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MONOTONICITY THEOREMS AND INEQUALITIES FOR THE GAMMA FUNCTION

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Abstract. In this paper monotonicity results concerning the gamma function are deduced. These results lead to inequalities which improve some known bounds for the Γ function.

1. Introduction

In [7] the author proved the following double inequality:

$$\frac{x^2+1}{x+1} \leqslant \Gamma(x+1) \leqslant \frac{x^2+2}{x+2}, \; x \in [0,1]. \tag{1}$$

The authors of [8], [9] improved this inequality proving that the function

$$f_1 : (0,1) \to \mathbb{R}, \quad f_1(x) = \frac{\ln \Gamma(x+1)}{\ln \frac{x^2+1}{x+1}}$$

is strictly increasing. This result implies the inequality

$$\left(\frac{x^2+1}{x+1}\right)^{2(1-\gamma)} \leqslant \Gamma(x+1) \leqslant \left(\frac{x^2+1}{x+1}\right)^{\gamma}, \; x \in [0,1]. \tag{2}$$

According to the authors of the same work it can be proved that:

$$f_6 : (0,1) \to \mathbb{R}, \quad f_6(x) = \frac{\ln \Gamma(x+1)}{\ln \frac{x^2+6}{x+6}}$$

is strictly decreasing. Motivated by these results they posed among others the following open problems:

(1) Determine the largest $\lambda > 1$ so that the function

$$f_\lambda : (0,1) \to \mathbb{R}, \quad f_\lambda(x) = \frac{\ln \Gamma(x+1)}{\ln \frac{x^2+\lambda}{x+\lambda}}$$

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to be strictly increasing.

(2) Determine the smallest $\lambda < 6$ so that the function

$$f_\lambda : (0,1) \to \mathbb{R}, \ \ f_\lambda(x) = \frac{\ln\Gamma(x+1)}{\ln\frac{x^2+\lambda}{x+\lambda}}$$

to be strictly decreasing. In the following we shall solve these open problems.

2. Preliminaries

In order to prove our main result we need the following lemmas.

LEMMA 1. [3] *Let $h,k : [a,b] \to \mathbb{R}$ be two continuous functions which are differentiable on (a,b). Further let $k'(x) \neq 0$, $x \in (a,b)$. If h'/k' is strictly increasing (resp. decreasing) on (a,b), then the functions*

$$x \longmapsto \frac{h(x)-h(a)}{k(x)-k(a)} \qquad x \longmapsto \frac{h(x)-h(b)}{k(x)-k(b)}$$

are also strictly increasing (resp. decreasing) on (a,b).

LEMMA 2. *Let λ and λ' be two real numbers such that: $0 < \lambda < \lambda'$. The function $\phi : (0,1) \to \mathbb{R}$ defined by*

$$\phi_{\lambda',\lambda}(x) = \frac{\ln(x^2+\lambda') - \ln(x+\lambda')}{\ln(x^2+\lambda) - \ln(x+\lambda)}$$

is strictly increasing.

Proof. We use the previous lemma in our proof. Let h and k be defined by $h(x) = \ln(x^2+\lambda') - \ln(x+\lambda')$ and $k(x) = \ln(x^2+\lambda) - \ln(x+\lambda)$. We have

$$\phi_{\lambda',\lambda}(x) = \frac{h(x)-h(0)}{k(x)-k(0)} = \frac{h(x)-h(1)}{k(x)-k(1)}.$$

Let $x_1 = \sqrt{\lambda^2+\lambda} - \lambda$ be the root of $k'(x) = 0$, and let $x_2 = \sqrt{\lambda'^2+\lambda'} - \lambda'$ be the root of $h'(x) = 0$ in the interval $(0,1)$. We have $0 < x_1 < x_2 < 1$. We shall prove that the function

$$\phi^*_{\lambda',\lambda}(x) = \frac{h'(x)}{k'(x)}$$

is strictly increasing on the intervals $(0,x_1)$ and $(x_2,1)$. It is simple to verify that the functions

$$\phi_1 = \frac{x^2+2\lambda' x-\lambda'}{x^2+2\lambda x-\lambda}, \ \phi_2(x) = \frac{x^2+\lambda}{x^2+\lambda'}, \ \phi_3(x) = \frac{x+\lambda}{x+\lambda'}$$

are strictly increasing, positive functions on the intervals $(0,x_1)$ and $(x_2,1)$ and $\phi^*_{\lambda',\lambda}(x) = \phi_1(x)\phi_2(x)\phi_3(x)$. Thus $\phi^*_{\lambda',\lambda}$ is strictly increasing on the intervals $(0,x_1)$ and $(x_2,1)$, because it is the product of strictly increasing positive functions. Hence, according to Lemma 1, $\phi_{\lambda',\lambda}$ is also strictly increasing on $(0,x_1)$ and $(x_2,1)$. On the other hand

$$\phi'_{\lambda',\lambda}(x)=\frac{\frac{x^2+2\lambda' x-\lambda'}{(x^2+\lambda')(x+\lambda')}\ln\frac{x^2+\lambda}{x+\lambda}-\frac{x^2+2\lambda x-\lambda}{(x^2+\lambda)(x+\lambda)}\ln\frac{x^2+\lambda'}{x+\lambda'}}{\ln^2\frac{x^2+\lambda}{x+\lambda}}>0,\ \ x\in(x_1,x_2).$$

Consequently, the function $\phi_{\lambda',\lambda}$ is also strictly increasing on (x_1,x_2). Since $\phi_{\lambda',\lambda}$ is strictly increasing on the intervals $(0,x_1)$; (x_1,x_2); $(x_2,1)$ and is continuous on $(0,1)$ it follows that it is strictly increasing on $(0,1)$.

LEMMA 3. *The following inequalities hold:*

$$0\leqslant\sum_{n=p}^{\infty}\frac{x}{(n+x)n}<\frac{1}{p},\ \ x\in[0,1],$$

$$0<\sum_{n=p}^{\infty}\frac{1}{(n+x)^2}<\frac{1}{p-1}.\ \ x\in[0,1],$$

$$0<\sum_{n=p}^{\infty}\frac{1}{(n+x)^3}<\frac{1}{2p(p-1)},\ \ x\in[0,1].$$

Proof. The function $\varphi_1:[0,1]\to\mathbb{R},\ \varphi_1(x)=\sum_{n=p}^{\infty}\frac{x}{(n+x)n}$ is strictly increasing. Thus, $0=\varphi_1(0)\leqslant\varphi_1(x)\leqslant\varphi_1(1)=\frac{1}{p},\ \ x\in[0,1]$.

In order to prove the second inequality, we observe that:

$$0<\sum_{n=p}^{\infty}\frac{1}{(n+x)^2}\leqslant\sum_{n=p}^{\infty}\frac{1}{n^2}<\sum_{n=p}^{\infty}\frac{1}{n(n-1)}=\frac{1}{p-1},\ \ x\in[0,1].$$

The proof of the third inequality is analogous to the previous one:

$$0<\sum_{n=p}^{\infty}\frac{1}{(n+x)^3}\leqslant\sum_{n=p}^{\infty}\frac{1}{n^3}<\sum_{n=p}^{\infty}\frac{1}{n(n^2-1)}=\frac{1}{2p(p-1)},\ \ x\in[0,1].$$

3. Main Result

THEOREM 1. *Let be $\lambda\in(0,\infty)$ a positive real number. The function*

$$f_\lambda:(0,1)\to\mathbb{R},\ \ f_\lambda(x)=\frac{\ln\Gamma(x+1)}{\ln\frac{x^2+\lambda}{x+\lambda}}$$

is strictly increasing if and only if $\lambda\in(0,\lambda_0)$, where $\lambda_0=\frac{\gamma}{\frac{\pi^2}{6}-2\gamma}$.

Proof. If f_{λ_0} is strictly increasing on $(0,1)$, and $0<\lambda<\lambda_0$, then f_λ is also strictly increasing because $f_\lambda(x)=f_{\lambda_0}(x)\phi_{\lambda_0,\lambda}(x)$, namely f_λ can be written as the product of two strictly increasing positive functions.

Thus, in order to show that f_λ is a strictly increasing function on $(0,1)$ for every $\lambda\in(0,\lambda_0]$, we have to prove that f_{λ_0} is strictly increasing on $(0,1)$.

We use Lemma 1 again. This time let h be defined by $h(x)=\ln\Gamma(x+1)$ and let k be defined by $k(x)=\ln\frac{x^2+\lambda_0}{x+\lambda_0}$. Thus $f_\lambda(x)=\frac{h(x)-h(1)}{k(x)-k(1)}=\frac{h(x)-h(0)}{k(x)-k(0)}$. We shall apply Lemma 1 on the intervals $(0,x_3)$, and $(x_3,1)$, where $x_3=\sqrt{\lambda_0^2+\lambda_0}-\lambda_0\approx 0.4237...$ is the root of the equation: $x^2+2\lambda_0x-\lambda_0=0$. According to Lemma 1, the function f_{λ_0} is strictly increasing on these intervals, if

$$f^*_{\lambda_0}:(0,1)\to\mathbb{R},\ f^*_{\lambda_0}(x)=\frac{h'(x)}{k'(x)}=\frac{\left(-\gamma+\sum_{n=1}^{\infty}\frac{x}{(x+n)n}\right)(x^3+\lambda_0x^2+\lambda_0x+\lambda_0^2)}{x^2+2\lambda_0x-\lambda_0}$$

is a strictly increasing function on the same intervals.

We calculate ${f^*_{\lambda_0}}'$:

$${f^*_{\lambda_0}}'(x)=\frac{S_{\lambda_0}(x)}{(x^2+2\lambda_0x-\lambda_0)^2},\ x\in(0,1),$$

where

$$S_{\lambda_0}(x)=(x^3+\lambda_0x^2+\lambda_0x+\lambda_0^2)(x^2+2\lambda_0x-\lambda_0)\sum_{n=1}^{\infty}\frac{1}{(n+x)^2}+$$
$$(\gamma-\sum_{n=1}^{\infty}\frac{x}{(x+n)n})(2x+2\lambda_0)(x^3+\lambda_0x^2+\lambda_0x+\lambda_0^2)-$$
$$(\gamma-\sum_{n=1}^{\infty}\frac{x}{(x+n)n})(x^2+2\lambda_0x-\lambda_0)(3x^2+2\lambda_0x+\lambda_0).$$

In order to prove the theorem, we have to show that $S_{\lambda_0}(x)>0,\ x\in(0,1)$. To do this, we discuss different cases.

Case I: $x\in[0,\frac{1}{4}]$.
Since $S_{\lambda_0}(0)=\lambda_0^2[\gamma-\lambda_0(\frac{\pi^2}{6}-2\gamma)]=0$, in order to prove the inequality $S_{\lambda_0}(x)>0,\ x\in(0,\frac{1}{4}]$, we shall prove that $S'_{\lambda_0}(x)>0,\ x\in(0,\frac{1}{4})$. We have

$$S'_{\lambda_0}(x)=2(x^3+\lambda_0x^2+\lambda_0x+\lambda_0^2)(-x^2-2\lambda_0x+\lambda_0)\sum_{n=1}^{\infty}\frac{1}{(n+x)^3}-$$
$$2(-x^2-2\lambda_0x+\lambda_0)(3x^2+2\lambda_0x+\lambda_0)\sum_{n=1}^{\infty}\frac{1}{(n+x)^2}$$
$$+2(\gamma-\sum_{n=1}^{\infty}\frac{x}{(x+n)n})(-x^2-2\lambda_0x+\lambda_0)(3x+\lambda_0)$$
$$+2(\gamma-\sum_{n=1}^{\infty}\frac{x}{(x+n)n})(x^3+\lambda_0x^2+\lambda_0x+\lambda_0^2).$$

We introduce the notations: $u_1(x)=\sum_{n=1}^{\infty}\frac{1}{(n+x)^3}$, $u_2(x)=x^3+\lambda_0x^2+\lambda_0x+\lambda_0^2$, $u_3(x)=-x^2-2\lambda_0x+\lambda_0$, $u_4(x)=3x^2+2\lambda_0x+\lambda_0$, $u_5(x)=\sum_{n=1}^{\infty}\frac{1}{(n+x)^2}$, $u_6(x)=\gamma-\sum_{n=1}^{\infty}\frac{x}{(x+n)n}$, $u_7(x)=x+\lambda_0$, $u_8(x)=3x+\lambda_0$. The functions u_1,u_3,u_5,u_6 are strictly decreasing and u_2,u_4,u_7,u_8 are strictly increasing on $(0,1)$. Furthermore, all these functions are positive on $(0,\frac{1}{4})$.

On the other hand, we have

$$S'_{\lambda_0}(x)=2u_1(x)u_2(x)u_3(x)-2u_3(x)u_4(x)u_5(x)+2u_6(x)u_3(x)u_8(x)+2u_6(x)u_2(x).$$

Let $t_k,\ k=\overline{0,10}$ be defined by $t_k=\frac{k}{40}$.

If $x\in[t_{k-1},t_k]$ then the monotonicity of the functions $u_i,\ i=\overline{1,7}$ implies

$$S'_{\lambda_0}(x)\geqslant\alpha_k=2u_1(t_k)u_2(t_{k-1})u_3(t_k)-2u_3(t_{k-1})u_4(t_k)u_5(t_{k-1})+ \\ 2u_6(t_k)u_3(t_k)u_8(t_{k-1})+2u_6(t_k)u_2(t_{k-1}),\ k=\overline{1,10}.$$

Now if we verify the conditions

$$\alpha_k>0,\ k=\overline{1,10}, \tag{3}$$

then the inequality $S'_{\lambda_0}(x)>0,\ x\in(0,\frac{1}{4})$ follows. Since $S_{\lambda_0}(0)=0$ we obtain that $S_{\lambda_0}(x)>0,\ x\in(0,\frac{1}{4}]$.

Case II: $x\in[\frac{1}{4},\frac{42}{100}]$.

This time we shall prove directly that $S_{\lambda_0}(x)>0,\ x\in[\frac{1}{4},\frac{42}{100}]$ using the previous idea. Let $v_k,\ k=\overline{0,10}$ be defined by $v_k=\frac{1}{4}+\frac{k}{10}\left(\frac{42}{100}-\frac{1}{4}\right)$. The equality

$$S_{\lambda_0}(x)=2u_2(x)u_6(x)u_7(x)+u_3(x)u_4(x)u_6(x)-u_3(x)u_2(x)u_5(x) \tag{4}$$

and the monotonicity of the functions $u_i,\ i=\overline{1,7}$ imply the following:

$$S_{\lambda_0}(x)>\delta_k=2u_2(v_{k-1})u_6(v_k)u_7(v_{k-1})+u_3(v_k)u_4(v_{k-1})u_6(v_k)- \\ u_3(v_{k-1})u_2(v_k)u_5(v_{k-1}),\ x\in[v_{k-1},v_k].$$

Therefore, if

$$\delta_k>0,\ k=\overline{1,10}, \tag{5}$$

then: $S_{\lambda_0}(x)>0,\ x\in[\frac{1}{4},\frac{42}{100}]$.

Case III: $x\in[\frac{42}{100},x_3]$.

This case must be discussed separately, because the equation $u_3(x)=0$ has the root $x_3=\sqrt{\lambda_0^2+\lambda_0}-\lambda_0\approx 0.4237...$ and $u_3(x)$ changes its sign in this point. The monotonicity of the functions $u_i,\ i=\overline{1,7}$ and the equality (4) imply:

$$S_{\lambda_0}(x)>\eta=2u_6\Big(\frac{43}{100}\Big)u_7\Big(\frac{42}{100}\Big)u_2\Big(\frac{42}{100}\Big)-u_5\Big(\frac{42}{100}\Big)u_3\Big(\frac{42}{100}\Big)u_2\Big(\frac{43}{100}\Big), \\ x\in\Big[\frac{42}{100},x_3\Big].$$

Thus the inequality

$$\eta > 0 \tag{6}$$

implies that $S_{\lambda_0}(x) > 0,\ x \in [\frac{42}{100}, x_3]$.

Case IV: $x \in (x_3, x_4)$
(where $x_4 \approx 0.461\ldots$ is the root of $\Gamma'(x+1) = 0$ and x_3 is the root of $x^2 + 2\lambda_0 x - \lambda_0^2 = 0$). In this case we have: $-u_3(x) > 0,\quad u_6(x) > 0,\ x \in (x_3, x_4)$ and

$$S'_{\lambda_0}(x) = -2u_3(x)\big[-u_1(x)u_2(x) + u_4(x)u_5(x)\big] + 2u_6(x)\big[u_3(x)u_7(x) + u_2(x)\big].$$

It is easily seen that Lemma 3 implies that: $-u_1(x)u_2(x) + u_4(x)u_5(x) > 0,\ x \in (x_3, x_4)$ and $u_3(x)u_7(x) + u_2(x) > 0,\ x \in (x_3, x_4)$. Thus $S'_{\lambda_0}(x) > 0,\ x \in (x_3, x_4)$ and consequently $S_{\lambda_0}(x)$ is a strictly increasing function. This implies that $S_{\lambda_0}(x) > 0,\ x \in (x_3, x_4)$, provided that condition (6) holds.

Case V: $x \in [x_4, \frac{47}{100}]$.
We have $u_6(x_4) = 0$ where $x_4 \in (\frac{46}{100}, \frac{47}{100})$, and $u_6(x)$ changes its sign in x_4. The monotonicity of the functions $u_i,\ i = \overline{1,7}$ and the equality (4) imply

$$S_{\lambda_0}(x) > \tau = 2u_6(\frac{47}{100})u_7(\frac{47}{100})u_2(\frac{47}{100}) - u_5(\frac{47}{100})u_2(\frac{46}{100})u_3(\frac{46}{100}),\\ x \in [\frac{46}{100}, \frac{47}{100}].$$

If

$$\tau > 0 \tag{7}$$

holds, then $S_{\lambda_0}(x) > 0,\ x \in [x_4, \frac{47}{100}]$ follows.

Case VI: $x \in [\frac{47}{100}, 1]$.
We shall prove directly that $S_{\lambda_0}(x) > 0,\ x \in [\frac{47}{100}, 1]$. Let $v_k,\ k = \overline{0,10}$ be defined by $v_k = \frac{47}{100} + \frac{53k}{1000},\ k = \overline{0,10}$. If $x \in \big[v_{k-1}, v_k\big]$ then

$$S_{\lambda_0}(x) \geqslant \beta_k = 2u_2(v_k)u_6(v_k)u_7(v_k) + u_3(v_{k-1})u_4(v_{k-1})u_6(v_{k-1}) -\\ u_2(v_{k-1})u_3(v_{k-1})u_5(v_k),\ k = \overline{1,10}.$$

We have to verify the conditions

$$\beta_k > 0,\ k = \overline{1,10}, \tag{8}$$

whence the inequality $S_{\lambda_0}(x) > 0,\ x \in [\frac{47}{100}, 1]$ follows. Consequently, if we verify the conditions (3), (5), (6), (7), (8) (an operation easily done by using a computer program) then we obtain $S_{\lambda_0}(x) > 0,\ x \in (0,1)$ and ${f^*_{\lambda_0}}'(x) > 0,\ x \in (0, x_3) \cup (x_3, 1)$. This means that $f^*_{\lambda_0}(x)$ is strictly increasing on $(0, x_3)$ and on $(x_3, 1)$. According to Lemma 1, $f_{\lambda_0}(x)$ is also strictly increasing on these intervals, but $f_{\lambda_0}(x)$ is continuous on $(0,1)$ and consequently it follows that $f_{\lambda_0}(x)$ is strictly increasing on $(0,1)$.
If $\lambda > \lambda_0$ then $f_\lambda^{*\prime}(0) < 0$ and the continuity of $f_\lambda^{*\prime}(x)$ in zero implies that there is an $\varepsilon > 0$ such that $f_\lambda^{*\prime}(x) < 0,\ x \in (0, \varepsilon)$. Thus $f(x)$ will be strictly decreasing on $(0, \varepsilon)$. This means that λ_0 is the biggest value with the given property.

COROLLARY 1. *The monotonicity of* f_{λ_0} *and*

$$\lim_{x\to 0} f_\lambda(x) = (1-\gamma)(1+\lambda), \qquad \lim_{x\to 1} f_\lambda(x) = \gamma\lambda, \tag{9}$$

implies the following inequalities

$$\left(\frac{x^2+\lambda_0}{x+\lambda_0}\right)^{(1-\gamma)(1+\lambda_0)} \leqslant \Gamma(x+1) \leqslant \left(\frac{x^2+\lambda_0}{x+\lambda_0}\right)^{\gamma\lambda_0}, \quad x\in[0,1].$$

THEOREM 2. *The function*

$$f_\lambda : (0,1) \to (0,\infty), \;\; f_\lambda(x) = \frac{\ln\Gamma(x+1)}{\ln\frac{x^2+\lambda}{x+\lambda}}$$

is strictly decreasing if and only if $\lambda \in [\lambda_1,\infty)$, *where* $\lambda_1 = \frac{\frac{\pi^2}{6}-\gamma}{3-\frac{\pi^2}{6}-2\gamma}$.

Proof. According to Lemma 2, the inequality $\lambda_1 < \lambda$ implies that the function $\Psi_\lambda : (0,1) \to (0,\infty), \Psi_\lambda(x) = \frac{\log(x^2+\lambda_1)-\log(x+\lambda_1)}{\log(x^2+\lambda)-\log(x+\lambda)}$ is strictly decreasing. If $f_{\lambda_1} : (0,1) \to (0,\infty)$ is strictly decreasing, then the equality $f_\lambda(x) = f_{\lambda_1}(x)\Psi_\lambda(x)$ implies that $f_\lambda(x)$ is strictly decreasing too. We shall prove in the following that

$$f_{\lambda_1} : (0,1) \to (0,\infty), \;\; f_{\lambda_1}(x) = \frac{\ln\Gamma(x+1)}{\ln\frac{x^2+\lambda_1}{x+\lambda_1}}$$

is strictly decreasing. The idea of the proof is similar to the one used in the previous theorem. We use Lemma 1 again, this time on the intervals $(0,x_5)$ and $(x_5,1)$, where $x_5 = \sqrt{\lambda_1^2+\lambda_1} - \lambda_1 = 0.478...$ is the root of the equation $x^2+2\lambda_1 x - \lambda_1 = 0$ situated in $(0,1)$. We shall show that

$$f^*_{\lambda_1} : (0,1) \to \mathbb{R}, \; f^*_{\lambda_1}(x) = \frac{h'(x)}{k'(x)} = \frac{\left(-\gamma+\sum_{n=1}^{\infty}\frac{x}{(x+n)n}\right)(x^3+\lambda_1x^2+\lambda_1x+\lambda_1^2)}{x^2+2\lambda_1x-\lambda_1}$$

is a strictly decreasing function on the same intervals.
We have:

$$f^{*\prime}_{\lambda_1}(x) = \frac{S_{\lambda_1}(x)}{(x^2+2\lambda_1x-\lambda_1)^2}, \; x\in(0,1),$$

where

$$S_{\lambda_1}(x) = (x^3+\lambda_1x^2+\lambda_1x+\lambda_1^2)(x^2+2\lambda_1x-\lambda_1)\sum_{n=1}^{\infty}\frac{1}{(n+x)^2} + \\ (\gamma - \sum_{n=1}^{\infty}\frac{x}{(x+n)n})(2x+2\lambda_1)(x^3+\lambda_1x^2+\lambda_1x+\lambda_1^2) - \\ (\gamma - \sum_{n=1}^{\infty}\frac{x}{(x+n)n})(x^2+2\lambda_1x-\lambda_1)(3x^2+2\lambda_1x+\lambda_1).$$

In order to prove the theorem, we have to show that $S_{\lambda_1}(x) < 0,\ x \in (0,1)$. We shall achieve that in five steps.

Case I: $x \in [\frac{3}{4}, 1]$.
Since $S_{\lambda_1}(1) = (1+\lambda_1)^2[\frac{\pi^2}{6} - \gamma - \lambda_1(3 - \frac{\pi^2}{6} - 2\gamma)] = 0$, in order to prove the inequality $S_{\lambda_1}(x) < 0,\ x \in (\frac{3}{4}, 1)$, we will prove that $S'_{\lambda_1}(x) > 0,\ x \in (\frac{3}{4}, 1)$. We have

$$S'_{\lambda_1}(x) = 2(x^3+\lambda_1 x^2+\lambda_1 x+\lambda_1^2)(-x^2-2\lambda_1 x+\lambda_1)\sum_{n=1}^{\infty}\frac{1}{(n+x)^3} -$$
$$2(-x^2-2\lambda_1 x+\lambda_1)(3x^2+2\lambda_1 x+\lambda_1)\sum_{n=1}^{\infty}\frac{1}{(n+x)^2}$$
$$+2(\gamma - \sum_{n=1}^{\infty}\frac{x}{(x+n)n})(-x^2-2\lambda_1 x+\lambda_1)(3x+\lambda_1)$$
$$+2(\gamma - \sum_{n=1}^{\infty}\frac{x}{(x+n)n})(x^3+\lambda_1 x^2+\lambda_1 x+\lambda_1^2).$$

We use again the notations introduced in the proof of the previous theorem: $u_1(x) = \sum_{n=1}^{\infty}\frac{1}{(n+x)^3}$, $u_5(x) = \sum_{n=1}^{\infty}\frac{1}{(n+x)^2}$, $u_6(x) = \gamma - \sum_{n=1}^{\infty}\frac{x}{(x+n)n}$, and we introduce the notations: $u_{21}(x) = x^3+\lambda_1 x^2+\lambda_1 x+\lambda_1^2$, $u_{31}(x) = -x^2-2\lambda_1 x+\lambda_1$, $u_{41}(x) = 3x^2+2\lambda_1 x+\lambda_1$, $u_{71}(x) = x+\lambda_1$, $u_{81}(x) = 3x+\lambda_1$. The functions u_1, u_{31}, u_5, u_6 are strictly decreasing and $u_{21}, u_{41}, u_{71}, u_{81}$ are strictly increasing on $(0,1)$. Using the introduced notations $S'_{\lambda_1}(x)$ can be rewritten as follows:

$$S'_{\lambda_1}(x) = 2u_1(x)u_{21}(x)u_{31}(x) - 2u_{31}(x)u_{41}(x)u_5(x) + 2u_6(x)u_{31}(x)u_{81}(x) + 2u_6(x)u_{21}(x). \quad (10)$$

Let $x_k,\ k=\overline{1,10}$ be defined by $x_k = \frac{3}{4} + \frac{k}{10}(1-\frac{3}{4}),\ k=\overline{0,10}$. If $x \in [x_{k-1}, x_k]$, then the monotonicity of the functions u_j imply:

$$S'_{\lambda_1}(x) > \alpha'_k = 2u_1(x_{k-1})u_{21}(x_k)u_{31}(x_k) - 2u_{31}(x_{k-1})u_{41}(x_{k-1})u_5(x_k) + 2u_6(x_{k-1})u_{31}(x_{k-1})u_{81}(x_{k-1}) + 2u_6(x_k)u_{21}(x_k).$$

Now if the conditions

$$\alpha'_k > 0, \quad k=\overline{1,10} \quad (11)$$

hold, then the inequality $S'_{\lambda_1}(x) > 0,\ x \in (\frac{3}{4}, 1)$ follows, and this together with $S_{\lambda_1}(1) = 0$ imply that $S_{\lambda_1}(x) < 0,\ x \in [\frac{3}{4}, 1)$.

Case II: $x \in [0.48, 0.75]$.
Let $y_k,\ k=\overline{0,10}$ be defined by $y_k = 0.48 + \frac{k}{10}(0.75-0.48),\ k=\overline{0,10}$. If $x \in [y_{k-1}, y_k]$ then the equality $S_{\lambda_1}(x) = 2u_{21}(x)u_6(x)u_{71}(x) + u_{31}(x)u_{41}(x)u_6(x) - u_{31}(x)u_{21}(x)u_5(x)$ and the monotonicity of u_j imply that

$$S_{\lambda_1}(x) < \beta'_k = 2u_{21}(y_{k-1})u_6(y_{k-1})u_{71}(y_{k-1}) + u_{31}(y_k)u_{41}(y_k)u_6(y_k) - u_{31}(y_k)u_{21}(y_k)u_5(y_{k-1}).$$

Consequently, in order to prove the inequality $S_{\lambda_1}(x) < 0,\ x \in [0.48, \frac{3}{4}]$, we have to check that:

$$\beta_k' < 0,\ k = \overline{0,10}. \tag{12}$$

Case III: $x \in [0.47, 0.48]$.
The equation $x^2 + 2\lambda_1 x - \lambda_1 = 0$ has the root $x_5 = 0.478...$ in $(0.47, 0.48)$, and u_{31} changes its sign. If $x \in [x_5, 0.48]$, then

$$S_{\lambda_1}(x) < \gamma' = 2u_{21}(0.47)u_6(0.47)u_{71}(0.47) + u_{31}(0.48)u_{41}(0.48)u_6(0.48) - u_{31}(0.48)u_{21}(0.48)u_5(0.47).$$

If $x \in [0.47, x_5]$, then $S_{\lambda_1}(x) < 0$. because every term of $S_{\lambda_1}(x)$ is negative. In order to prove the inequality $S_{\lambda_1}(x) < 0,\ x \in [0.47, 0.48]$, we have to verify

$$\gamma' \leqslant 0. \tag{13}$$

Case IV: $x \in [0.46, 0.47]$.
This time the equation $u_6(x) = 0$ has the root $x_6 = 0.461... \in (0.46, 0.47)$. If $x \in [x_6, 0.47]$, then $S_{\lambda_1}(x) < 0$ because every term of $S_{\lambda_1}(x)$ is negative.
If $x \in [0.46, x_6]$, then:

$$S_{\lambda_1}(x) < \delta' = 2u_{21}(0.47)u_6(0.46)u_{71}(0.47) + u_{31}(0.46)u_{41}(0.47)u_6(0.46) - u_{31}(0.47)u_{21}(0.46)u_5(0.47).$$

In order to finish the proof of this case, we have to verify that:

$$\delta' < 0. \tag{14}$$

Case V: $x \in [0, 0.46]$.
Let t_k be defined by $t_k = \frac{0.46}{k},\ k = \overline{0,100}$. If $x \in [t_{k-1}, t_k]$, then

$$S_{\lambda_1}(x) < \varepsilon_k' = 2u_{21}(t_k)u_6(t_{k-1})u_{71}(t_k) + u_{31}(t_{k-1})u_{41}(t_k)u_6(t_{k-1}) - u_{31}(t_k)u_{21}(t_{k-1})u_5(t_k).$$

If the inequalities

$$\varepsilon_k' < 0,\ \ k = \overline{1,100} \tag{15}$$

hold, then $S_{\lambda_1}(x) < 0,\ \ x \in [0, 0.46]$ follows. The conditions (11), (12), (13), (14), (15) can be easily verified with a computer program. If $\lambda < \lambda_1$ then $f_\lambda^{*\prime}(1) > 0$ and the continuity of $f_\lambda^{*\prime}(x)$ in $x_0 = 1$ implies that there is an $\varepsilon > 0$ such that $f_\lambda^{*\prime}(x) > 0,\ x \in (1-\varepsilon, 1)$. Thus $f(x)$ will be strictly increasing on $(1-\varepsilon, 1)$. This means that λ_1 is the smallest value with the given property.

COROLLARY 2. *The monotonicity of f_{λ_1} and (9) implies the following inequalities:*

$$\left(\frac{x^2+\lambda_1}{x+\lambda_1}\right)^{\gamma\lambda_1} \leqslant \Gamma(x+1) \leqslant \left(\frac{x^2+\lambda_1}{x+\lambda_1}\right)^{(1-\gamma)(1+\lambda_1)},\ \ x \in [0,1].$$

REMARK 1. We used the estimations proved in Lemma 3 in order to approximate the functions $u_1(x)$, $u_5(x)$, $u_6(x)$ in the proofs of Theorem 1 and Theorem 2.

REMARK 2. In the following we shall compare the inequalities obtained from our monotonicity results with some inequalities published earlier concerning the Γ function. In [1], page 145, it was proved that

$$x^{\alpha(x-1)-\gamma} < \Gamma(x) < x^{\beta(x-1)-\gamma}, \quad x \in (0,1), \tag{16}$$

with the best possible constants $\alpha = 1-\gamma$ and $\beta = \frac{1}{2}\left(\frac{\pi^2}{6}-\gamma\right)$.
In [2] p. 780, the following inequality was established: if $x \in (0,\infty)$ then

$$\sqrt{2\pi}x^x \exp\left[-x-\frac{1}{2}\psi(x+\alpha)\right] < \Gamma(x) < \sqrt{2\pi}x^x \exp\left[-x-\frac{1}{2}\psi(x+\beta)\right] \tag{17}$$

with the best possible constants $\alpha = \frac{1}{3}$ and $\beta = 0$. Another inequality which we are interested in, was published in [6], page 3:

$$\frac{x^{x[1-\ln x+\psi(x)]}}{e^x} < \Gamma(x) < \frac{x^{x[1-\ln x+\psi(x)]}}{e^{x-1}}, \quad x \in (0,1]. \tag{18}$$

In [5] the following inequalities were published: if $x > 0$, then

$$\sqrt{2}\left(x+\frac{1}{2}\right)^{x+\frac{1}{2}} e^{-x} \leqslant \Gamma(x+1) < e^{\frac{\gamma}{e^\gamma}}\left(x+\frac{1}{e^\gamma}\right)^{x+\frac{1}{e^\gamma}} e^{-x}, \tag{19}$$

$$\sqrt{2e}\left(\frac{x+1/2}{e}\right)^{x+\frac{1}{2}} \leqslant \Gamma(x+1) < \sqrt{2\pi}\left(\frac{x+1/2}{e}\right)^{x+\frac{1}{2}}, \tag{20}$$

$$\sqrt{2x+1}x^x \exp\left[-\left(x+\frac{1}{6x+9/8}-\frac{4}{9}\right)\right] < \Gamma(x+1)$$
$$< \sqrt{\pi(2x+1)}x^x \exp\left[-\left(x+\frac{1}{6x+9/8}\right)\right]. \tag{21}$$

It is obvious that Corollary 1 improves (2). Numerical approaches show that:
1. the second inequality of Corollary 2 improves the second inequality of (16),
2. the first inequality of Corollary 1 improves the first inequality of (18),
3. Corollary 2 improves (18),
4. the first inequality of Corollary 2 improves the first inequality of (19),
5. the second inequality of Corollary 1 improves the second inequality of (20),
6. Corollary 2 improves (20),
7. the first inequality of Corollary 1 improves the first inequality of (21),
8. the first inequality of Corollary 2 improves the first inequality of (21).

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