theorems for fourth order and more general higher order Hardy-Hénon systems of inequalities. The test function method is applied to show the nonexistence of nontrivial nonnegative global solutions, which improve and extend some recent results.

1. Introduction

This paper is devoted to the study of nonnegative solutions for a fourth order parabolic system of inequalities with the biharmonic operator as leading principal part

$$\begin{cases} u_t + \triangle^2 u \geqslant |x|^a v^p, \\ v_t + \triangle^2 v \geqslant |x|^b u^q, \end{cases} \tag{1}$$

and the following more general higher order systems of inequalities

$$\begin{cases} u_t + L[u] \geqslant |x|^a v^p, \\ v_t + M[v] \geqslant |x|^b u^q, \end{cases} \tag{2}$$

where $p,q>1$, $a,b\in\mathbb{R}$, $L=L(x,t,D_x)$, $M=M(x,t,D_x)$ are differential operators of order $l,h\,(l,h\geqslant 4)$ respectively,

$$L[u] \triangleq -\sum_{|\alpha|=l} D^{\alpha}\left(a_{\alpha}(x,t,u)u\right), \qquad M[u] \triangleq -\sum_{|\beta|=h} D^{\beta}\left(b_{\beta}(x,t,u)u\right),$$

and $a_\alpha(x,t,u)$, $b_\beta(x,t,u)$ are bounded functions. We are mainly concerned with the Liouville type theorems, i.e. the nonexistence of nontrivial nonnegative solutions. Below we refer to such solutions as entire solutions, that is, solutions defined for all $(x,t)\in\mathbb{S}\triangleq\mathbb{R}^N\times(0,+\infty)$ for $N\geqslant 1$.

As it is well known, Liouville-type theorems have proved very useful in many aspects. For example, they can be efficiently used to obtain a priori bounds, singularity,

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decay estimates and blow up rates for solutions, see [20, 21, 9] and the references therein for more detailed discussions.

We start with a short description of related elliptic system of the form

$$\begin{cases}(-\triangle)^m u = |x|^a v^p, \\ (-\triangle)^m v = |x|^b u^q.\end{cases} \tag{3}$$

The most studied case is when $m=1$, especially for $a=b=0$. In the single equation case (i.e. $a=b$, $p=q$, $u=v$),

$$-\triangle u = |x|^a u^p \tag{4}$$

is traditionally called the Hénon (resp. Hardy, or Lane-Emden) equation for $a>0$ (resp. $a<0$, $a=0$). The case $a=0$ has been very widely studied, see for example [11]. We now briefly recall the known results about $a \neq 0$. It is known that Equation (4) has no positive solutions on any domain containing the origin whenever $a \leqslant -2$ [11, 5]. For the case of $a>-2$, Bidaut-Véron and Giacomini [2] showed that (4) has no positive classical radial solution on $\mathbb{R}^N$ if $p < \frac{N+2+2a}{N-2}$ for $N \geqslant 3$. Actually, as a direct consequence of Theorem 1.7 in [2], the conclusion can be proved for general solutions (non-radial solutions) provided $a \leqslant 0$ and $N \geqslant 3$. Recently, Phan and Souplet [19] showed that the Liouville property also holds when $a>0$ and $N=3$ in the case of bounded solutions. The case of $a>0$ and $N \geqslant 4$ is still open.

For the second order system, we first recall the following famous conjecture:

$$m=1, \quad \frac{N+a}{p+1}+\frac{N+b}{q+1} > N-2 \Rightarrow$$
$$\text{system (3) has no positive solutions in } \mathbb{R}^N.$$

When $a=b=0$, this conjecture was proved to be true for radial solutions in any dimension [16]. Partial results are known for non-radial case, we refer to [16, 17] for $N=1,2$ and [20] for $N=3$. Recently, Souplet [22] fully solved the Lane-Emden conjecture in $N=4$ and established a new region of non-existence for $N \geqslant 5$.

Comparing to the Lane-Emden case, less is known about $a \neq 0$ or $b \neq 0$. Some special cases such as radial solutions, supersolutions have been discussed. In [2], Bidaut-Véron and Giacomini proved that there is no positive radial classical solution to system (3) in $\mathbb{R}^N$ for $N \geqslant 3$, $a,b > -2$ if and only if $\frac{N+a}{p+1}+\frac{N+b}{q+1} > N-2$. The nonexistence of supersolutions can be obtained in [17] and [1]: If $pq \leqslant 1$, or $pq > 1$ and

$$\max\left\{\frac{2(p+1)+a+bp}{pq-1}, \frac{2(q+1)+b+aq}{pq-1}\right\} \geqslant N-2,$$

the system (3) admits no positive supersolutions in $\mathbb{R}^N$ for $N \geqslant 3$ and $a,b > -2$. It should be noticed that, almost at the same time, Fazly, Ghoussoub in [8] and Phan in [18] showed that there is no positive bounded solution to system (3) for $\frac{N+a}{p+1}+\frac{N+b}{q+1} > N-2$ in $\mathbb{R}^3$ provided $a,b > -2$. We stress the fact that the conclusion remains true for higher dimension $N \geqslant 4$ if stronger condition holds, see [17, 18] for more details.

On the other hand, (non) existence of positive solutions for the system (3) in a bounded domain were discussed, for example, in [18, 3].

In terms of fourth order Equation (i.e. $m=1$, $a=0$ and $p=1$)

$$\triangle^2 u = |x|^b u^q, \tag{5}$$

the case $b=0$ has studied by many authors (see for examples, [23]), while the case $b \neq 0$ is less completely understood. Fazly and Ghoussoub [8] first considered the fourth order Hénon-Lane-Emden equation (5), and proved that (5) has no positive finite Morse index solutions if $1 < q < \frac{N+4+2b}{N-4}$ for $N \geqslant 5$. When $b > 0$ and $N = 5$, the conclusion for the case of positive bounded classical solution was established by Cowan in [4].

We now come to the higher order elliptic system (3) ($m \geqslant 2$) and the coupled parabolic system

$$\begin{cases} u_t + (-\triangle)^m u = |x|^a v^p, \\ v_t + (-\triangle)^m v = |x|^b u^q. \end{cases} \tag{6}$$

To the best of our knowledge, the known Liouville type results of solutions for both systems are for the case $a=b=0$ only. We refer readers to [24] and the references therein for the elliptic case. Concerning the initial value problem of parabolic equations

$$u_t + (-\triangle)^m u = |x|^a u^p,$$

it is well known that the critical Fujita exponent is $p_c = 1 + \frac{2m}{N}$. In fact, for $p > p_c$, equation admits a class of small global solutions. For $1 < p \leqslant p_c$, if bounded integrable initial data $u_0 \not\equiv 0$ and $\int_{\mathbb{R}^N} u_0(x)\,dx \geqslant 0$, then the solution blows up at a finite time $T > 0$ in $L^\infty(\mathbb{R}^N)$ in the sense that

$$\|u(\cdot,t)\|_\infty > [D(p-1)](T-t)^{-\frac{1}{p-1}} \to \infty, \quad \text{as } t \to T^-,$$

where $D = D(m,N)$ is a constant [10].

Without taking their traces on the hyperplane $t=0$ into account, the global nonexistence results for the corresponding (second and higher order) inequality problem with $1 < p \leqslant p_c$ were also obtained in [13] and [6]. For the Cauchy problem of system (6), the following results are essentially known [7, 17]: For $m \geqslant 1$, $p > 1$ and $q > 1$, if $\frac{N}{2m} \leqslant \max\{\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\}$, then every solution of (6) with initial data having positive average value does not exist globally in time; if $\frac{N}{2m} > \max\{\frac{p+1}{pq-1}, \frac{q+1}{pq-1}\}$, then globally solutions of (6) with small initial data exist.

Let us next turn to the inequality problem (1). The case of $a=b=0$ has been studied. Most recently, Jiang and Zheng [12] used the completely similar ideas and techniques in [13] to show that if $\frac{N}{4} \leqslant \max(\frac{p+1}{pq-1}, \frac{q+1}{pq-1})$, the inequality system (1) with $a=b=0$ does not admit nontrivial nonnegative global solutions in $\mathbb{S}$ without considering their initial traces.

The purposes of this paper are mainly to extend and improve the result to the case of $a, b \neq 0$, and then establish a similar Liouville type theorem for more general higher order inequality system (2). It is well known that higher-order equations are different from second-order equations mainly in the lack of maximum principles, which makes necessary the development of new analytical tools and different theoretic techniques. For instance, entropy and entropy dissipation methods have been demonstrated to be efficient for the understanding of the structure of equations and the qualitative behavior of their solutions, see [15, 14] and the references therein. In this note, we apply the "(rescaled) test function method" (cf. e.g. [17]) to get the Liouville type theorems for higher order parabolic Hardy-Hénon inequality system (1) and (2), without taking into account their traces on the hyperplane $t = 0$. We remark that the problems considered in [6] do not cover our case, due to singularity and degeneracy occur in right hand terms of our problem.

This paper is organized as follows. In Section 2, our main results of this paper are presented. We establish a new Liouville-type theorem for problem (1) in Theorem 1. Theorem 2 focuses on the nonexistence of solutions which are bounded below by a positive constant on $\mathbb{S}$. Moreover, we also prove the Liouville properties of solutions to more general higher order inequality system (2), this result is given in Theorem 3. Section 3 are devoted to the proofs of our main results. In the following sections, we denote by C positive generic constants, which may change from line to line even if in the same inequality.

2. Preliminaries and Main results

In this section, the main results of this paper are described. We first consider the problem (1) and start with our precise definition of solutions.

DEFINITION 1. We say a pair of functions (u,v) is a (weak) solution of (1) if $u, v \in L^1_{loc}(\mathbb{S})$ and the following are satisfied:

i) $|x|^a v^p \in L^1_{loc}(\mathbb{S})$ and $|x|^b u^q \in L^1_{loc}(\mathbb{S})$;
ii) for any positive function $\phi \in C_0^\infty(\mathbb{S})$, there hold

$$\iint_{\mathbb{S}} (-u\phi_t + u\triangle^2\phi)\,\mathrm{d}x\,\mathrm{d}t \geqslant \iint_{\mathbb{S}} |x|^a v^p \phi\,\mathrm{d}x\,\mathrm{d}t \tag{7}$$

and

$$\iint_{\mathbb{S}} (-v\phi_t + v\triangle^2\phi)\,\mathrm{d}x\,\mathrm{d}t \geqslant \iint_{\mathbb{S}} |x|^b u^q \phi\,\mathrm{d}x\,\mathrm{d}t. \tag{8}$$

We will show the following Liouville-type theorem.

THEOREM 1. *If* $\max\{\frac{a}{p-1}, \frac{b}{q-1}\} < N \leqslant \max\left\{\frac{4(p+1)+(a+bp)}{pq-1}, \frac{4(q+1)+(b+aq)}{pq-1}\right\}$, *then any nonnegative global solution* (u,v) *of system (1) is trivial, i.e.* $u = v = 0$ *a.e. on* $\mathbb{S}$.

In addition to Theorem 1, we also prove that for $p,q > 1$ satisfying weak assumptions inequality system (1) has no solutions bounded below by a positive constant on $\mathbb{S}$. We need the following definition.

DEFINITION 2. We say a pair of functions (u,v) is to be bounded below by a positive constant on $\mathbb{S}$, if there exists a positive constant C such that $u,v \geqslant C$ a.e. on $\mathbb{S}$.

Then we have

THEOREM 2. *Assume $(a+4)q+(4+b)>0$, $(b+4)p+(4+a)>0$ for $p,q>1$. Then (1) admits no solutions which are bounded below by a positive constant on $\mathbb{S}$.*

We remark that if $a=b=0$, the assumption in Theorem 2 is trivial. Our next result of this paper handles the Liouville properties of solutions to more general higher order inequality system. Similarly, we consider solutions of system (2) in the following sense.

DEFINITION 3. We say a pair of functions (u,v) is a solution of (2) if $u,v \in L^1_{loc}(\mathbb{S})$ and the following are satisfied:

i) $|x|^a v^p \in L^1_{loc}(\mathbb{S})$ and $|x|^b u^q \in L^1_{loc}(\mathbb{S})$;
ii) for any positive function $\phi \in C_0^\infty(\mathbb{S})$, there hold

$$\iint_{\mathbb{S}} (-u\phi_t - uL^*[\phi])\,\mathrm{d}x\,\mathrm{d}t \geqslant \iint_{\mathbb{S}} |x|^a v^p \phi\,\mathrm{d}x\,\mathrm{d}t \tag{9}$$

and

$$\iint_{\mathbb{S}} (-v\phi_t - vM^*[\phi])\,\mathrm{d}x\,\mathrm{d}t \geqslant \iint_{\mathbb{S}} |x|^b u^q \phi\,\mathrm{d}x\,\mathrm{d}t, \tag{10}$$

where

$$L^*[\phi] \triangleq \sum_{|\alpha|=l} a_\alpha(x,t,u)(-D)^\alpha \phi, \qquad M^*[\phi] \triangleq \sum_{|\beta|=h} b_\beta(x,t,v)(-D)^\beta \phi.$$

Now we present a Liouville type theorem for system (2).

THEOREM 3. *Suppose (u,v) is a nonnegative global solution of system (2), then $u=v=0$ a.e. on $\mathbb{S}$, provided $N > \max\{\frac{a}{p-1}, \frac{b}{q-1}\}$ and $(p,q) \in \Gamma_1 \cup \Gamma_2$ for*

$$\Gamma_1 = \left\{ (p,q) | N \leqslant \max\left\{ \frac{\min\{l,h\}(p+1)+a+bp}{pq-1}, \frac{\min\{l,h\}(q+1)+aq+b}{pq-1} \right\} \right\},$$
$$\Gamma_2 = \left\{ (p,q) | N + \max\{l,h\} \leqslant \max\left\{ \frac{p(b+h)+a+lpq}{pq-1}, \frac{q(a+l)+b+hpq}{pq-1} \right\} \right\}.$$

REMARK 1. As we noted above, the results obtained are new and they generalize and improve recent non-existence results [12]. It is easy to see that our conclusion especially for $a=b=0$ agrees with that in [12]. Furthermore, if $l=h=4$ in Theorem 3, simple calculations show that $\Gamma_2 \subset \Gamma_1$ and in this case, the result is consistent with that in Theorem 1.

REMARK 2. As an application of the above Liouville theorems, we can obtain the universal estimates for the following elliptic system

$$\triangle^2 u = |x|^a v^p, \quad \triangle^2 v = |x|^b u^q \quad \text{in } \Omega, \tag{11}$$

where $\Omega = \{x \in \mathbb{R}^N : 0 < |x| < \rho\}$ $(\rho > 0)$ and $p, q > 1$, $a, b \in \mathbb{R}$. Based on Theorem 1, using doubling lemma in [20] and proceeding very similarly as the proof of Theorem 1.3 in [18], we have: if $N > \max\{\frac{a}{p-1}, \frac{b}{q-1}\}$ and

$$N \leqslant \max\left\{\frac{4(p+1)+(a+bp)}{pq-1}, \frac{4(q+1)+(b+aq)}{pq-1}, \frac{4(p+1)}{pq-1}, \frac{4(q+1)}{pq-1}\right\},$$

there exists $C > 0$ depending only on a, b, p, q and N, such that any positive classical solution (u, v) of (11) satisfies

$$u(x) \leqslant C|x|^{\frac{a+4+(b+4)p}{1-pq}}, \qquad v(x) \leqslant C|x|^{\frac{b+4+(a+4)q}{1-pq}}, \quad \text{for any } 0 < |x| < \rho/2.$$

3. The proofs of Theorems

In this section, we address the proofs of our results. The main method is the test function method. In what follows, we denote $P(R) \triangleq \{(x,t) \in \mathbb{S} : |x|^4 + t < R\}$ for any fixed $R > 0$.

3.1. Proof of Theorem 1

The proof of Theorem 1 is heavily inspired by ideals of Kartsatos and Kurta in [13], which are also used in [12] to prove the Liouville type theorem for the case of $a = b = 0$.

Suppose the hypotheses of Theorem 1 are satisfied and (u,v) is a nonnegative solution to (1). For $0 < R < +\infty$, let $\xi : \mathbb{R}^N \times \mathbb{R}^+ \to [0,1]$ be a C^∞ function, which equals to 1 on $P(R/2)$ and 0 outside $P(R)$. Furthermore, for $0 < \tau < +\infty$, set $\eta : [0,+\infty) \to [0,1]$ be a C^∞ function such that the derivative $\eta' \geqslant 0$ and

$$\begin{cases} \eta(t) = 0, & \text{if } t \in [0,\tau], \\ \eta(t) = 1, & \text{if } t \in [2\tau,+\infty). \end{cases}$$

Let $\phi(x,t) = \xi^s(x,t)\eta^2(t)$ with $s > 4$ to be determined and substitute $\phi(x,t)$ into (7), we deduce

$$-s\iint_{P(R)} u\xi^{s-1}\xi_t\eta^2\,\mathrm{d}x\,\mathrm{d}t + \left(-2\iint_{P(R)} u\xi^s\eta\eta'\,\mathrm{d}x\,\mathrm{d}t\right) + \iint_{P(R)} u(\triangle^2\xi^s)\eta^2\,\mathrm{d}x\,\mathrm{d}t \geqslant \iint_{P(R)} |x|^a v^p\xi^s\eta^2\,\mathrm{d}x\,\mathrm{d}t. \tag{12}$$

Since $\eta'(t) \geqslant 0$ for all $t > 0$, the second integral on the left-hand side of (12) is nonpositive. So

$$s\iint_{P(R)} u\xi^{s-1}|\xi_t|\eta^2 \,\mathrm{d}x\,\mathrm{d}t + \iint_{P(R)} u|\triangle^2\xi^s|\eta^2 \,\mathrm{d}x\,\mathrm{d}t \geqslant \iint_{P(R)} |x|^a v^p \xi^s \eta^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{13}$$

Notice that

$$\begin{aligned}\triangle^2\xi^s =& s(s-1)(s-2)(s-3)\xi^{s-4}|\nabla\xi|^4 + 4s(s-1)(s-2)\xi^{s-3}\nabla\xi\cdot\nabla^2\xi\cdot\nabla^T\xi \\ &+ 2s(s-1)(s-2)\xi^{s-3}|\nabla\xi|^2\triangle\xi + 2s(s-1)\xi^{s-2}\sum_{i,j=1}^{n}\left(\frac{\partial^2\xi}{\partial x_i\partial x_j}\right)^2 \\ &+ 4s(s-1)\xi^{s-2}\nabla\xi\nabla\triangle\xi + s(s-1)\xi^{s-2}(\triangle\xi)^2 + s\xi^{s-1}\triangle^2\xi,\end{aligned}$$

where $\nabla^2\xi$ denotes the Hessian matrix of ξ.

It follows easily from (13) and Hölder's inequality that,

$$\begin{aligned}&\left(\iint_{P(R)\setminus P(\frac{R}{2})} |x|^b u^q \xi^s \eta^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{q}} \Bigg[\left(\iint_{P(R)} |x|^{-\frac{b}{q-1}}|\xi_t|^{\frac{q}{q-1}}\xi^{s-\frac{q}{q-1}}\eta^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{q-1}{q}} \\ &+ \left(\iint_{P(R)} |x|^{-\frac{b}{q-1}}|\nabla\xi|^{\frac{4q}{q-1}}\xi^{s-\frac{4q}{q-1}}\eta^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{q-1}{q}} \\ &+ \left(\iint_{P(R)} |x|^{-\frac{b}{q-1}}(\nabla\xi\cdot\nabla^2\xi\cdot\nabla^T\xi)^{\frac{q}{q-1}}\xi^{s-\frac{3q}{q-1}}\eta^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{q-1}{q}} \\ &+ \left(\iint_{P(R)} |x|^{-\frac{b}{q-1}}(|\nabla\xi|^2\triangle\xi)^{\frac{q}{q-1}}\xi^{s-\frac{3q}{q-1}}\eta^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{q-1}{q}} \\ &+ \left(\iint_{P(R)} |x|^{-\frac{b}{q-1}}\left(\sum_{i,j=1}^{n}\left(\frac{\partial^2\xi}{\partial x_i\partial x_j}\right)^2\right)^{\frac{q}{q-1}}\xi^{s-\frac{2q}{q-1}}\eta^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{q-1}{q}} \\ &+ \left(\iint_{P(R)} |x|^{-\frac{b}{q-1}}(|\nabla\xi||\nabla\triangle\xi|)^{\frac{q}{q-1}}\xi^{s-\frac{2q}{q-1}}\eta^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{q-1}{q}} \\ &+ \left(\iint_{P(R)} |x|^{-\frac{b}{q-1}}|\triangle\xi|^{\frac{2q}{q-1}}\xi^{s-\frac{2q}{q-1}}\eta^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{q-1}{q}} \\ &+ \left(\iint_{P(R)} |x|^{-\frac{b}{q-1}}|\triangle^2\xi|^{\frac{q}{q-1}}\xi^{s-\frac{q}{q-1}}\eta^2 \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{q-1}{q}}\Bigg] \\ &\geqslant C\iint_{P(R)} |x|^a v^p \xi^s \eta^2 \,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{14}$$

Now we let $\xi(x,t)=\psi(\frac{|x|^4+t}{R})$ in (14), where $\psi:[0,+\infty)\to[0,1]$ is a monotonically decreasing C^∞ function satisfying $\psi=1$ on $[0,1/2]$ and $\psi=0$ on $[1,+\infty)$. According to the definition of ξ, we have

$$|\xi_t|\leqslant CR^{-1},\qquad |\xi_x^{(j)}|\leqslant CR^{-\frac{j}{4}}\quad \text{for } 1\leqslant j\leqslant 4.$$

On the other hand, simple calculations show that for $N>\frac{b}{q-1}$, there holds

$$\iint_{\{|x|^4+t<R\}}|x|^{-\frac{b}{q-1}}\,\mathrm{d}x\,\mathrm{d}t\leqslant CR^{\frac{N+4}{4}-\frac{b}{4(q-1)}},$$

here C is a positive constant which depends on N,b,q only. Hence, noting that $s>\frac{4q}{q-1}$, we conclude from (14) that

$$\left(\iint_{P(R)\setminus P(\frac{R}{2})}|x|^b u^q\xi^s\eta^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{q}}\left(R^{-\frac{q}{q-1}+\frac{N+4}{4}-\frac{b}{4(q-1)}}\right)^{\frac{q-1}{q}}\geqslant C\iint_{P(R)}|x|^a v^p\xi^s\eta^2\,\mathrm{d}x\,\mathrm{d}t. \tag{15}$$

Similarly, for $N>\frac{a}{p-1}$, we can get

$$\left(\iint_{P(R)\setminus P(\frac{R}{2})}|x|^a v^p\xi^s\eta^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}}\left(R^{-\frac{p}{p-1}+\frac{N+4}{4}-\frac{a}{4(p-1)}}\right)^{\frac{p-1}{p}}\geqslant C\iint_{P(R)}|x|^b u^q\xi^s\eta^2\,\mathrm{d}x\,\mathrm{d}t. \tag{16}$$

Integrating (15) with (16), we arrive at

$$\begin{aligned}C&\iint_{P(R)}|x|^a v^p\xi^s\eta^2\,\mathrm{d}x\,\mathrm{d}t\\&\leqslant\left(\iint_{P(R)\setminus P(\frac{R}{2})}|x|^a v^p\xi^s\eta^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{pq}}R^{\left(\frac{N+4}{4}-\frac{q}{q-1}-\frac{b}{4(q-1)}\right)\frac{q-1}{q}+\left(\frac{N+4}{4}-\frac{p}{p-1}-\frac{a}{4(p-1)}\right)\frac{p-1}{pq}}\\&=\left(\iint_{P(R)\setminus P(\frac{R}{2})}|x|^a v^p\xi^s\eta^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{pq}}R^{\frac{N(pq-1)-4(p+1)-(a+bp)}{4pq}}\end{aligned} \tag{17}$$

and

$$C\iint_{P(R)}|x|^b u^q\xi^s\eta^2\,\mathrm{d}x\,\mathrm{d}t\leqslant\left(\iint_{P(R)\setminus P(\frac{R}{2})}|x|^b u^q\xi^s\eta^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{pq}}R^{\frac{N(pq-1)-4(q+1)-(b+aq)}{4pq}}. \tag{18}$$

Hence, from (17), we obtain

$$CR^{\frac{4(p+1)+(a+bp)-N(pq-1)}{4pq}}\left(\iint_{P(R)}|x|^a v^p\xi^s\eta^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1-\frac{1}{pq}}\leqslant 1,$$

and so

$$CR^{\frac{4(p+1)+(a+bp)-N(pq-1)}{4pq}}\left(\iint_{P(\frac{R}{2})}|x|^a v^p\eta^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1-\frac{1}{pq}}\leqslant 1. \tag{19}$$

In a similar way, we have

$$CR^{\frac{4(q+1)+(b+aq)-N(pq-1)}{4pq}}\left(\iint_{P(\frac{R}{2})}|x|^b u^q\eta^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1-\frac{1}{pq}}\leqslant 1. \tag{20}$$

Next we consider the following four possibilities.

Case 1. $N<\frac{4(p+1)+(a+bp)}{pq-1}$.

Let $R\to\infty$ in (19), we deduce

$$\iint_{\mathbb{S}}|x|^a v^p\eta^2\,\mathrm{d}x\,\mathrm{d}t=0.$$

Since the function $\eta(t)\equiv 1$ on $[2\tau,\infty)$ and τ is arbitrary, we obtain $v(x,t)=0$ a.e. on $\mathbb{S}$. $u(x,t)=0$ a.e. on $\mathbb{S}$ then follows from (16).

Case 2. $N<\frac{4(q+1)+(b+aq)}{pq-1}$.

Let $R\to\infty$ in (20), we get

$$\iint_{\mathbb{S}}|x|^b u^q\eta^2\,\mathrm{d}x\,\mathrm{d}t=0.$$

According to the definition of $\eta(t)$, we have $u(x,t)=0$ a.e. on $\mathbb{S}$, and consequently, $v(x,t)=0$ a.e. on $\mathbb{S}$ by (15).

Case 3. $N=\frac{4(p+1)+(a+bp)}{pq-1}$.

In this case, (19) implies $\iint_{\mathbb{S}}|x|^a v^p\eta^2\,\mathrm{d}x\,\mathrm{d}t$ is bounded. So for any sequence $r_k\to\infty$,

$$\iint_{P(r_k)\setminus P(r_k/2)}|x|^a v^p\eta^2\,\mathrm{d}x\,\mathrm{d}t\to 0$$

holds. Combining with (17), we infer

$$\lim_{r_k\to\infty}\iint_{P(r_k/2)}|x|^a v^p\eta^2\,\mathrm{d}x\,\mathrm{d}t=0,$$

which in turn leads that

$$\iint_{\mathbb{S}}|x|^a v^p\eta^2\,\mathrm{d}x\,\mathrm{d}t=0.$$

So as we discussed in Case 1, $u(x,t)=v(x,t)=0$ a.e. on $\mathbb{S}$.

Case 4. $N=\frac{4(q+1)+(b+aq)}{pq-1}$.

Similarly to Case 3, there holds

$$\iint_{\mathbb{S}}|x|^b u^q\eta^2\,\mathrm{d}x\,\mathrm{d}t=0.$$

Hence we deduce $u(x,t)=v(x,t)=0$ a.e. on $\mathbb{S}$ as in Case 2 and complete the proof of Theorem 1. □

3.2. Proof of Theorem 2

The proof is by contradiction. Let $(a+4)q+(4+b)>0$, $(b+4)p+(4+a)>0$ for $p,q>1$.

Suppose inequality system (1) admits a solution (u,v) such that $u,v\geqslant C>0$ on $\mathbb{S}$. Then let ξ, η, ψ as in the proof of Theorem 1, we have

$$C\iint_{P(R)}|x|^a v^p\xi^s\eta^2\,\mathrm{d}x\,\mathrm{d}t\leqslant\left(\iint_{P(R)\setminus P(\frac{R}{2})}|x|^a v^p\xi^s\eta^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{pq}}R^{\frac{N(pq-1)-4(p+1)-(a+bp)}{4pq}} \tag{21}$$

and

$$C\iint_{P(R)}|x|^b u^q\xi^s\eta^2\,\mathrm{d}x\,\mathrm{d}t\leqslant\left(\iint_{P(R)\setminus P(\frac{R}{2})}|x|^b u^q\xi^s\eta^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{pq}}R^{\frac{N(pq-1)-4(q+1)-(b+aq)}{4pq}}. \tag{22}$$

Therefore,

$$\iint_{P(\frac{R}{2})}|x|^a v^p\eta^2\,\mathrm{d}x\,\mathrm{d}t\leqslant CR^{\frac{N(pq-1)-4(p+1)-(a+bp)}{4(pq-1)}},$$
$$\iint_{P(\frac{R}{2})}|x|^b u^q\eta^2\,\mathrm{d}x\,\mathrm{d}t\leqslant CR^{\frac{N(pq-1)-4(q+1)-(b+aq)}{4(pq-1)}}.$$

Now we pass to the limit as $\tau\to 0$ to get that

$$\begin{aligned}&\iint_{P(\frac{R}{2})}|x|^a v^p\,\mathrm{d}x\,\mathrm{d}t\leqslant CR^{\frac{N}{4}-\frac{4(p+1)+(a+bp)}{4(pq-1)}},\\&\iint_{P(\frac{R}{2})}|x|^b u^q\,\mathrm{d}x\,\mathrm{d}t\leqslant CR^{\frac{N}{4}-\frac{4(q+1)+(b+aq)}{4(pq-1)}}.\end{aligned} \tag{23}$$

Next we proceed by estimating the integral $\iint_{P(\frac{R}{2})}|x|^a\,\mathrm{d}x\,\mathrm{d}t$. If $a\leqslant 0$, then

$$\iint_{P(\frac{R}{2})}|x|^a\,\mathrm{d}x\,\mathrm{d}t\geqslant(\frac{R}{2})^{\frac{a}{4}}\iint_{P(\frac{R}{2})}\mathrm{d}x\,\mathrm{d}t\geqslant CR^{\frac{a+N+4}{4}},$$

where C depends on N only. If $a>0$, it is easy to show that for $N+a>0$, which is trivial, there also holds

$$\iint_{P(\frac{R}{2})}|x|^a\,\mathrm{d}x\,\mathrm{d}t\geqslant CR^{\frac{a+N+4}{4}}$$

for C depends on N and a. By the assumption $u,v\geqslant C$, we then deduce from (23) that

$$R^{\frac{p[(a+4)q+(4+b)]}{4(pq-1)}}\leqslant C,\qquad R^{\frac{q[(b+4)p+(4+a)]}{4(pq-1)}}\leqslant C,$$

for any $p,q>1$ and $R>0$. Let R tends to infinity, we have thus shown that the assumption leads to a contradiction, which proves the theorem. □

3.3. Proof of Theorem 3

Suppose (u,v) is a nonnegative global solution of system (2). Take $\phi\in C_0^\infty(\mathbb{R})$, $\phi\geqslant 0$, $\phi(s)=1$ for $s\leqslant 1$, and $\phi(s)=0$ for $s\geqslant 2$. We may further assume that

$$\begin{aligned} |\phi'(s)| &\leqslant C\phi^{\frac{1}{p}}(s), & |\phi'(s)| &\leqslant C\phi^{\frac{1}{q}}(s),\\ |\phi^{(h)}(s)| &\leqslant C\phi^{\frac{1}{p}}(s), & |\phi^{(l)}(s)| &\leqslant C\phi^{\frac{1}{q}}(s). \end{aligned} \tag{24}$$

In fact, such a function exists, for example, one can choose $\phi(s)=(2-s)^\rho$ with $\rho>\max\{\frac{hp}{p-1},\frac{lq}{q-1}\}$ for $3/2<s<2$. We remark here that such a test function was first used in [6].

Let us introduce

$$\psi_R(x,t)=\phi\left(\frac{|x|^2+t^{2/\sigma}}{R^2}\right)$$

for $R>0$ and $\sigma>0$ to be determined below. Then we set $\phi=\psi_R(x,t)$ in (9) and (10) to obtain that

$$\begin{aligned} \iint_{\mathbb{S}}|x|^a v^p\psi_R\,\mathrm{d}x\,\mathrm{d}t &\leqslant \iint_{\mathbb{S}}(-u\tfrac{\partial\psi_R}{\partial t}-uL^*[\psi_R])\,\mathrm{d}x\,\mathrm{d}t,\\ \iint_{\mathbb{S}}|x|^b u^q\psi_R\,\mathrm{d}x\,\mathrm{d}t &\leqslant \iint_{\mathbb{S}}(-v\tfrac{\partial\psi_R}{\partial t}-vM^*[\psi_R])\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{25}$$

Denote by I_1, I_2 the left hand sides of the above two inequalities respectively, we now in a position to estimate the integrals on the right hand sides. Notice that for $N>\frac{b}{q-1}$, there holds

$$\iint_{\{|x|^2+t^{2/\sigma}<2R^2\}}|x|^{-\frac{b}{q-1}}\,\mathrm{d}x\,\mathrm{d}t\leqslant CR^{N+\sigma-\frac{b}{q-1}},$$

and $\operatorname{supp}\frac{\partial\psi_R}{\partial t}\subseteq\{(x,t)\in\mathbb{S}:|x|^2+t^{2/\sigma}<2R^2\}$, then applying (24) and Young's inequality, we deduce that

$$\begin{aligned} &\iint_{\mathbb{S}}-u\frac{\partial\psi_R}{\partial t}\,\mathrm{d}x\,\mathrm{d}t=-C\iint_{\operatorname{supp}\frac{\partial\psi_R}{\partial t}}\phi' R^{-2}t^{\frac{2-\sigma}{\sigma}}u\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant CR^{-\sigma}\iint_{\operatorname{supp}\frac{\partial\psi_R}{\partial t}}\psi_R^{1/q}u\,\mathrm{d}x\,\mathrm{d}t\\ &\leqslant C\left(\iint_{\operatorname{supp}\frac{\partial\psi_R}{\partial t}}|x|^b u^q\psi_R\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{q}}\left(\iint_{\operatorname{supp}\frac{\partial\psi_R}{\partial t}}R^{-\frac{\sigma q}{q-1}}|x|^{-\frac{b}{q-1}}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{q-1}{q}}\\ &\leqslant CI_2^{1/q}\left(R^{-\frac{\sigma q}{q-1}-\frac{b}{q-1}+N+\sigma}\right)^{\frac{q-1}{q}}, \end{aligned}$$

and, similarly, also

$$-\iint_{\mathbb{S}}uL^*[\psi_R]\,\mathrm{d}x\,\mathrm{d}t\leqslant CI_2^{1/q}R^{-l-\frac{b}{q}+\frac{(N+\sigma)(q-1)}{q}}.$$

Therefore, it follows from (25) that

$$
\begin{aligned}
I_1 &\leqslant C I_2^{1/q} R^{-\frac{b}{q}+\frac{(N+\sigma)(q-1)}{q}} \left(R^{-\sigma}+R^{-l}\right)\\
&\leqslant C I_2^{1/q} R^{\frac{(N+\sigma)(q-1)-b}{q}-\min\{\sigma,l\}}.
\end{aligned} \tag{26}
$$

In a similar way, for $N > \frac{a}{p-1}$,

$$
I_2 \leqslant C I_1^{1/p} R^{\frac{(N+\sigma)(p-1)-a}{p}-\min\{\sigma,h\}} \tag{27}
$$

holds.

Set $\lambda \triangleq \frac{(N+\sigma)(q-1)-b}{q} - \min\{\sigma, l\}$ and $\omega \triangleq \frac{(N+\sigma)(p-1)-a}{p} - \min\{\sigma, h\}$, then combining (26) and (27), we get that

$$
I_1 \leqslant C R^{\frac{p(\lambda q+\omega)}{pq-1}}, \qquad I_2 \leqslant C R^{\frac{q(p\omega+\lambda)}{pq-1}}. \tag{28}
$$

Next, taking $\sigma = \min\{l, h\}$, it is easy to show that $\lambda q + \omega \leqslant 0$ is equivalent to $N \leqslant \frac{\min\{l,h\}(p+1)+(a+bp)}{pq-1}$, and

$$
\omega p + \lambda \leqslant 0 \Leftrightarrow N \leqslant \frac{\min\{l, h\}(q+1) + (aq+b)}{pq-1}.
$$

If $\sigma = \max\{l, h\}$, simple calculations give that

$$
\begin{aligned}
&\lambda q + \omega \leqslant 0 \Leftrightarrow N + \max\{l, h\} \leqslant l + \tfrac{p(b+h)+a+l}{pq-1},\\
&\omega p + \lambda \leqslant 0 \Leftrightarrow N + \max\{l, h\} \leqslant h + \tfrac{q(a+l)+b+h}{pq-1}.
\end{aligned} \tag{29}
$$

Hence, as is discussed in the last part of Section 3.1, we learn from (28) that $u = v = 0$ a.e. on $\mathbb{S}$ whenever $(p,q) \in \Gamma_1 \cup \Gamma_2$ by letting $R \to +\infty$. □

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