A NEW MULTIPLE HALF–DISCRETE HILBERT–TYPE INEQUALITY

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(*Communicated by I. Perić*)

Abstract. By using the way of weight functions and technique of real analysis, a new multiple half-discrete Hilbert-type inequality with the best constant factor is given. As applications, the equivalent forms, operator expressions as well as some reverse inequalities are also considered.

1. Introduction

Suppose that a_m, $b_n \geqslant 0$, $a = \{a_m\}_{m=1}^{\infty}$, $b = \{b_n\}_{n=1}^{\infty} \in l^2$, $||a|| = \{\sum_{m=1}^{\infty} a_m^2\}^{\frac{1}{2}} > 0$, $||b|| > 0$, we have the following well known discrete Hilbert's inequality (cf. [1])

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \pi ||a|| \, ||b||, \tag{1}$$

where the constant factor π is the best possible. Moreover, for $f(x)$, $g(y) \geqslant 0$, $f, g \in L^2(\mathbf{R}_+), ||f|| = \{\int_0^{\infty} f^2(x)dx\}^{\frac{1}{2}} > 0, ||g|| > 0$, we still have the following Hilbert's integral inequality

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{x+y}dxdy < \pi ||f|| \, ||g||, \tag{2}$$

with the same best constant factor π. Inequalities (1) and (2) are important in analysis and its applications (cf. [2]–[8]). Hilbert's inequality may be classified into several types (discrete, integral, half-discrete etc.). On half-discrete Hilbert-type inequality, Yang [9] gave the following inequality with the best factor π recently:

$$\int_0^{\infty} f(x) \sum_{n=1}^{\infty} \frac{a_n}{x+n}dx < \pi ||f|| \, ||a||, \tag{3}$$

It is significant to generalize Hilbert-type inequalities into multiple forms. In recent years, some results have been obtained ([10]–[17]). In 2005, Yang [18] gave a multiple Hilbert-type integral inequality as follows: If $n \in \mathbf{N} \setminus \{1\}$, p_i, $r_i > 1$, $\sum_{i=1}^{n}(1/p_i) =$

$\sum_{i=1}^{n}(1/r_i) = 1$, $\lambda > 0$, $0 < \int_0^\infty x_i^{p_i(1-\lambda/r_i)-1} f_i^{p_i}(x_i)dx_i < \infty$ $(i = 1,\cdots,n)$, then

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{(\sum_{i=1}^n x_i)^\lambda} \prod_{i=1}^n f_i(x_i)dx_1 \cdots dx_n$$

$$< \frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{r_i}\right) \left\{ \int_0^\infty x^{p_i(1-\frac{\lambda}{r_i})-1} f_i^{p_i}(x)dx \right\}^{\frac{1}{p_i}}, \tag{4}$$

where the constant factor $\frac{1}{\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(\frac{\lambda}{r_i}\right)$ is the best possible.

Until now, we have obtained only few multiple Hilbert-type inequalities. It is not an easy job to find new forms of such inequalities. In this paper, by using the way of weight functions and the technique of real analysis, a new multiple half-discrete Hilbert-type inequality is given. As applications, the equivalent forms, operator expressions as well as some reverse inequalities are also considered.

2. Some Lemmas

LEMMA 1. (cf. [19], (9.1.1)) If $m \in \mathbf{N}$, $p_i \neq 0, 1$, $(i = 1, \cdots, m+1)$, $\sum_{i=1}^{m+1} \frac{1}{p_i} = 1$, then

$$A := \prod_{i=1}^{m+1} \left[x_i^{p_i-1} \prod_{j=1(j\neq i)}^{m+1} x_j^{-1} \right]^{\frac{1}{p_i}} = 1. \tag{5}$$

Proof. We find

$$A = \prod_{i=1}^{m+1} \left(x_i^{p_i-1+1} \prod_{j=1}^{m+1} x_j^{-1} \right)^{\frac{1}{p_i}} = \prod_{i=1}^{m+1} (x_i^{p_i})^{\frac{1}{p_i}} \left(\prod_{j=1}^{m+1} x_j^{-1} \right)^{\frac{1}{p_i}}$$

$$= \prod_{i=1}^{m+1} x_i \left(\prod_{j=1}^{m+1} x_j^{-1} \right)^{\sum_{i=1}^{m+1} \frac{1}{p_i}} = \prod_{i=1}^{m+1} x_i \prod_{j=1}^{m+1} x_j^{-1} = 1.$$

Hence we have (5). □

LEMMA 2. If $c > 0$, $n \in \mathbf{N}$, then

$$\int_0^c (\ln x)^n dx = c \sum_{i=0}^n (-1)^{n-i} \frac{n!}{i!} (\ln c)^i. \tag{6}$$

Proof. We prove (6) by mathematical induction. For $n = 1$,

$$\int_0^c \ln x dx = (x \ln x - x) |_{0+}^c = c(\ln c - 1).$$

Assume that (6) is valid for $n = k-1$, i.e.

$$\int_0^c (\ln x)^{k-1} dx = c \sum_{i=0}^{k-1} (-1)^{k-1-i} \frac{(k-1)!}{i!} (\ln c)^i.$$

Then for $n = k$, we have

$$\int_0^c (\ln x)^k\, dx = x\,(\ln x)^k\,|_{0^+}^c - k\int_0^c (\ln x)^{k-1}\, dx$$

$$= c\,(\ln c)^k - ck\sum_{i=0}^{k-1}(-1)^{k-1-i}\frac{(k-1)!}{i!}(\ln c)^i$$

$$= c\sum_{i=0}^k (-1)^{k-i}\frac{k!}{i!}(\ln c)^i.$$

Hence (6) is valid. □

LEMMA 3. *Suppose that* $s,t \in \mathbf{R}_+$, $k \in \mathbf{N}$, $a = \min\{s,1\}$, $b = \max\{s,1\}$, *then*

$$\int_0^\infty \frac{\min\{s,t,1\}}{\max\{s,t,1\}}\left(\ln\frac{\min\{s,t,1\}}{\max\{s,t,1\}}\right)^k t^{-1}dt$$

$$= \frac{a}{b}\left[-\left(\ln\frac{a}{b}\right)^{k+1} + 2\sum_{i=0}^k(-1)^{k-i}\frac{k!}{i!}\left(\ln\frac{a}{b}\right)^i\right]. \tag{7}$$

Proof. In view of (6), we have

$$\int_0^\infty \frac{\min\{s,t,1\}}{\max\{s,t,1\}}\left(\ln\frac{\min\{s,t,1\}}{\max\{s,t,1\}}\right)^k t^{-1}dt$$

$$= \int_0^{\min\{s,1\}} \frac{t}{\max\{s,1\}}\left(\ln\frac{t}{\max\{s,1\}}\right)^k t^{-1}dt$$

$$+ \int_{\min\{s,1\}}^{\max\{s,1\}} \frac{\min\{s,1\}}{\max\{s,1\}}\left(\ln\frac{\min\{s,1\}}{\max\{s,1\}}\right)^k t^{-1}dt$$

$$+ \int_{\max\{s,1\}}^\infty \frac{\min\{s,1\}}{t}\left(\ln\frac{\min\{s,1\}}{t}\right)^k t^{-1}dt$$

$$= \frac{1}{b}\int_0^a \left(\ln\frac{t}{b}\right)^k dt + \frac{a}{b}\left(\ln\frac{a}{b}\right)^k\int_a^b t^{-1}dt + a\int_b^\infty\left(\ln\frac{a}{t}\right)^k t^{-2}dt$$

$$= \frac{2}{b}\int_0^a \left(\ln\frac{t}{b}\right)^k dt - \frac{a}{b}\left(\ln\frac{a}{b}\right)^{k+1}$$

$$= 2\int_0^{a/b} (\ln x)^k\, dx - \frac{a}{b}\left(\ln\frac{a}{b}\right)^{k+1}$$

$$= \frac{a}{b}\left[-\left(\ln\frac{a}{b}\right)^{k+1} + 2\sum_{i=0}^k(-1)^{k-i}\frac{k!}{i!}\left(\ln\frac{a}{b}\right)^i\right].$$

Hence (7) is valid. □

LEMMA 4. *For* $m,n \in \mathbf{N}$, *define the weight functions* $\omega_i(x_i)$ $(i = 1,\cdots,m)$ *and*

$\omega_{m+1}(n)$ *as follows:*

$$\omega_i(x_i) := \sum_{n=1}^{\infty} n^{-1} \int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}}$$

$$\times \prod_{j=1(j\neq i)}^{m} x_j^{-1} dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_m, \qquad (8)$$

$$\omega_{m+1}(n) := \int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} \prod_{j=1}^{m} x_j^{-1} dx_1 \cdots dx_m. \qquad (9)$$

Then we have

$$\omega_{m+1}(n) = (m+1)!, \qquad (10)$$

$$0 < (m+1)!(1 - \theta_i(x_i)) < \omega_i(x_i) < (m+1)!, \qquad (11)$$

$$\theta_i(x_i) := \frac{1}{(m+1)!} \int_0^{1/x_i} u_{m+1}^{-1} \left[\int_0^{\infty} \cdots \int_0^{\infty} \frac{\min\{u_1, \cdots u_{i-1}, 1, u_{i+1}, \cdots, u_{m+1}\}}{\max\{u_1, \cdots u_{i-1}, 1, u_{i+1}, \cdots, u_{m+1}\}} \right.$$

$$\left. \times \prod_{j=1(j\neq i)}^{m} x_j^{-1} \, du_1 \cdots du_{i-1} du_{i+1} \cdots du_m \right] du_{m+1} > 0 \quad (i = 1, \cdots, m).$$

$$(12)$$

Proof. Setting $x_j = n u_j$ $(j = 1, \cdots, m)$ in the integral of (9), we find

$$\omega_{m+1}(n) = \int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{u_j, 1\}}{\max_{1 \leqslant j \leqslant m}\{u_j, 1\}} \prod_{j=1}^{m} u_j^{-1} du_1 \cdots du_m.$$

We prove (10) by mathematical induction. For $m = 1$,

$$\omega_2(n) = \int_0^{\infty} \frac{\min\{u_1, 1\}}{\max\{u_1, 1\}} u_1^{-1} du_1$$

$$= \int_0^1 du_1 + \int_1^{\infty} u_1^{-2} du_1 = 2 \int_0^1 du_1 = 2!.$$

Assuming that for $m = k - 1$ $(k \geqslant 3)$, we have

$$\omega_k(n) = \int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant k-1}\{u_j, 1\}}{\max_{1 \leqslant j \leqslant k-1}\{u_j, 1\}} \prod_{j=1}^{k-1} u_j^{-1} du_1 \cdots du_{k-1}$$

$$= \int_0^{\infty} \left(\int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant k-1}\{u_j, 1\}}{\max_{1 \leqslant j \leqslant k-1}\{u_j, 1\}} \prod_{j=2}^{k-1} u_j^{-1} du_2 \cdots du_{k-1} \right) u_1^{-1} du_1$$

$$= (k-1)! \int_0^{\infty} \frac{\min\{u_1, 1\}}{\max\{u_1, 1\}} \left[1 + \frac{(-1)^{k-2}}{(k-1)!} \left(\ln \frac{\min\{u_1, 1\}}{\max\{u_1, 1\}} \right)^{k-2} \right.$$

$$\left. + \sum_{i=0}^{k-4} \frac{(-1)^{i+1}}{i!} \left(\frac{1}{i+1} - \frac{1}{k-1} \right) \left(\ln \frac{\min\{u_1, 1\}}{\max\{u_1, 1\}} \right)^{i+1} \right] \frac{du_1}{u_1}$$

$$= 2(k-1)! \int_0^1 \left[1 + \frac{(-1)^{k-2}}{(k-1)!} (\ln x)^{k-2} + \sum_{i=0}^{k-4} \frac{(-1)^{i+1}}{i!} \left(\frac{1}{i+1} - \frac{1}{k-1} \right) (\ln x)^{i+1} \right] dx$$

$$= 2(k-1)! \left[1 + \frac{(k-2)!}{(k-1)!} + \sum_{i=0}^{k-4} \frac{(i+1)!}{i!} \left(\frac{1}{i+1} - \frac{1}{k-1} \right) \right] = k!,$$

then for $m = k$, in view of the assumption and (7), we have

$$\omega_{k+1}(n) = \int_0^\infty \cdots \int_0^\infty \frac{\min_{1 \leqslant j \leqslant k} \{u_j, 1\}}{\max_{1 \leqslant j \leqslant k} \{u_j, 1\}} \prod_{j=1}^k u_j^{-1} du_1 \cdots du_k$$

$$= \int_0^\infty \left(\int_0^\infty \cdots \int_0^\infty \frac{\min_{1 \leqslant j \leqslant k} \{u_j, 1\}}{\max_{1 \leqslant j \leqslant k} \{u_j, 1\}} \prod_{j=2}^k u_j^{-1} du_2 \cdots du_k \right) u_1^{-1} du_1$$

$$= \int_0^\infty \left[\int_0^\infty \left(\int_0^\infty \cdots \int_0^\infty \frac{\min_{1 \leqslant j \leqslant k} \{u_j, 1\}}{\max_{1 \leqslant j \leqslant k} \{u_j, 1\}} \prod_{j=3}^k u_j^{-1} du_3 \cdots du_k \right) \frac{du_2}{u_2} \right] \frac{du_1}{u_1}$$

$$= \int_0^\infty \left\{ \int_0^\infty (k-1)! \frac{\min\{u_1, u_2, 1\}}{\max\{u_1, u_2, 1\}} \left[1 + \frac{(-1)^{k-2}}{(k-1)!} \left(\ln \frac{\min\{u_1, u_2, 1\}}{\max\{u_1, u_2, 1\}} \right)^{k-2} \right. \right.$$

$$\left. \left. + \sum_{i=0}^{k-4} \frac{(-1)^{i+1}}{i!} \left(\frac{1}{i+1} - \frac{1}{k-1} \right) \left(\ln \frac{\min\{u_1, u_2, 1\}}{\max\{u_1, u_2, 1\}} \right)^{i+1} \right] \frac{du_2}{u_2} \right\} \frac{du_1}{u_1}$$

$$= (k-1)! \int_0^\infty \frac{\min\{u_1, 1\}}{\max\{u_1, 1\}} \left\{ \left(2 - \ln \frac{\min\{u_1, 1\}}{\max\{u_1, 1\}} \right) + \frac{(-1)^{k-2}}{(k-1)!} \right.$$

$$\times \left[-(\ln \frac{\min\{u_1, 1\}}{\max\{u_1, 1\}})^{k-1} + 2 \sum_{i=0}^{k-2} (-1)^{k-i} \frac{(k-2)!}{i!} (\ln \frac{\min\{u_1, 1\}}{\max\{u_1, 1\}})^i \right]$$

$$+ \sum_{i=0}^{k-4} \frac{(-1)^{i+1}}{i!} \left(\frac{1}{i+1} - \frac{1}{k-1} \right) \left[-\left(\ln \frac{\min\{u_1, 1\}}{\max\{u_1, 1\}} \right)^{i+2} \right.$$

$$\left. \left. + 2 \sum_{j=0}^{i+1} (-1)^{i+1-j} \frac{(i+1)!}{j!} \left(\ln \frac{\min\{u_1, 1\}}{\max\{u_1, 1\}} \right)^j \right] \right\} \frac{du_1}{u_1}$$

$$= 2(k-1)! \int_0^1 \left\{ (2 - \ln x) + \frac{(-1)^{k-2}}{(k-1)!} \left[-(\ln x)^{k-1} \right. \right.$$

$$\left. + 2 \sum_{i=0}^{k-2} (-1)^{k-i} \frac{(k-2)!}{i!} (\ln x)^i \right] + \sum_{i=0}^{k-4} \frac{(-1)^{i+1}}{i!} \left(\frac{1}{i+1} - \frac{1}{k-1} \right)$$

$$\times \left[-(\ln x)^{i+2} + 2 \sum_{j=0}^{i+1} (-1)^{i+1-j} \frac{(i+1)!}{j!} (\ln x)^j \right] \right\} dx$$

$$= 2(k-1)! \left\{ 3 + \frac{(-1)^{k-2}}{(k-1)!} \left[(-1)^k (k-1)! + 2 \sum_{i=0}^{k-2} (-1)^k (k-2)! \right] \right.$$

$$\left. + \sum_{i=0}^{k-4} \frac{(-1)^{i+1}}{i!} \left(\frac{1}{i+1} - \frac{1}{k-1} \right) \left[(-1)^{i+1} (i+2)! + 2 \sum_{j=0}^{i+1} (-1)^{i+1} (i+1)! \right] \right\}$$

$$= 2(k-1)! \left[6 + 3\sum_{i=0}^{k-4}(i+1)(i+2)\left(\frac{1}{i+1} - \frac{1}{k-1}\right) \right]$$
$$= (k+1)!.$$

and then (10) follows. $\square$

Since $\frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} \cdot n^{-1}$ is decreasing with respect to $n \in \mathbf{N}$, then for any $i = 1, \cdots, m$, it follows

$$\omega_i(x_i) \quad < \quad \int_0^\infty x_{m+1}^{-1} \int_0^\infty \cdots \int_0^\infty \frac{\min_{1 \leqslant j \leqslant m+1}\{x_j\}}{\max_{1 \leqslant j \leqslant m+1}\{x_j\}} dx_{m+1}$$
$$\times \prod_{j=1(j\neq i)}^m x_j^{-1} dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_m$$
$$\overset{u_j = x_j/x_i(j\neq i)}{=} \quad \omega_{m+1}(n) = (m+1)!.$$
$$\omega_i(x_i) \quad > \quad \int_1^\infty x_{m+1}^{-1} \left[\int_0^\infty \cdots \int_0^\infty \frac{\min_{1 \leqslant j \leqslant m+1}\{x_j\}}{\max_{1 \leqslant j \leqslant m+1}\{x_j\}} \right.$$
$$\left. \times \prod_{j=1(j\neq i)}^m x_j^{-1} dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_m \right] dx_{m+1}$$
$$\overset{u_j = x_j/x_i(j\neq i)}{=} \quad \int_{1/x_i}^\infty u_{m+1}^{-1} \left[\int_0^\infty \cdots \int_0^\infty \frac{\min\{u_1, \cdots u_{i-1}, 1, u_{i+1}, \cdots u_{m+1}\}}{\max\{u_1, \cdots u_{i-1}, 1, u_{i+1}, \cdots u_{m+1}\}} \right.$$
$$\left. \times \prod_{j=1(j\neq i)}^m u_j^{-1} du_1 \cdots du_{i-1} du_{i+1} \cdots du_m \right] du_{m+1}$$
$$= \quad (k+1)!(1 - \theta_i(x_i)) > 0,$$

where $\theta_i(x_i)$ is defined by (12). $\square$

LEMMA 5. *If* $m \in \mathbf{N}, p_i \neq 0, 1 (i = 1, \cdots, m+1), \frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i} = 1 - \frac{1}{p_{m+1}}$, *then for* $0 < \varepsilon < |p_{m+1}|$,

$$I(\varepsilon) := \varepsilon \sum_{n=1}^\infty n^{-\frac{\varepsilon}{p_{m+1}}-1} \int_1^\infty \cdots \int_1^\infty \frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} \prod_{j=1}^m x_j^{-\frac{\varepsilon}{p_j}-1} dx_1 \cdots dx_m$$
$$= (m+1)! + o(1)(\varepsilon \to 0^+). \tag{13}$$

Proof. Since $\frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} n^{-\frac{\varepsilon}{p_{m+1}}-1}$ is strictly decreasing with respect to $n \in \mathbf{N}$, then by the decreasing property and setting $u_j = x_j/x_{m+1}$ $(j = 1, \cdots, m)$, we find

$$I(\varepsilon) \geqslant \varepsilon \int_1^\infty x_{m+1}^{-\frac{\varepsilon}{p_{m+1}}-1} \int_1^\infty \cdots \int_1^\infty \frac{\min_{1 \leqslant j \leqslant m+1}\{x_j\}}{\max_{1 \leqslant j \leqslant m+1}\{x_j\}} \prod_{j=1}^m x_j^{-\frac{\varepsilon}{p_j}-1}$$
$$\times dx_1 \cdots dx_m dx_{m+1} = \varepsilon \int_1^\infty x_{m+1}^{-\varepsilon-1} \left[\int_{\frac{1}{x_{m+1}}}^\infty \cdots \int_{\frac{1}{x_{m+1}}}^\infty \frac{\min_{1 \leqslant j \leqslant m}\{u_j, 1\}}{\max_{1 \leqslant j \leqslant m}\{u_j, 1\}} \right.$$

$$\times \left[\int_{\frac{1}{x_{m+1}}}^{\infty} \cdots \int_{\frac{1}{x_{m+1}}}^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{u_j, 1\}}{\max_{1 \leqslant j \leqslant m}\{u_j, 1\}} \prod_{j=1}^{m} u_j^{-\frac{\varepsilon}{p_j}-1} du_1 \cdots du_m \right] dx_{m+1}$$

$$\geqslant \varepsilon \int_1^{\infty} x_{m+1}^{-\varepsilon-1} \left[\int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{u_j, 1\}}{\max_{1 \leqslant j \leqslant m}\{u_j, 1\}} \prod_{j=1}^{m} u_j^{-\frac{\varepsilon}{p_j}-1} du_1 \cdots du_m \right] dx_{m+1}$$

$$- \varepsilon \sum_{i=1}^{m} \int_1^{\infty} x_{m+1}^{-1} A_i(x_{m+1}) dx_{m+1}$$

$$= (m+1)! + o(1) - \varepsilon \sum_{i=1}^{m} \int_1^{\infty} x_{m+1}^{-1} A_i(x_{m+1}) dx_{m+1} (\varepsilon \to 0^+),$$

$$A_i(x_{m+1}) := \int_0^{\infty} \cdots \int_0^{\infty} \int_0^{1/x_{m+1}} \int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{u_j, 1\}}{\max_{1 \leqslant j \leqslant m}\{u_j, 1\}}$$

$$\prod_{j=1}^{m} u_j^{-\frac{\varepsilon}{p_j}-1} du_1 \cdots du_{i-1} du_i du_{i+1} \cdots dx_m.$$

Without loss of generality, we estimate $\int_1^{\infty} x_{m+1}^{-1} A_m(x_{m+1}) dx_{m+1}$ as follows:

$$\int_1^{\infty} x_{m+1}^{-1} A_m(x_{m+1}) dx_{m+1}$$

$$= \int_1^{\infty} x_{m+1}^{-1} \left[\int_0^{1/x_{m+1}} \int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{u_j, 1\}}{\max_{1 \leqslant j \leqslant m}\{u_j, 1\}} \right.$$

$$\left. \times \prod_{j=1}^{m} u_j^{-\frac{\varepsilon}{p_j}-1} du_1 \cdots du_{m-1} du_m \right] dx_{m+1}$$

$$= \int_0^1 \left(\int_1^{1/u_m} x_{m+1}^{-1} dx_{m+1} \right) \int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{u_j, 1\}}{\max_{1 \leqslant j \leqslant m}\{u_j, 1\}}$$

$$\times \prod_{j=1}^{m} u_j^{-\frac{\varepsilon}{p_j}-1} du_1 \cdots du_{m-1} du_m$$

$$= \int_0^1 (-\ln u_m) \int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{u_j, 1\}}{\max_{1 \leqslant j \leqslant m}\{u_j, 1\}}$$

$$\times \prod_{j=1}^{m} u_j^{-\frac{\varepsilon}{p_j}-1} du_1 \cdots du_{m-1} du_m.$$

Setting $\alpha > 0$ small enough, since $\lim_{u_m \to 0^+} u_m^{\alpha}(-\ln u_m) = 0$, there exists a constant $M_m > 0$, and $0 < u_m^{\alpha}(-\ln u_m) < M_m, u_m \in (0, 1]$. We obtain

$$0 < \int_1^{\infty} x_{m+1}^{-1} A_m(x_{m+1}) dx_{m+1}$$

$$\leqslant M_m \int_0^{\infty} \int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{u_j, 1\}}{\max_{1 \leqslant j \leqslant m}\{u_j, 1\}}$$

$$\times \prod_{j=1}^{m-1} u_j^{-\frac{\varepsilon}{p_j}-1} u_m^{-\frac{\alpha p_m + \varepsilon}{p_m}-1} du_1 \cdots du_{m-1} du_m$$

$$\to M_m O_m(\alpha) < \infty (\varepsilon \to 0^+),$$

namely, $\sum_{i=1}^{\infty} \int_1^{\infty} x_{m+1}^{-1} A_i(x_{m+1}) dx_{m+1} = O(1)$. Hence

$$I(\varepsilon) \geqslant (m+1)! + o(1) - \varepsilon O(1)(\varepsilon \to 0^+).$$

We still have

$$I(\varepsilon) \leqslant \varepsilon \sum_{n=1}^{\infty} n^{-\varepsilon-1} \left[n^{\varepsilon - \frac{\varepsilon}{p_{m+1}}} \int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} \prod_{j=1}^{m} x_j^{-\frac{\varepsilon}{p_j}-1} dx_1 \cdots dx_m dx_m \right]$$

$$= \varepsilon \left(1 + \sum_{n=2}^{\infty} n^{-\varepsilon-1} \right) [(m+1)! + o(1)]$$

$$\leqslant \varepsilon \left(1 + \int_1^{\infty} y^{-\varepsilon-1} dy \right) [(m+1)! + o(1)]$$

$$= (\varepsilon+1)[(m+1)! + o(1)](\varepsilon \to 0^+).$$

Hence (13) is valid. $\square$

LEMMA 6. *If* $m \in \mathbf{N}$, $p_i \neq 0, 1$ $(i = 1, \cdots, m+1)$, $\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i} = 1 - \frac{1}{p_{m+1}}$, $f_i(x_i)$ $(i = 1, \cdots, m)$ *are non-negative measurable functions in* $\mathbf{R}_+$, *then (i) for* $p_i > 1$ $(i = 1, \cdots, m+1)$, *we have the following inequality*

$$J := \left\{ \sum_{n=1}^{\infty} n^{-1} \left(\int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} \prod_{i=1}^{m} f_i(x_i) dx_1 \cdots dx_m \right)^p \right\}^{1/p}$$

$$\leqslant [(m+1)!]^{\frac{1}{p_{m+1}}} \prod_{i=1}^{m} \left\{ \int_0^{\infty} \omega_i(x_i) x_i^{p_i - 1} f_i(x_i) dx_i \right\}^{1/p_i}; \tag{14}$$

(ii) for $0 < p_1 < 1$, $p_i < 0$ $(i = 2, \cdots, m+1)$, *or* $p_i < 0$ $(i = 1, \cdots, m)$, $0 < p_{m+1} < 1$, *we have the reverse of (14).*

Proof. For $p_i > 1$ $(i = 1, \cdots, m+1)$ $(p > 1)$, by Hölder's inequality with weight (cf. [21]) and (6), we have

$$\left(\int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} \prod_{i=1}^{m} f_i(x_i) dx_1 \cdots dx_m \right)^p$$

$$= \left\{ \int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} \prod_{i=1}^{m} \left[x_i^{p_i - 1} n^{-1} \prod_{j=1(j \neq i)}^{m} x_j^{-1} \right]^{1/p_i} f_i(x_i) \right.$$

$$\left. \times \left[n^{p_{m+1} - 1} \prod_{j=1}^{m} x_j^{-1} \right]^{1/p_{m+1}} dx_1 \cdots dx_m \right\}^p$$

$$\leqslant \int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} \prod_{i=1}^{m} \left[x_i^{p_i - 1} n^{-1} \prod_{j=1(j \neq i)}^{m} x_j^{-1} \right]^{\frac{p}{p_i}} f_i^p(x_i) dx_1 \cdots dx_m$$

$$\times \left\{ n^{p_{m+1} - 1} \int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} \prod_{j=1}^{m} x_j^{-1} dx_1 \cdots dx_m \right\}^{\frac{p}{p_{m+1}}}$$

$$
= \frac{[\omega_{m+1}(n)]^{\frac{p}{p_{m+1}}}}{n^{-1}} \int_0^\infty \cdots \int_0^\infty \frac{\min_{1\leqslant j\leqslant m}\{x_j,n\}}{\max_{1\leqslant j\leqslant m}\{x_j,n\}} n^{-1}
$$
$$
\times \prod_{i=1}^m \left[x_i^{p_i-1} \prod_{j=1(j\neq i)}^m x_j^{-1} \right]^{\frac{p}{p_i}} f_i^p(x_i)dx_1\cdots dx_m.
$$

Since $\omega_{m+1}(n) = (m+1)!$, we obtain

$$
J \leqslant [(m+1)!]^{\frac{1}{p_{m+1}}} \left\{ \sum_{n=1}^\infty \int_0^\infty \cdots \int_0^\infty \frac{\min_{1\leqslant j\leqslant m}\{x_j,n\}}{\max_{1\leqslant j\leqslant m}\{x_j,n\}} n^{-1} \right.
$$
$$
\left. \times \prod_{i=1}^m \left[x_i^{p_i-1} \prod_{j=1(j\neq i)}^m x_j^{-1} \right]^{\frac{p}{p_i}} f_i^p(x_i)dx_1\cdots dx_m \right\}^{\frac{1}{p}}
$$
$$
= [(m+1)!]^{\frac{1}{p_{m+1}}} \left\{ \int_0^\infty \cdots \int_0^\infty \frac{\min_{1\leqslant j\leqslant m}\{x_j,n\}}{\max_{1\leqslant j\leqslant m}\{x_j,n\}} \sum_{n=1}^\infty n^{-1} \right.
$$
$$
\left. \times \prod_{i=1}^m \left[x_i^{p_i-1} \prod_{j=1(j\neq i)}^m x_j^{-1} \right]^{\frac{p}{p_i}} f_i^p(x_i)dx_1\cdots dx_m \right\}^{\frac{1}{p}}.
$$

Since $\sum_{n=1}^\infty \frac{1}{p_i/p} = 1$, applying Hölder's inequality, it follows

$$
J \leqslant [(m+1)!]^{\frac{1}{p_{m+1}}} \prod_{i=1}^m \left\{ \int_0^\infty \left[\sum_{n=1}^\infty n^{-1} \int_0^\infty \cdots \int_0^\infty \frac{\min_{1\leqslant j\leqslant m}\{x_j,n\}}{\max_{1\leqslant j\leqslant m}\{x_j,n\}} \right. \right.
$$
$$
\left. \left. \times \prod_{j=1(j\neq i)}^m x_j^{-1} dx_1\cdots dx_{i-1}dx_{i+1}\cdots dx_m \right] x_i^{p_i-1} f_i^p(x_i)dx_i \right\}^{\frac{1}{p_i}}
$$
$$
= [(m+1)!]^{\frac{1}{p_{m+1}}} \prod_{i=1}^m \left\{ \int_0^\infty \omega_i(x_i) x_i^{p_i-1} f_i^p(x_i)dx_i \right\}^{\frac{1}{p_i}},
$$

and (14) follows.

(ii) For $0 < p_1 < 1$, $p_i < 0$ $(i = 2,\cdots,m+1)$, or $p_i < 0$ $(i = 1,\cdots,m)$, $0 < p_{m+1} < 1$, in view of the assumptions and by the same way, we obtain the reverse of (14). $\square$

3. Main results and applications

In this section, for $p_i > 1$ $(i = 1,\cdots,m+1)$, we define the following functions and spaces:

$$
\varphi_i(x_i) := x_i^{p_i-1}(x_i \in \mathbf{R}_+), \quad \psi(n) := n^{p_{m+1}-1}(n \in \mathbf{N}),
$$
$$
L_{p,\varphi_i}(\mathbf{R}_+) := \left\{ f; ||f||_{p,\varphi_i} = \left\{ \int_0^\infty x_i^{p_i-1}|f(x_i)|^{p_i}dx \right\}^{\frac{1}{p_i}} < \infty \right\}, \quad (x_i \in \mathbf{R}_+, i = 1,\cdots,m),
$$

$$l_{p_{m+1},\psi} := \left\{ a = \{a_n\}_{n=1}^{\infty}; \ ||a||_{p_{m+1},\Psi} = \left\{ \sum_{n=1}^{\infty} n^{p_{m+1}-1}|a_n|^{p_{m+1}} \right\}^{\frac{1}{p_{m+1}}} < \infty \right\}.$$

THEOREM 1. *If* $m \in \mathbf{N}$, $p_i > 1$ $(i = 1, \cdots, m+1)$, $\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i} = 1 - \frac{1}{p_{m+1}}$, $f_i(x_i)$, $a_n \geq 0$, $f_i \in L_{p_i,\varphi_i}(\mathbf{R}_+)$ $(i = 1, \cdots, m)$, $a = \{a_n\}_{n=1}^{\infty} \in l_{p_{m+1},\psi}$, $||f_i||_{p_i,\varphi_i}$, $||a||_{p_{m+1},\Psi} > 0$, *then we have the following equivalent inequalities*

$$I := \sum_{n=1}^{\infty} a_n \int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leq j \leq m}\{x_j, n\}}{\max_{1 \leq j \leq m}\{x_j, n\}} \prod_{i=1}^{m} f_i(x_i) dx_1 \cdots dx_m$$

$$< (m+1)! ||a||_{p_{m+1},\psi} \prod_{i=1}^{m} ||f_i||_{p_i,\varphi_i}, \tag{15}$$

$$J = \left\{ \sum_{n=1}^{\infty} n^{-1} \left(\int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leq j \leq m}\{x_j, n\}}{\max_{1 \leq j \leq m}\{x_j, n\}} \prod_{i=1}^{m} f_i(x_i) dx_1 \cdots dx_m \right)^p \right\}^{1/p}$$

$$< (m+1)! \prod_{i=1}^{m} ||f_i||_{p_i,\varphi_i}, \tag{16}$$

where the constant factor $(m+1)!$ *in the above inequalities are the best possible.*

Proof. By (14), for $\omega_i(x_i) < (m+1)!$ and the assumptions of Theorem, we have (16).

By Hölder's inequality, we obtain

$$I = \sum_{n=1}^{\infty} \left[(\psi(n))^{\frac{-1}{p_{m+1}}} \int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leq j \leq m}\{x_j, n\}}{\max_{1 \leq j \leq m}\{x_j, n\}} \right.$$

$$\left. \times \prod_{i=1}^{m} f_i(x_i) dx_1 \cdots dx_m \right] \left[(\psi(n))^{\frac{1}{p_{m+1}}} a_n \right] \leq J ||a||_{p_{m+1},\psi}. \tag{17}$$

Then by (16), we have (15). Assuming (15) is valid, setting

$$a_n := n^{-1} \left(\int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leq j \leq m}\{x_j, n\}}{\max_{1 \leq j \leq m}\{x_j, n\}} \prod_{i=1}^{m} f_i(x_i) dx_1 \cdots dx_m \right)^{p-1}, \quad n \in \mathbf{N},$$

then we find $J^{p-1} = ||a||_{p_{m+1},\psi}$. By (14), we have $J < \infty$. If $J = 0$, then (16) is valid trivially; if $J > 0$, then by (15), it follows

$$||a||_{p_{m+1},\psi}^{p_{m+1}} = J^p = I < (m+1)! ||a||_{p_{m+1},\psi} \prod_{i=1}^{m} ||f_i||_{p_i,\varphi_i},$$

$$||a||_{p_{m+1},\psi}^{p_{m+1}-1} = J < (m+1)! \prod_{i=1}^{m} ||f_i||_{p_i,\varphi_i},$$

and (16) follows. Hence (15) and (16) are equivalent.

It remains to prove that the constants appearing in inequalities (15) and (16) are the best possible. For $\varepsilon > 0$, define $\tilde{f}_i(x_i)$ and $\tilde{a}_n$ as follows:

$$\tilde{f}_i(x_i) := \begin{cases} 0, & 0 < x_i < 1, \\ x_i^{-\frac{\varepsilon}{p_i}-1}, & x_i \geq 1 \end{cases} \quad (i = 1, \cdots, m),$$

$\tilde{a}_n := n^{-\frac{\varepsilon}{p_{m+1}}-1}$, $n \in \mathbf{N}$. If there exists a constant $K \leqslant (m+1)!$, such that (15) is still valid as we replace $(m+1)!$ by K, then by (13), we have

$$
\begin{aligned}
(m+1)! + o(1) &= I(\varepsilon) \\
&= \varepsilon \sum_{n=1}^{\infty} \tilde{a}_n \int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} \prod_{i=1}^{m} \tilde{f}_i(x_i) dx_1 \cdots dx_m \\
&< \varepsilon K \|\tilde{a}\|_{p_{m+1},\psi} \prod_{i=1}^{m} \|\tilde{f}_i\|_{p_i,\varphi_i} \\
&= \varepsilon K \left(1 + \sum_{n=2}^{\infty} \frac{1}{n^{1+\varepsilon}} \right)^{\frac{1}{p_{m+1}}} \prod_{i=1}^{m} \left(\frac{1}{\varepsilon} \right)^{\frac{1}{p_i}} \\
&< \varepsilon K \left(1 + \int_1^{\infty} \frac{dy}{y^{1+\varepsilon}} \right)^{\frac{1}{p_{m+1}}} \prod_{i=1}^{m} \left(\frac{1}{\varepsilon} \right)^{\frac{1}{p_i}} = K(1+\varepsilon)^{\frac{1}{p_{m+1}}},
\end{aligned}
$$

and then $(m+1)! \leqslant K(\varepsilon \to 0^+)$. Hence $(m+1)! = K$ is the best constant factor in (15).

By the equivalence, the constant factor $(m+1)!$ in (16) is still the best possible. Otherwise, we can come to a contradiction by (17) that the constant in (15) is not the best possible. $\square$

REMARK 1. With the assumptions of Theorem 1, we define a half-discrete Hilbert-type operator $T : \prod_{i=1}^{m} L_{p_i,\varphi_i}(\mathbf{R}_+) \to l_{p,\psi^{1-p}}$ as follows:

For any $f = (f_1, \cdots f_m) \in \prod_{i=1}^{m} L_{p_i,\varphi_i}(\mathbf{R}_+)$, we define a Tf, satisfying for any $n \in \mathbf{N}$,

$$
(Tf)(n) = \int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} \prod_{i=1}^{m} f_i(x_i) dx_1 \cdots dx_m. \tag{18}
$$

Then by (16), it follows

$$
\|Tf\|_{p,\psi^{1-p}} \leqslant (m+1)! \prod_{i=1}^{m} \|f_i\|_{p_i,\varphi_i},
$$

and consequently $Tf \in l_{p,\psi^{1-p}}$. Hence T is a bounded linear operator with $\|T\| \leqslant (m+1)!$. Since the constant factor in (16) is the best possible, we have

$$
\|T\| := \sup_{f(\neq \theta) \in \Pi_{i=1}^{m} L_{p_i,\varphi_i}(\mathbf{R}_+)} \frac{\|Tf\|_{p,\psi^{1-p}}}{\Pi_{i=1}^{m} \|f_i\|_{p_i,\varphi_i}} = (m+1)!. \tag{19}
$$

In order to obtain reverse inequalities that correspond to those in Theorem 1, we define the function $\Phi_1(x_1) := (1 - \theta_1(x_1))\varphi_1(x_1)$. For $p_i < 1$ $(p_i \neq 0)$, the spaces $L_{p_i,\varphi_i}(\mathbf{R}_+)$ and $l_{p_{m+1},\psi}$ are not normal spaces, but we still use then as the formal symbols in the following.

THEOREM 2. *Suppose that $m \in \mathbf{N}$, $0 < p_1 < 1$, $p_i < 0$ $(i = 2, \cdots, m+1)$, $\frac{1}{p} = \sum_{i=1}^m \frac{1}{p_i} = 1 - \frac{1}{p_{m+1}}$, $f_i(x_i)$, $a_n \geqslant 0$, $f_i \in L_{p_i, \varphi_i}(\mathbf{R}_+)$ $(i = 1, \cdots, m)$, $a = \{a_n\}_{n=1}^\infty \in l_{p_{m+1}, \psi}$, $||f_i||_{p_i, \varphi_i}$, $||a||_{p_{m+1}, \Psi} > 0$, then we have the following equivalent inequalities*

$$I = \sum_{n=1}^\infty a_n \int_0^\infty \cdots \int_0^\infty \frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} \prod_{i=1}^m f_i(x_i) dx_1 \cdots dx_m$$

$$> (m+1)! ||a||_{p_{m+1}, \psi} ||f_1||_{p_1, \Phi_1} \prod_{i=2}^m ||f_i||_{p_i, \varphi_i}, \tag{20}$$

$$J = \left\{ \sum_{n=1}^\infty n^{-1} \left(\int_0^\infty \cdots \int_0^\infty \frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} \prod_{i=1}^m f_i(x_i) dx_1 \cdots dx_m \right)^p \right\}^{1/p}$$

$$> (m+1)! ||f_1||_{p_1, \Phi_1} \prod_{i=2}^m ||f_i||_{p_i, \varphi_i}, \tag{21}$$

where the constant factor $(m+1)!$ in the above inequalities is the best possible.

Proof. By the reverse of (14), for $\omega_1(x_1) > (m+1)![1 - \theta_1(x_1)]$, $\omega_i(x_i) < (m+1)!$ and the assumptions of Theorem, we have (21). By Hölder's inequality, we obtain

$$I = \sum_{n=1}^\infty \left[(\psi(n))^{\frac{-1}{p_{m+1}}} \int_0^\infty \cdots \int_0^\infty \frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} \right.$$

$$\left. \times \prod_{i=1}^m f_i(x_i) dx_1 \cdots dx_m \right] \left[(\psi(n))^{\frac{1}{p_{m+1}}} a_n \right] \geqslant J ||a||_{p_{m+1}, \psi}. \tag{22}$$

Then by (21), we have (20). Assuming that (20) is valid, setting

$$a_n := n^{-1} \left(\int_0^\infty \cdots \int_0^\infty \frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} \prod_{i=1}^m f_i(x_i) dx_1 \cdots dx_m \right)^{p-1}, n \in \mathbf{N},$$

we find that $J^{p-1} = ||a||_{p_{m+1}, \psi}$. By the reverse of (14), we have $J > 0$. If $J = \infty$, then (21) is valid trivially; if $J < \infty$, then by (20), it follows

$$||a|_{p_{m+1}, \psi}^{p_{m+1}} = J^p = I > (m+1)! ||a||_{p_{m+1}, \psi} ||f_1||_{p_1, \Phi_1} \prod_{i=2}^m ||f_i||_{p_i, \varphi_i},$$

$$||a|_{p_{m+1}, \psi}^{p_{m+1}-1} = J > (m+1)! ||f_1||_{p_1, \Phi_1} \prod_{i=2}^m ||f_i||_{p_i, \varphi_i},$$

namely, (21) follows. Hence (20) and (21) are equivalent.

Now we prove that inequalities (20) and (21) include the best constants. For $0 < \varepsilon < |p_{m+1}|$, we set $\tilde{f}_i(x_i)$ and $\tilde{a}_n$ as Theorem 1. If there exists a constant $K \geqslant (m+1)!$, such that (20) is still valid as we replace $(m+1)!$ by K, then by (13), we have

$$(m+1)! + o(1) = I(\varepsilon)$$

$$= \varepsilon \sum_{n=1}^\infty \tilde{a}_n \int_0^\infty \cdots \int_0^\infty \frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} \prod_{i=1}^m \tilde{f}_i(x_i) dx_1 \cdots dx_m$$

$$> \varepsilon K ||\tilde{a}||_{p_{m+1}, \psi} |\tilde{f}_1||_{p_1, \Phi_1} \prod_{i=2}^m ||\tilde{f}_i||_{p_i, \varphi_i}.$$

For $0 < \alpha < 1$, define

$$F(u_2, \cdots, u_{m+1}) := u_{m+1}^{-\alpha} \frac{\frac{\min\{1, u_2, \cdots, u_{m+1}\}}{\max\{1, u_2, \cdots, u_{m+1}\}}}{\frac{\min\{1, u_2, \cdots, u_m\}}{\max\{1, u_2, \cdots, u_m\}}}$$

$$(u = (u_2, \cdots, u_{m+1}) \in \mathbf{R}_+^m).$$

Since we find

$$\lim_{u \to 0^+} F(u_2, \cdots, u_{m+1}) = \lim_{u \to \infty} F(u_2, \cdots, u_{m+1}) = 0,$$

there exists a constant $M_m > 0$, such that for any $(u_2, \cdots, u_{m+1}) \in \mathbf{R}_+^m$, $F(u_2, \cdots, u_{m+1}) \leqslant M_m$, namely,

$$\frac{\min\{1, u_2, \cdots, u_{m+1}\}}{\max\{1, u_2, \cdots, u_{m+1}\}} \leqslant M_m u_{m+1}^{\alpha} \frac{\min\{1, u_2, \cdots, u_m\}}{\max\{1, u_2, \cdots, u_m\}}.$$

Hence, in view of (12), it follows that

$$\begin{aligned}
\theta_1(x_1) &= \frac{1}{(m+1)!} \int_0^{1/x_1} u_{m+1}^{-1} \left[\int_0^\infty \cdots \int_0^\infty \frac{\min\{1, u_2, \cdots, u_{m+1}\}}{\max\{1, u_2, \cdots, u_{m+1}\}} \right. \\
&\quad \left. \times \prod_{j=2}^m x_j^{-1} du_2 \cdots, du_m \right] du_{m+1} \\
&\leqslant \frac{M_m}{(m+1)!} \int_0^{1/x_1} u_{m+1}^{\alpha-1} \left[\int_0^\infty \cdots \int_0^\infty \frac{\min\{1, u_2, \cdots, u_m\}}{\max\{1, u_2, \cdots, u_m\}} \right. \\
&\quad \left. \times \prod_{j=2}^m x_j^{-1} du_2 \cdots, du_m \right] du_{m+1} \\
&= \frac{M_m}{m+1} \int_0^{1/x_1} u_{m+1}^{\alpha-1} du_{m+1} = O\left(\frac{1}{x_1^\alpha}\right).
\end{aligned}$$

Hence we find

$$\begin{aligned}
&\varepsilon K ||\tilde{a}||_{p_{m+1}, \Psi} |\tilde{f}_1||_{p_1, \Phi_1} \prod_{i=2}^m ||\tilde{f}_i||_{p_i, \varphi_i} \\
&= \varepsilon K \left(\sum_{n=1}^\infty \frac{1}{n^{1+\varepsilon}} \right)^{\frac{1}{p_{m+1}}} \left[\int_1^\infty (1 - O\left(\frac{1}{x_1^\alpha}\right) \frac{dx_1}{x_1^{1+\varepsilon}} \right]^{\frac{1}{p_1}} \prod_{i=2}^m \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p_i}} \\
&> \varepsilon K \left(1 + \int_1^\infty \frac{dy}{y^{1+\varepsilon}} \right)^{\frac{1}{p_{m+1}}} \left[\frac{1}{\varepsilon} - O(1) \right]^{\frac{1}{p_1}} \prod_{i=2}^m \left(\frac{1}{\varepsilon}\right)^{\frac{1}{p_i}} \\
&= K(1+\varepsilon)^{\frac{1}{p_{m+1}}} [1 - \varepsilon O(1)]^{\frac{1}{p_1}},
\end{aligned}$$

and then $(m+1)! \geqslant K (\varepsilon \to 0^+)$. Hence $(m+1)! = K$ is the best constant factor in (20).

By the equivalence, the constant factor $(m+1)!$ in (21) is still the best possible. Otherwise, we can come to a contradiction by (22) that the constant in (20) is not the best possible. □

In the same way, we still have

THEOREM 3. *Assuming that* $m \in \mathbf{N}$, $p_i < 0$ $(i = 1, \cdots, m)$, $0 < p_{m+1} < 1$, $\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i} = 1 - \frac{1}{p_{m+1}}$, $f_i(x_i), a_n \geqslant 0$, $f_i \in L_{p_i, \varphi_i}$ $(\mathbf{R}_+)$ $(i = 1, \cdots, m)$, $a = \{a_n\}_{n=1}^{\infty} \in l_{p_{m+1}, \psi}$, $||f_i||_{p_i, \varphi_i}$, $||a||_{p_{m+1}, \psi} > 0$, then we have the following equivalent inequalities*

$$\sum_{n=1}^{\infty} a_n \int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} \prod_{i=1}^{m} f_i(x_i) dx_1 \cdots dx_m$$

$$> (m+1)! ||a||_{p_{m+1}, \psi} \prod_{i=1}^{m} ||f_i||_{p_i, \varphi_i}, \tag{23}$$

$$\left\{ \sum_{n=1}^{\infty} n^{-1} \left(\int_0^{\infty} \cdots \int_0^{\infty} \frac{\min_{1 \leqslant j \leqslant m}\{x_j, n\}}{\max_{1 \leqslant j \leqslant m}\{x_j, n\}} \prod_{i=1}^{m} f_i(x_i) dx_1 \cdots dx_m \right)^p \right\}^{1/p}$$

$$> (m+1)! \prod_{i=1}^{m} ||f_i||_{p_i, \varphi_i}, \tag{24}$$

where the constant factor $(m+1)!$ *in the above inequalities is the best possible.*

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