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# ISOPERIMETRIC INEQUALITIES FOR POSITIVE SOLUTION OF P–LAPLACIAN

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*Abstract.* In this paper, we prove some isoperimetric inequalities and give a explicit bound for the positive solution of P-Laplacian.

## 1. Introduction and main result

In this paper, we consider the following problem

$$\begin{cases} -\operatorname{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = u^q, \ x \in \Omega, \\ u > 0, \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega. \end{cases} \tag{1.1}$$

where $p \geqslant 2$, $0 < q < p-1$. $\Omega \subset R^n$ denotes a bounded domain whose boundary $\partial\Omega$ is assumed to be of Lipschitz type.

The purpose of this paper is to prove some isoperimetric inequalities and give an explicit bound for the solution of problem (1.1) by making use of symmetrization method.

There are a lot of material related to isoperimetric inequality linking with solutions of equations, for example [1]–[20]. The first result on isoperimetric inequality for eigenfunctions of Laplace operator was derived by Payne and Rayner in [13] in 1972: If $\Omega$ is a bounded domain in $R^2$ whose boundary $\partial\Omega$ is assumed to be of Lipschitz type. $\lambda_1(\Omega)$ and $\varphi_1(x)$ are the first eigenvalue and the corresponding eigenfunction of the problem

$$\begin{cases} -\Delta\varphi = \lambda\varphi \ \text{ in } \Omega, \\ \varphi = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

then the following inequality holds

$$\left(\int_\Omega |\varphi_1| dA\right)^2 \geqslant \frac{4\pi}{\lambda_1(\Omega)} \int_\Omega \varphi_1^2 dA, \tag{1.3}$$

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with equality if and only if $\Omega$ is a disk.

The method used by Payne and Rayner in [13] is integral transformation and the Co-area formula. However, the method they used works only for the case $n = 2$. Kohnler-Jobin [18, 20] and G. Chiti [15] generalized the Payne and Rayner's inequality (1.3) to arbitrary dimension $n$ by employing the Schwarz symmetrization method. It is by now well known that the Schwarz symmetrization method is very useful for the estimate of sharp bound of solutions to elliptic and parabolic equations and has been extensively studied since the pioneer work of Weinberger [21], Talenti [22] and Bandle [23]. See for example [24, 25, 26, 27] for more details.

Worthy of mention is that much attention had been paid to the isoperimetric inequality related to the eigenvalue and eigenfunction of Laplacian, we refer to [1], [2], [16], [28], [29], [30], [31] and the same kind of inequality for P-Laplacian (see however [3], [4], [17], [32]). It is easy to find the fact that almost all of the above results are either valid for equation which is homogeneous or related to eigenvalue. It is also worth to point out that all arguments we mentioned above have been extend to more general elliptic equations, even include the following nonlinear problem

$$\begin{cases} -\sum_{ij=1}^{n} \dfrac{\partial}{\partial x_j}\left(a_{ij}(x)|\nabla u|^{p-2}\dfrac{\partial u}{\partial x_i}\right) = f(x,u) \quad x \in G, \\ \hfill u = 0 \quad x \in \partial G, \end{cases}$$

See for example [24]. However, they required that the nonlinear term $f(x,u)$ can be replaced by a linear function. A typical condition imposed on $f(x,u)$ is as

$$(f(x,u),\ f(x,0))u \leqslant 0 \qquad \text{for all } (x,u). \tag{1.4}$$

which is given in [24].

Contrast to the eigenvalue problem, there are some other results on the isoperimetric inequalities for solutions of quasi-linear elliptic problems. This is the motivation of our study of the isoperimetric inequalities for the solution of problem (1.1). Our result is valid for equation of (1.1) that is nonhomogeneous and has nothing to do with eigenvalue. What's more, the nonlinear term in (1.1) does not satisfies (1.4) and is different from the results of [33, 34].

Our method is symmetrization which is adapted from G. Chiti's paper [15]. The basic idea in the use of the symmetrization method is to compare the original problem with an auxiliary problem defined on a suitable ball.

To state our results, we introduce the following auxiliary problem

$$\begin{cases} -\mathrm{div}\Big(|\nabla h|^{p-2}\nabla h\Big) = |h|^{q-1}h & x \in \Omega^*, \\ h > 0 & x \in \Omega^*, \\ h = 0 & x \in \partial\Omega^*. \end{cases} \tag{1.5}$$

where $\Omega^*$ is the Schwarz symmetrization of $\Omega$. ($\Omega^*$ is a ball in $R^n$ with center at 0 such that $|\Omega^*| = |\Omega|$.)

Let $\sigma_{p,1} = \frac{(1+q)[kp+n(p-1-q)]}{np-n+p-(n-p)q}$ and $\sigma_{p,2} = \frac{(1+q)p}{np-n+p-(n-p)q}$ be fixed. Then our main result can be stated as

THEOREM 1. *Let $u(x)$ be the unique solution of problem (1.1) and $h(x)$ be the unique solution of problem (1.5). Then for any $k \geqslant q+1$, we have*

$$\int_\Omega u^k(x)dx \leqslant C(q,p,\Omega^*)\|u\|_{L^{q+1}(\Omega)}^{\sigma_{p,1}}. \tag{1.6}$$

*Consequentely*

$$\max_{x\in\Omega} u(x) \leqslant C(q,\Omega^*)\|u\|_{L^{q+1}(\Omega)}^{\sigma_{p,2}}, \tag{1.7}$$

*where* $C(q,p,\Omega^*) = \int_{\Omega^*} h^k(x)dx/\|h\|_{L^{q+1}(\Omega^*)}^{\sigma_{p,1}}$ *and* $C(q,\Omega^*) = \max_{x\in\Omega^*} h(x)/\|h\|_{L^{q+1}(\Omega^*)}^{\sigma_{p,2}}$. *Moreover, the equality holds in each of inequalities (1.6) and (1.7) if and only if $\Omega$ is a ball.*

By theorem 1 and a Faber-Krahn type inequality proved in Lemma 3.2 of Section 3, we have

COROLLARY 1. *Let $u(x)$ be the unique solution of problem (1.1) and $h(x)$ be the unique solution of problem (1.5). Then for any $k \geqslant q+1$, we have*

$$\int_\Omega u^k(x)dx \leqslant \int_{\Omega^*} h^k(x)dx, \tag{1.8}$$

*and*

$$\max_{x\in\Omega} u(x) \leqslant \max_{x\in\Omega^*} h(x). \tag{1.9}$$

*Moreover, the equality holds in each of inequalities (1.8) and (1.9) if $\Omega$ is a ball.*

Thanks to Corollary 1 and an explicit bound of solution of problem (1.5), we have

COROLLARY 2. *Let $u(x)$ be the unique solution of problem (1.1), then the following estimate holds*

$$\max_{x\in\Omega} u(x) \leqslant \left[\frac{|\Omega|}{\omega_n[n(\frac{p}{p-1})^{p-1}]^{\frac{n}{p}}}\right]^{\frac{p}{n(p-1-q)}}. \tag{1.10}$$

REMARK 1. Let $u(x)$ be the unique solution of problem (1.1). If $|\Omega| < \omega_n[n(\frac{p}{p-1})^{p-1}]^{\frac{n}{p}}$, then it follows from Corollary 2 that $u(x) \to 0$ uniformly on $\Omega$ when $q \to (p-1)^-$. It is interest to know the asymptotic behavior of $u(x)$ when $|\Omega| \geqslant \omega_n[n(\frac{p}{p-1})^{p-1}]^{\frac{n}{p}}$ and $q \to (p-1)^-$. It is also interest to know the asymptotic behavior of $u(x)$ when $q \to 0^+$.

The paper is organized as follows: Preliminary is contained in Section 2. In Section 3, we prove a Chiti type comparison result which is essential to the proof of our main results. The proofs of Theorem 1, Corollary 1 and Corollary 2 are given in Section 4.

## 2. Preliminary

In this section, we give some notations and some lemmas which are essential to our results.

Let $\Omega$ be a bounded domain in $R^n$. The Schwarz symmetrization $\Omega^*$ of $\Omega$ is a ball in $R^n$ with radius $R^*$ and centered at 0 such that $|\Omega^*| = |\Omega|$. Here, $|\Omega|$ denotes the Lebesgue measure of $\Omega$. If we denote by $\omega_n$ the volume of unit ball in $R^n$, then it is easy to see

$$R^* = \left(\frac{|\Omega|}{\omega_n}\right)^{\frac{1}{n}}.$$

Let $f: \Omega \mapsto R$ be a nonnegative measurable function. For any $t \geqslant 0$. The level set $\Omega_t$ of $f$ at the level $t$ is defined by

$$\Omega_t \doteq \{x \in \Omega: f(x) > t\}, \quad t \geqslant 0.$$

The distribution function of $f$ is given by

$$\mu_f(t) = |\Omega_t| = \text{meas}\{x \in \Omega: f(x) > t\}, \quad t \geqslant 0.$$

Let $\Omega$ be a bounded domain in $R^n$, $f: \Omega \mapsto R$ be a nonnegative measurable function. Then the decreasing rearrangement $f^*$ and the decreasing Schwarz symmetrization $f^\star$ of $f$ are defined by

$$f^*(s) = \begin{cases} \text{ess.}\sup.f & \text{for } s = 0, \\ \inf\{t > 0 | \mu_f(t) < s\} & \text{for } s > 0; \end{cases}$$

and

$$f^\star(x) = f^*(\omega_n |x|^n), \quad \text{for } x \in \Omega^*.$$

LEMMA 2.1. ([35]) *Let $M$, $\alpha$, $\beta$ be real numbers such that $0 < \alpha \leqslant \beta$ and $M > 0$. Let $f$, $g$ be real functions in $L^\beta([0, M])$. If the decreasing rearrangements of $f$ and $g$ satisfy the inequality*

$$\int_0^s f^{*^\alpha}(t)dt \leqslant \int_0^s g^{*^\alpha}(t)dt \quad \textit{for} \quad s \in [0, M],$$

*then*

$$\int_0^M f^{*^\beta}(t)dt \leqslant \int_0^M g^{*^\beta}(t)dt.$$

The following result may be well known. Readers can see [36] for details.

LEMMA 2.2. *Problem (1.1) has a unique solution.*

## 3. Chiti type comparison result

Let $\Omega$ be a bounded domain in $R^n$, and $\|\cdot\|_{L^{q+1}(\Omega)}$ denote the norm of space $L^{q+1}(\Omega)$. We define

$$S_{p,q}(\Omega) = \inf_{v\in W_0^{1,p}(\Omega)} \{\int_\Omega |\nabla v|^p dx \Big| \, \|v\|^p_{L^{q+1}(\Omega)} = 1\}.$$

It is easy to prove that $S_{p,q}(\Omega)$ can be achieved by a unique positive function $v(x)$. Moreover, $v(x)$ satisfies

$$\begin{cases} -\mathrm{div}(|\nabla v(x)|^{p-2}\nabla v(x)) = S_{p,q}(\Omega) v^q(x), & x\in\Omega, \\ v(x) > 0, & x\in\Omega, \\ v(x) = 0, & x\in\partial\Omega, \\ \int_\Omega v^{q+1}(x)dx = 1. \end{cases} \tag{3.1}$$

In this section, we prove a Chiti type comparison result for problem (3.1). To this end, we need some lemmas first.

LEMMA 3.1. *For any $\lambda > 0$ and $\lambda \neq S_{p,q}(\Omega)$, the following problem has no solution*

$$\begin{cases} -div(|\nabla f(x)|^{p-2}\nabla f(x)) = \lambda f^q(x), & x\in\Omega, \\ f(x) > 0, & x\in\Omega, \\ f(x) = 0, & x\in\partial\Omega, \\ \int_\Omega f^{q+1}(x)dx = 1. \end{cases} \tag{3.2}$$

*Proof.* We prove Lemma 3.1 by contradiction. Assume that problem (3.2) has a solution $f_{\lambda_0}$ for some $\lambda_0 > 0$ and $\lambda_0 \neq S_{p,q}(\Omega)$. Then, it is easy to check that $\widetilde{f} = \lambda_0^{\frac{1}{q+1-p}} f_{\lambda_0}$ is a solution of problem (1.1) which satisfies

$$\int_\Omega \widetilde{f}^{q+1}(x)dx = \lambda_0^{\frac{q+1}{q+1-p}}.$$

On the other hand, if we denote by $v(x)$ the minimizer of $S_{p,q}(\Omega)$, then $\widetilde{v} = S_q^{\frac{1}{q+1-p}}(\Omega)v(x)$ is also a solution of problem (1.1) which satisfies

$$\int_\Omega \widetilde{v}^{q+1}(x)dx = S_q^{\frac{q+1}{q+1-p}}(\Omega).$$

It is obvious that $\widetilde{v} \neq \widetilde{f}$ due to $\lambda_0 \neq S_{p,q}(\Omega)$. Hence problem (1.1) has at least two solutions $\widetilde{v}$ and $\widetilde{f}$. This contradicts Lemma 2.2. □

LEMMA 3.2. ([29]) *$S_{p,q}(\Omega) \geqslant S_{p,q}(\Omega^*)$ with equality if and only if $\Omega$ is a ball.*

Let $\sigma_{p,3} = \frac{q+1}{np-n+p-(n-p)q}$, then the following lemma holds

LEMMA 3.3. *Let $v(x)$ be the minimizer of $S_{p,q}(\Omega^*)$ and $r_* = \left(\frac{S_{p,q}(\Omega^*)}{S_{p,q}(\Omega)}\right)^{\sigma_{p,3}} R^*$. Then $S_{p,q}(B_{r_*}(0)) = S_{p,q}(\Omega)$ and the minimizer of $S_{p,q}(B_{r_*}(0))$ is $z(y) = \left(\frac{R^*}{r_*}\right)^{\frac{n}{q+1}} v(\frac{R^*}{r_*}y)$ for $y \in B_{r_*}(0)$.*

*Proof.* Since $v(x)$ is the minimizer of $S_{p,q}(\Omega^*)$, $v(x)$ satisfies

$$\begin{cases} -\mathrm{div}(|\nabla v(x)|^{p-2}\nabla v(x)) = S_{p,q}(\Omega^*)v^q(x), & x \in \Omega^*, \\ v(x) > 0, & x \in \Omega^*, \\ v(x) = 0, & x \in \partial\Omega^*, \\ \int_{\Omega^*} v^{q+1}(x)dx = 1. \end{cases}$$

Let $x = \frac{R^*}{r_*}y$ and $H(y) = v(\frac{R^*}{r_*}y)$. Then

$$\frac{\partial H}{\partial y_i} = \frac{R^*}{r_*}\frac{\partial v}{\partial x_i}.$$

Hence

$$\begin{aligned} -\mathrm{div}(|\nabla H(y)|^{p-2}\nabla H(y)) &= -\left(\frac{R^*}{r_*}\right)^p \mathrm{div}(|\nabla v(x)|^{p-2}\nabla v(x)) \\ &= \left(\frac{R^*}{r_*}\right)^p S_{p,q}(\Omega^*)H^q(y), \quad y \in B_{r_*}(0). \end{aligned}$$

Noting that

$$\begin{aligned} 1 = \int_{\Omega^*} v^{q+1}(x)dx &= \left(\frac{R^*}{r_*}\right)^n \int_{B_{r_*}(0)} H^{q+1}(y)dy \\ &= \int_{B_{r_*}(0)} \left[\left(\frac{R^*}{r_*}\right)^{\frac{n}{q+1}} H(y)\right]^{q+1} dy. \end{aligned}$$

If we let $z(y) = (\frac{R^*}{r_*})^{\frac{n}{q+1}} H(y) = (\frac{R^*}{r_*})^{\frac{n}{q+1}} v(\frac{R^*}{r_*}y)$, then $z(y)$ satisfies

$$\begin{cases} -\mathrm{div}(|\nabla z(y)|^{p-2}\nabla z(y)) = \left(\frac{R^*}{r_*}\right)^{\frac{1}{\sigma_{p,3}}} S_{p,q}(\Omega^*)z^q(y), & y \in B_{r_*}(0), \\ z(y) > 0, & y \in B_{r_*}(0), \\ z(y) = 0, & y \in \partial B_{r_*}(0), \\ \int_{B_{r_*}(0)} z^{q+1}(y)dy = 1. \end{cases}$$

Hence, by Lemma 3.1, we have

$$S_{p,q}(B_{r_*}(0)) = \left(\frac{R^*}{r_*}\right)^{\frac{1}{\sigma_{p,3}}} S_{p,q}(\Omega^*) = S_{p,q}(\Omega).$$

and the minimizer of $S_{p,q}(B_{r_*}(0))$ is $z(y) = \left(\frac{R^*}{r_*}\right)^{\frac{n}{q+1}} v(\frac{R^*}{r_*}y)$. This completes the proof of Lemma 3.3. □

REMARK 2. By Lemma 3.2 and the definition of $r_*$, we have $B_{r_*}(0) \subseteq \Omega^*$ with equality if and only if $\Omega$ is a ball. Let $M = |\Omega|$ and $M_* = |B_{r_*}(0)|$, then $M_* \leqslant M$.

The main result of this section is the following Chiti type comparison result.

THEOREM 3.1. *Let $u(x)$ be the minimizer of $S_{p,q}(\Omega)$ and $z(x)$ be the minimizer of $S_{p,q}(B_{r_*}(0))$. If we denote by $u^*(s)$ the decreasing rearrangement of $u(x)$, and $z^*(s)$ the decreasing rearrangement of $z(x)$, then there exists a unique point $s_0 \in (0,\ M_*)$ such that*

$$\begin{cases} u^*(s) \leqslant z^*(s) & \text{for } s \in [0,\ s_0), \\ u^*(s) > z^*(s) & \text{for } s \in (s_0,\ M_*]. \end{cases}$$

*Proof.* Since $u(x)$ is the minimizer of $S_{p,q}(\Omega)$, it is easy to see that $u(x)$ satisfies

$$\begin{cases} -\mathrm{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = S_{p,q}(\Omega)u^q(x), & x \in \Omega, \\ u(x) > 0, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{3.3}$$

From this, we can prove that the decreasing rearrangement $u^*(s)$ of $u(x)$ satisfies

$$-\frac{du^*(s)}{ds} \leqslant S_{p,q}^{\frac{1}{p-1}} n^{-\frac{p}{p-1}} \omega_n^{-\frac{p}{n(p-1)}} s^{-\frac{(n-1)p}{n(p-1)}} \left(\int_0^s u^*(t)^q dt\right)^{\frac{1}{p-1}} \quad a.e. \text{ in } [0,\ M], \tag{3.4}$$

In fact, integrating the first equation in (3.3) over $\Omega_t = \{x \in \Omega \mid u(x) > t\}$, we have

$$-\int_{\Omega_t} \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) dA = S_{p,q}(\Omega)\int_{\Omega_t} u^q dA. \tag{3.5}$$

Since $\partial\Omega_t = \{x \in \Omega \mid u(x) = t\}$, we have

$$\int_{\partial\Omega_t} |\nabla u|^{p-1} = -\int_{\partial\Omega_t} |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = S_{p,q}(\Omega)\int_{\Omega_t} u^q dA. \tag{3.6}$$

Noting that

$$\left(\int_{\partial\Omega_t} \frac{1}{|\nabla u|} d\sigma\right)^{\frac{p-1}{p}} \left(\int_{\partial\Omega_t} |\nabla u|^{p-1} d\sigma\right)^{\frac{1}{p}} \geqslant \int_{\partial\Omega_t} \frac{1}{|\nabla u|^{\frac{p-1}{p}}} |\nabla u|^{\frac{p-1}{p}} d\sigma = \int_{\partial\Omega_t} d\sigma.$$

It follows from the isoperimetric inequality

$$\left(\int_{\partial\Omega_t}\frac{1}{|\nabla u|}d\sigma\right)^{\frac{p-1}{p}}\left(\int_{\partial\Omega_t}|\nabla u|^{p-1}d\sigma\right)^{\frac{1}{p}}\geqslant n\omega_n^{\frac{1}{n}}|\Omega_t|^{\frac{n-1}{n}}. \tag{3.7}$$

By Co-area formula, we have

$$\mu(t)=|\Omega_t|=\int_{\Omega_t}dx=\int_t^{+\infty}\int_{\partial\Omega_t}\frac{ds}{|\nabla u|}.$$

Consequently,

$$\frac{d\mu(t)}{dt}=-\int_{\partial\Omega_t}\frac{ds}{|\nabla u|}. \tag{3.8}$$

From (3.5), (3.6), (3.7) and (3.8), we obtain

$$n\omega_n^{\frac{1}{n}}\mu(t)^{\frac{n-1}{n}}\leqslant(-\mu'(t))^{\frac{p-1}{p}}\left(S_{p,q}(\Omega)\int_{\Omega_t}u^q dA\right)^{\frac{1}{p}}. \tag{3.9}$$

Since $\Omega_t\subset\Omega$, we have

$$\int_{\Omega_t}u^q dx\leqslant\int_0^{|\Omega_t|}(u^q)^*(\tau)d\tau=\int_0^{\mu(t)}(u^*(\tau))^q d\tau. \tag{3.10}$$

Combing (3.9) with (3.10), we obtain

$$\left(\frac{1}{-\mu'(t)}\right)^{p-1}\leqslant\frac{S_{p,q}(\Omega)\int_0^{\mu(t)}u^{*q}(\tau)d\tau}{\left(n\omega_n^{\frac{1}{n}}\mu(t)^{\frac{n-1}{n}}\right)^p}.$$

Noticing that $u^*(s)$ is essentially an inverse of $\mu(t)$, we have

$$-\frac{du^*(s)}{ds}\leqslant S_{p,q}^{\frac{1}{p-1}}(\Omega)n^{-\frac{p}{p-1}}\omega_n^{-\frac{p}{n(p-1)}}s^{-\frac{(n-1)p}{n(p-1)}}\left(\int_0^s u^*(t)^q dt\right)^{\frac{1}{p-1}}$$

This is just the desired conclusion of (3.4).
Since, $S_{p,q}(B_{r_*}(0))=S_{p,q}(\Omega)$, the minimizer $z(x)$ of $S_{p,q}(B_{r_*}(0))$ satisfies

$$\begin{cases}-\mathrm{div}(|\nabla z(x)|^{p-2}\nabla z(x))=S_{p,q}(\Omega)z^q(x), & x\in B_{r_*}(0),\\ z(x)>0, & x\in B_{r_*}(0),\\ z(x)=0, & x\in\partial B_{r_*}(0).\end{cases} \tag{3.11}$$

Noticing that uniqueness result valid for (3.11), it is trivial to see that $z$ is radial symmetry. That is $z(x)=z(|x|)$. Moreover, as a function of $s=\omega_n|x|^n$, $z(s)$ is decreasing. Hence, from (3.11) and a similar argument to that used to derive (3.4), we can obtain

$$-\frac{dz^*(s)}{ds}=S_{p,q}^{\frac{1}{p-1}}(\Omega)n^{-\frac{p}{p-1}}\omega_n^{-\frac{p}{n(p-1)}}s^{-\frac{(n-1)p}{n(p-1)}}\left(\int_0^s z^*(t)^q dt\right)^{\frac{1}{p-1}}\quad a.e.\text{ in }[0,\,M_*], \tag{3.12}$$

To prove Theorem 3.1, we first prove that $z^*(0) \geqslant u^*(0)$.

If $u^*(0) > z^*(0)$, then from

$$\int_\Omega u^{q+1}(x)dx = \int_0^M (u^*)^{q+1}(s)ds = 1 = \int_0^{M_*} (z^*(s))^{q+1}ds = \int_{B_{r_*}(0)} z^{q+1}(x)dx \quad (3.13)$$

and $M > M_*$, we know that there exists a point $s_1 \in (0, M_*)$ such that

$$\begin{cases} u^*(s) > z^*(s), & s \in (0,\ s_1), \\ u^*(s_1) = z^*(s_1). \end{cases}$$

Let

$$w(s) = \begin{cases} u^*(s), & s \in [0,\ s_1); \\ z^*(s), & s \in [s_1,\ M_*]. \end{cases}$$

Then, it is easy to verify that $w(s)$ satisfies

$$\begin{cases} -\dfrac{dw(s)}{ds} \leqslant S_{p,q}^{\frac{1}{p-1}}(\Omega) n^{-\frac{p}{p-1}} \omega_n^{-\frac{p}{n(p-1)}} s^{-\frac{(n-1)p}{n(p-1)}} \left( \int_0^s w^*(t)^q dt \right)^{\frac{1}{p-1}}, & a.e. \text{ in } [0,\ M_*], \\ w(s) > 0, & s \in (0,\ M_*), \\ w(M_*) = 0, & \\ \|w\|_{L^{q+1}(0,\ M_*)} \geqslant 1. & \end{cases} \quad (3.14)$$

Define

$$\eta(x) = \frac{w(\omega_n |x|^n)}{\|w(\omega_n |x|^n)\|_{q+1(B_{r_*}(0))}}.$$

Then, $\eta(x) \in W_0^{1,\ p}(B_{r_*}(0))$ and $\|\eta(x)\|_{q+1(B_{r_*}(0))} = 1$. Since problem (1.1) has a unique solution, $\eta(x) \neq z(x)$ and $z(x)$ is the minimizer of $S_{p,q}(B_{r_*}(0))$ that satisfies problem (3.11), thus $\eta(x)$ is not the minimizer of $S_{p,q}(B_{r_*}(0))$, we have

$$S_{p,q}(\Omega) = S_{p,q}(B_{r_*}(0)) < \int_{B_{r_*}(0)} |\nabla \eta(x)|^p dx.$$

Since

$$\begin{aligned} \int_{B_{r_*}(0)} |\nabla \eta(x)|^p dx &= n^p \omega_n^{\frac{p}{n}} \int_0^{M_*} |\eta'(s)|^p s^{\frac{p(n-1)}{n}} ds \\ &= \frac{n^p \omega_n^{\frac{p}{n}}}{\|w\|_{q+1(B_{r_*}(0))}^p} \int_0^{M_*} |w'(s)|^p s^{\frac{p(n-1)}{n}} ds, \end{aligned}$$

and

$$
\begin{aligned}
n^p \omega_n^{\frac{p}{n}} \int_0^{M_*} |w'(s)|^p s^{\frac{p(n-1)}{n}} ds &= n^p \omega_n^{\frac{p}{n}} \int_0^{M_*} (-w'(s))(-w'(s))^{p-1} s^{\frac{p(n-1)}{n}} ds \\
&\leqslant S_{p,q}(\Omega) \int_0^{M_*} (-w'(s)) \int_0^s w^q(\tau) d\tau ds \\
&= S_{p,q}(\Omega) \int_0^{M_*} w^{q+1}(s) ds \\
&= S_{p,q}(B_{r_*}(0)) \|w\|_{q+1(B_{r_*}(0))}^{q+1}
\end{aligned}
$$

We have

$$
\int_{B_{r_*}(0)} |\nabla \eta(x)|^p dx \leqslant S_{p,q}(B_{r_*}(0)) \|w\|_{q+1(B_{r_*}(0))}^{q+1-p} = S_{p,q}(B_{r_*}(0)) \|w\|_{q+1(B_{r_*}(0))}^{q+1-p}.
$$

Thus

$$
S_{p,q}(B_{r_*}(0)) < \int_{B_{r_*}(0)} |\nabla \eta(x)|^p dx \leqslant S_{p,q}(B_{r_*}(0)) \|w\|_{q+1(B_{r_*}(0))}^{q+1-p}.
$$

Noticing that $\|w\|_{q+1(B_{r_*}(0))} \geqslant 1$ and $q+1-p<0$, we obtain

$$
S_{p,q}(B_{r_*}(0)) < S_{p,q}(B_{r_*}(0)).
$$

This is a contradiction. Thus $z^*(0) \geqslant u^*(0)$.

Next, we prove Theorem 3.1 by contradiction. To this end, we first observe that by the assumption $M_* < M$ in Theorem 3.1, $u^*(M_*) > 0 = z^*(M_*)$ and (3.13) we can choose $s_0 \in (0, M_*)$ such that

$$
\begin{cases}
u^*(s) > z^*(s), & s \in (s_0, M_*), \\
u^*(s_0) = z^*(s_0).
\end{cases}
$$

In fact, $s_0$ can be defined as

$$
s_0 = \inf\left\{ s \,\middle|\, u^*(\tau) > z^*(\tau),\ \tau \in (s,\ M_*) \right\}
$$

Hence, to prove Theorem 3.1, what we want to do is to prove that $z^*(s) \geqslant u^*(s)$ for all $s \in [0, s_0]$. If this is not true, then from (3.13) we know that there exists at least one interval $I \subseteq [0,\ s_0]$ such that $u^*(s) > z^*(s)$ for $s \in I$. This and $z^*(0) \geqslant u^*(0)$ imply that we can choose an interval $[s_1, s_2] \subset [0, s_0]$ such that

$$
\begin{cases}
z^*(s) \geqslant u^*(s), & s \in [0,\ s_1], \\
u^*(s) > z^*(s), & s \in (s_1,\ s_2), \\
u^*(s_i) = z^*(s_i), & i = 1;2.
\end{cases}
\tag{3.15}
$$

Now, two cases $s_1 = 0$ and $s_1 > 0$ have to be considered.

If $s_1 = 0$, we let

$$w(s) = \begin{cases} u^*(s), & s \in [0,\ s_2); \\ z^*(s), & s \in [s_2,\ M_*]. \end{cases}$$

If $s_1 > 0$, we let

$$w(s) = \begin{cases} z^*(s), & s \in [0,\ s_1]; \\ u^*(s), & s \in (s_1,\ s_2); \\ z^*(s), & s \in [s_2,\ M_*]. \end{cases}$$

Then in any case, it is easy to prove that $w(s)$ satisfies (3.14). For the reader's convenience, we give a proof for the case $s_1 > 0$ as below.

At first, by the definition of $w(s)$, we can easily see that $w(s) > 0$ in $(0, M_*)$, $w(M_*) = 0$ and $\|w\|_{L^{q+1}(0,\ M_*)} \geqslant 1$. Hence, what we should do is to prove that $w(s)$ satisfies the differential inequality in (3.14). This can be done as the following.

If $s \in [0,\ s_1]$, then $w(s) = z^*(s)$. So, the differential inequality is automatically satisfied due to (3.12).

If $s \in [s_1,\ s_2]$, then $w(s) = u^*(s)$. Hence, by (3.4), (3.12) and (3.15), we have

$$\begin{aligned}
-\frac{dw(s)}{ds} &= -\frac{du^*(s)}{ds} \\
&\leqslant S_{p,q}^{\frac{1}{p-1}}(\Omega) n^{-\frac{p}{p-1}} \omega_n^{-\frac{p}{n(p-1)}} s^{-\frac{(n-1)p}{n(p-1)}} \left( \int_0^s u^*(t)^q dt \right)^{\frac{1}{p-1}} \\
&= S_{p,q}^{\frac{1}{p-1}}(\Omega) n^{-\frac{p}{p-1}} \omega_n^{-\frac{p}{n(p-1)}} s^{-\frac{(n-1)p}{n(p-1)}} \left( \int_0^{s_1} u^*(t)^q dt + \int_{s_1}^s u^*(t)^q dt \right)^{\frac{1}{p-1}} \\
&\leqslant S_{p,q}^{\frac{1}{p-1}}(\Omega) n^{-\frac{p}{p-1}} \omega_n^{-\frac{p}{n(p-1)}} s^{-\frac{(n-1)p}{n(p-1)}} \left( \int_0^{s_1} z^*(t)^q dt + \int_{s_1}^s u^*(t)^q dt \right)^{\frac{1}{p-1}} \\
&= S_{p,q}^{\frac{1}{p-1}}(\Omega) n^{-\frac{p}{p-1}} \omega_n^{-\frac{p}{n(p-1)}} s^{-\frac{(n-1)p}{n(p-1)}} \left( \int_0^s w^q(t) dt \right)^{\frac{1}{p-1}}.
\end{aligned}$$

Similarly, we can check the differential inequality in (3.14) for $w(s)$ when $s \in [s_2,\ M_*]$.

Finally, following the same argument as that used in the proof of $z^*(0) \geqslant u^*(0)$, we can get a contradiction. This completes the proof of Theorem 3.1. □

COROLLARY 3.1. *Let $u(x)$ be the minimizer of $S_{p,q}(\Omega)$ and $z(x)$ be the minimizer of $S_{p,q}(B_{r_*}(0))$. Then for any $k \geqslant q+1$, there holds*

$$\int_\Omega u^k dx \leqslant \int_{B_{r_*}(0)} z^k(x) dx.$$

*It follows that*

$$\sup_{x\in\Omega} u(x) \leqslant \sup_{x \in B_{r_*}(0)} z(x).$$

*Proof.* By the proposition of rearrangement, we have

$$\int_0^M (u^*)^{q+1}(s) ds = 1 = \int_0^{M_*} (z^*(s))^{q+1} ds.$$

Hence

$$\int_0^{M_*} (u^*)^{q+1}(s)ds \leqslant \int_0^{M_*} (z^*(s))^{q+1} ds. \tag{3.16}$$

Let $s_0$ be the point in $(0, M_*)$ determined in Theorem 3.1, then

$$\int_{s_0}^{M_*} (u^*)^{q+1}(s)ds - \int_{s_0}^{M_*} (z^*)^{q+1}(s)ds \leqslant \int_0^{s_0} \left((z^*)^{q+1} - (u^*)^{q+1}\right)(s)ds.$$

Since $u^*(s) \geqslant z^*(s)$ for any $s \in [s_0, M_*]$. It follows that for any $s \in [s_0, M_*]$, there holds

$$\int_{s_0}^{s} \left((u^*)^{q+1} - (z^*)^{q+1}\right)(s)ds \leqslant \int_0^{s_0} \left((z^*)^{q+1} - (u^*)^{q+1}\right)(s)ds.$$

Consequently,

$$\int_0^s (u^*)^{q+1}(\tau)d\tau \leqslant \int_0^s (z^*)^{q+1}(\tau)d\tau \quad \text{for any } s \in (0, M_*).$$

By the definition of $z^*(s)$, we have $z^*(s) = 0$ for $s \geqslant M_*$. Hence

$$\int_0^s (u^*)^{q+1}(\tau)d\tau \leqslant \int_0^s (z^*)^{q+1}(\tau)d\tau \quad \text{for any } s \in (0, M).$$

From this and Lemma 2.1, we have

$$\int_0^M (u^*)^k(s)ds \leqslant \int_0^{M_*} (z^*)^k(s)ds$$

for any $k \geqslant q+1$.

Noticing that

$$\int_\Omega u^k(x)dx = \int_0^M (u^*)^k(s)ds,$$

$$\int_{B_{r_*}(0)} z^k(x)dx = \int_0^{M_*} (z^*)^k(s)ds.$$

We obtain

$$\int_\Omega u^k(x)dx \leqslant \int_{B_{r_*}(0))} z^k(x)dx$$

for any $k \geqslant q+1$. When $k = +\infty$, then

$$\sup_{x\in\Omega} u(x) \leqslant \sup_{x\in B_{r_*}(0)} z(x).$$

This complete the proof of Corollary 3.1. □

## 4. Proofs of Theorem 1, Corollary 1 and Corollary 2

In this section, we prove Theorem 1, Corollary 1 and Corollary 2. For simplicity, we always use the notations $\sigma_{p,1}, \sigma_{p,2}$ and $\sigma_{p,3}$ introduced in Section 1 and Section 3 here.

First, we give the proof of Theorem 1.

*Proof.* Let $u(x)$ be the solution of problem (1.1) and $v(x) = \frac{u(x)}{\|u\|_{L^{q+1}(\Omega)}}$, then $v(x)$ satisfies

$$\begin{cases} -\operatorname{div}(|\nabla v(x)|^{p-2}\nabla v(x)) = \|u\|^{q+1-p}_{L^{q+1}(\Omega)} v^q(x), & x \in \Omega, \\ v(x) > 0, & x \in \Omega, \\ v(x) = 0, & x \in \partial\Omega, \\ \int_\Omega v^{q+1}(x)dx = 1. \end{cases}$$

Hence, by Lemma 3.1, we have $S_{p,q}(\Omega) = \|u\|^{q+1-p}_{L^{q+1}(\Omega)}$ and the minimizer of $S_{p,q}(\Omega)$ is $v(x)$.

Similarly, if $h(x)$ is the unique solution of problem (1.5). Then $S_{p,q}(\Omega^*) = \|h\|^{q+1-p}_{L^{q+1}(\Omega^*)}$ and the minimizer of $S_{p,q}(\Omega^*)$ is $\frac{h(x)}{\|h\|_{L^{q+1}(\Omega^*)}}$.

By the definition of $r_*$, we have $r_* = \left[\frac{\|h\|_{L^{q+1}(\Omega^*)}}{\|u\|_{L^{q+1}(\Omega)}}\right]^{(q+1-p)\sigma_{p,3}} R^*$. Moreover, by Lemma 3.3, we know that the minimizer of $S_{p,q}(B_{r_*}(0))$ is

$$z(x) = \left(\frac{R^*}{r_*}\right)^{\frac{n}{q+1}} \frac{h(\frac{R^*}{r_*}x)}{\|h\|_{L^{q+1}(\Omega^*)}}.$$

Applying Corollary 3.1 to $v(x)$ and $z(x)$, we have, for any $k \geqslant q+1$, that

$$\begin{aligned} \int_\Omega u^k(x)dx &\leqslant \frac{\|u\|^k_{L^{q+1}(\Omega)}}{\|h\|^k_{L^{q+1}(\Omega^*)}} \int_{B_{r_*}(0)} \left(\frac{R^*}{r_*}\right)^{\frac{nk}{q+1}} h^k(\frac{R^*}{r_*}x)dx \\ &= \frac{\|u\|^k_{L^{q+1}(\Omega)}}{\|h\|^k_{L^{q+1}(\Omega^*)}} \left(\frac{R^*}{r_*}\right)^{\frac{nk}{q+1}-n} \int_{\Omega^*} h^k(x)dx, \end{aligned}$$

with equality if and only if $\Omega$ is a ball.

Since

$$\left(\frac{R^*}{r_*}\right)^{\frac{nk}{q+1}-n} = \left(\frac{R^*}{r_*}\right)^{\frac{(k-q-1)n}{q+1}} = \left[\frac{\|h\|_{L^{q+1}(\Omega^*)}}{\|u\|_{L^{q+1}(\Omega)}}\right]^{-\frac{(k-q-1)(q+1-p)n}{np-n+p-(n-p)q}},$$

we have

$$\int_\Omega u^k(x)dx \leqslant \frac{\|u\|^k_{L^{q+1}(\Omega)}}{\|h\|^k_{L^{q+1}(\Omega^*)}} \left[\frac{\|h\|_{L^{q+1}(\Omega^*)}}{\|u\|_{L^{q+1}(\Omega)}}\right]^{-\frac{(k-q-1)(q+1-p)n}{np-n+p-(n-p)q}} \int_{\Omega^*} h^k(x)dx.$$

If we set

$$C(q,k,\Omega^*) = \int_{\Omega^*} h^k(x)dx \Big/ \|h\|_{L^{q+1}(\Omega^*)}^{\sigma_{p,1}},$$

then

$$\int_\Omega u^k(x)dx \leqslant C(q,k,\Omega^*)\|u\|_{L^{q+1}(\Omega)}^{\sigma_{p,1}}$$

and the equality holds if and only $\Omega$ is a ball.

If we set

$$C(q,\Omega^*) = \operatorname{ess.sup}_{x\Omega^*} h(x) \Big/ \|h\|_{L^{q+1}(\Omega^*)}^{\sigma_{p,2}},$$

then we can obtain

$$\operatorname{ess.sup}_{x\Omega} u(x) \leqslant C(q,\Omega^*) \cdot \|u\|_{L^{q+1}(\Omega)}^{\sigma_{p,2}}.$$

and the equality holds if and only if $\Omega$ is a ball. This complete the proof of Theorem 1. □

Then, we give the proof of Collorary 1.

*Proof.* Following the argument of theorem 1, we know that for any $k \geqslant q+1$,

$$\int_\Omega u^k(x)dx \leqslant \frac{\|u\|_{L^{q+1}(\Omega)}^k}{\|h\|_{L^{q+1}(\Omega^*)}^k} \left[\frac{\|h\|_{L^{q+1}(\Omega^*)}}{\|u\|_{L^{q+1}(\Omega)}}\right]^{-\frac{(k-q-1)(q+1-p)n}{np-n+p-(n-p)q}} \int_{\Omega^*} h^k(x)dx. \tag{4.1}$$

Since $S_{p,q}(\Omega) = \|u\|_{L^{q+1}(\Omega)}^{q+1-p}$ and $S_{p,q}(\Omega^*) = \|h\|_{L^{q+1}(\Omega^*)}^{q+1-p}$, we have

$$\int_\Omega u^k(x)dx \leqslant \frac{S_q^{\frac{k}{q+1-p}}(\Omega)}{S_q^{\frac{k}{q+1-p}}(\Omega^*)} \left[\frac{S_q^{\frac{1}{q+1-p}}(\Omega^*)}{S_q^{\frac{1}{q+1-p}}(\Omega)}\right]^{-\frac{(k-q-1)(q+1-p)n}{np-n+p-(n-p)q}} \int_{\Omega^*} h^k(x)dx. \tag{4.2}$$

Noting that $0 < q < p-1$, it follows from Lemma 3.2 that

$$\int_\Omega u^k(x)dx \leqslant \int_{\Omega^*} h^k(x)dx.$$

Consequently

$$\max_{x\in\Omega} u(x) \leqslant \max_{x\in\Omega^*} h(x).$$

If the equality holds, then it follows that $S_{p,q}(\Omega) = S_{p,q}(\Omega^*)$. Consequently, $\Omega$ is a ball due to Lemma 3.2. This completes the proof of Corollary 1. □

At last, we give the proof of Collorary 2.

*Proof.* As P-Laplace operator has no Green's representation formula in $\Omega^*$, we can not use the same method as [19] to prove the Corollary.

We will apply comparison principle to achieve our goal here.

Assume that $h(x)$ is the unique solution to (1.5). Let $\max_{x\in\Omega^*} h(x) = \beta$.

Suppose that $v(x)$ is the unique solution to

$$\begin{cases} -\operatorname{div}(|\nabla v(x)|^{p-2}\nabla v(x)) = \beta^q, & x \in \Omega^*, \\ v(x) > 0, & x \in \Omega^*, \\ v(x) = 0, & x \in \partial\Omega^*. \end{cases} \tag{4.3}$$

On one hand, easy computation implies that

$$v(x) = \frac{p-1}{p}\left(\frac{\beta^q}{n}\right)^{\frac{1}{p-1}}\left(R^{\frac{p-1}{p}} - |x|^{\frac{p-1}{p}}\right)$$

and

$$\max_{x\in\Omega^*} v(x) = \frac{p-1}{p}\left(\frac{\beta^q}{n}\right)^{\frac{1}{p-1}} R^{\frac{p-1}{p}} = \frac{p-1}{p}\left(\frac{\beta^q}{n}\right)^{\frac{1}{p-1}}\left(\frac{|\Omega|}{\omega_n}\right)^{\frac{p-1}{np}}.$$

On the other hand, $h(x) \leqslant \max_{x\in\Omega^*} h(x) = \beta$, $h^q(x) \leqslant (\max_{x\in\Omega^*} h(x))^q = \beta^q$. Thus

$$\begin{cases} -\operatorname{div}(|\nabla v(x)|^{p-2}\nabla v(x)) = \beta^q \geqslant h^q(x) = -\operatorname{div}(|\nabla h(x)|^{p-2}\nabla h(x)), & x \in \Omega^*, \\ v(x) > 0,\ h(x) > 0, & x \in \Omega^*, \\ v(x) = h(x) = 0, & x \in \partial\Omega^*. \end{cases}$$

By making use of Comparison principle we know that for any $x \in \Omega^*$, $v(x) \geqslant h(x)$. Consequently

$$\beta = \max_{x\in\Omega^*} h(x) \leqslant \max_{x\in\Omega^*} v(x) = \frac{p-1}{p}\left(\frac{\beta^q}{n}\right)^{\frac{1}{p-1}}\left(\frac{|\Omega|}{\omega_n}\right)^{\frac{p-1}{np}},$$

which implies that $\beta = \max_{x\in\Omega^*} h(x) \leqslant \left[\frac{|\Omega|}{\omega_n[n(\frac{p}{p-1})^{p-1}]^{\frac{n}{p}}}\right]^{\frac{p}{n(p-1-q)}}$.

Now, the conclusion of Corollary 2 follows from Corollary 1. This completes the proof of Corollary 2. □

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