

REVERSE HILBERT'S TYPE INTEGRAL INEQUALITIES

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(Communicated by J. Pečarić)

Abstract. In this paper, we are motivated by some newer Hilbert-Pachpatte inequalities, and that we derive some similar (but reverse) inequalities.

1. Introduction

The well-known Hilbert's integral inequality can be stated as follows ([1], p. 226).

THEOREM A. *Let $f(x), g(y) \geq 0$, $0 < \int_0^\infty f(x)^p dx < \infty$ and $0 < \int_0^\infty g(y)^q dy < \infty$. If $p > 1$ and $q = p/(p-1)$, then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \leq \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f(x)^p dx \right)^{1/p} \left(\int_0^\infty g(y)^q dy \right)^{1/q}.$$

Hilbert's integral inequalities were studied extensively and numerous variants, generalizations, and extensions appeared in the literature [2–14] and the references cited therein. Research on reverse Hilbert inequalities was published in [15–17]. In particular, Pachpatte [18] established a new integral inequality similar to the Hilbert's inequality as follows.

THEOREM B. *Let $h \geq 1$, $l \geq 1$ and $f(\sigma) \geq 0$, $g(\tau) \geq 0$ for $\sigma \in (0, x)$, $\tau \in (0, y)$, where x and y are positive real numbers and define $F(s) = \int_0^s f(\sigma) d\sigma$ and $G(t) = \int_0^t g(\tau) d\tau$, for $s \in (0, x)$, $t \in (0, y)$. Then*

$$\begin{aligned} \int_0^x \int_0^y \frac{F^h(s)G^l(t)}{s+t} ds dt &\leq \frac{1}{2} hl(xy)^{1/2} \left(\int_0^x (x-s) \left(F^{h-1}(s)f(s) \right)^2 ds \right)^{1/2} \\ &\quad \times \left(\int_0^y (y-t) \left(G^{l-1}(t)g(t) \right)^2 dt \right)^{1/2}. \end{aligned} \quad (1.1)$$

The first aim of the paper to establish an inequality similar to (1.1), which is a reverse inequality.

Mathematics subject classification (2010): 26D15.

Keywords and phrases: Hilbert's inequality, Pólya-Szegő's integral inequality, reverse Jensen's inequality.

Research is supported by National Natural Science Foundation of China (11371334).

Research is partially supported by a HKU Seed Grant for Basic Research.

THEOREM 1.1. For $i = 1, 2$, let $h_i \geq 1$ and $0 < r_i \leq f_i(\sigma_i) \leq R_i$ for $\sigma_i \in (0, x_i)$, where x_i are positive real numbers and define $F_i(s_i) = \int_0^{s_i} f_i(\sigma_i) d\sigma_i$, for $s_i \in (0, x_i)$. Then

$$\begin{aligned} & \int_0^{x_1} \int_0^{x_2} \frac{F_1^{h_1}(s_1) + F_2^{h_2}(s_2)}{\min\{s_1^{1/2}, s_2^{1/2}\}} ds_1 ds_2 \\ & \geq M \cdot \frac{1}{2} (h_1 h_2)^{1/2} (x_1 x_2)^{3/4} \left(\int_0^{x_1} (x_1 - s_1) \left(F_1^{h_1-1}(s_1) f_1(s_1) \right)^2 ds_1 \right)^{1/4} \\ & \quad \times \left(\int_0^{x_2} (x_2 - s_2) \left(F_2^{h_2-1}(s_2) f_2(s_2) \right)^2 ds_2 \right)^{1/4}, \end{aligned} \tag{1.2}$$

where

$$M = 16 \min \left\{ \left(\frac{R_1^{h_1/2}}{r_1^{h_1/2}} + \frac{r_1^{h_1/2}}{R_1^{h_1/2}} \right)^{-2}, \left(\frac{R_2^{h_2/2}}{r_2^{h_2/2}} + \frac{r_2^{h_2/2}}{R_2^{h_2/2}} \right)^{-2} \right\}.$$

REMARK 1.1. The reverse inequality in Theorem 1.1 is achieved. Moreover, the molecule of the integrand of the left side of inequality (1.2) becomes $F_1^{h_1}(s_1) + F_2^{h_2}(s_2)$, but inequality (1.1) in Theorem B is $F_1^{h_1}(s_1)F_2^{h_2}(s_2)$. This shows that we get a different precision result.

In [18], Pachpatte also established the following Hilbert’s type integral inequality for the sub-multiplicative functions. For a reader’s convenience, we include the definition of the sub-multiplicative function as follows:

A function $f(x)$ ($x \in [a, b]$) is called the sub-multiplicative function, if $f(x_1 x_2) \leq f(x_1)f(x_2)$ for any $x_1, x_2 \in [a, b]$.

THEOREM C. Let f, g, F, G be as in Theorem B. Let $p(\sigma)$ and $q(\tau)$ be two positive real function define for $\sigma \in (0, x)$, $\tau \in (0, y)$ and define $P(s) = \int_0^s p(\sigma) d\sigma$ and $Q(t) = \int_0^t q(\tau) d\tau$, for $s \in (0, x)$, $t \in (0, y)$, where x, y are positive real numbers. Let ϕ and ψ be two real-valued, nonnegative, convex, and sub-multiplicative functions defined on $R^+ = [0, \infty)$. Then

$$\begin{aligned} & \int_0^x \int_0^y \frac{\phi(F(s))\psi(G(t))}{s+t} ds dt \leq L(x, y) \left(\int_0^x (x-s) \left(p(s)\phi\left(\frac{f(s)}{p(s)}\right) \right)^2 ds \right)^{1/2} \\ & \quad \times \left(\int_0^y (y-t) \left(q(t)\psi\left(\frac{g(t)}{q(t)}\right) \right)^2 dt \right)^{1/2}, \end{aligned} \tag{1.3}$$

where

$$L(x, y) = \frac{1}{2} \left(\int_0^x \left(\frac{\phi(P(s))}{P(s)} \right)^2 ds \right)^{1/2} \left(\int_0^y \left(\frac{\psi(Q(t))}{Q(t)} \right)^2 dt \right)^{1/2}.$$

Another aim of this paper is to establish an inequality (but super-multiplicative functions) similar to (1.3), which is a reverse inequality. Here, we include the definition of the super-multiplicative function as follows:

A function $f(x)$ ($x \in [a, b]$) is called the super-multiplicative function, if $f(x_1x_2) \geq f(x_1)f(x_2)$ for any $x_1, x_2 \in [a, b]$.

THEOREM 1.2. For $i = 1, 2$, let $f_i(\sigma_i) \geq 0$ for $\sigma_i \in (0, x_i)$, where x_i are positive real numbers and define $F_i(s_i) = \int_0^{s_i} f_i(\sigma_i) d\sigma_i$, for $s_i \in (0, x_i)$. Let $0 < r_i \leq p_i(\sigma_i) \leq R_i$ and define $P_i(s_i) = \int_0^{s_i} p_i(\sigma_i) d\sigma_i$. If ϕ_i are real-valued nonnegative, convex, and super-multiplicative functions defined on $[0, b]$ and $0 \leq t_i \leq \phi_i \leq T_i$, then

$$\begin{aligned} & \int_0^{x_1} \int_0^{x_2} \frac{\phi_1(F_1(s_1)) + \phi_2(F_2(s_2))}{\min\{s_1^{1/2}, s_2^{1/2}\}} ds_1 ds_2 \\ & \geq N \cdot L(x_1, x_2) \left(\int_0^{x_1} (x_1 - s_1) \left(p_1(s_1) \phi_1 \left(\frac{f_1(s_1)}{p_1(s_1)} \right) \right)^2 ds_1 \right)^{1/4} \\ & \quad \times \left(\int_0^{x_2} (x_2 - s_2) \left(p_2(s_2) \phi_2 \left(\frac{f_2(s_2)}{p_2(s_2)} \right) \right)^2 ds_2 \right)^{1/4}, \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} N = 4 \min & \left\{ \frac{\phi_1(nb)}{m\phi_1(0) + n\phi_1(b)} \left(\frac{(R_1T_1)^{1/2}}{(r_1t_1)^{1/2}} + \frac{(r_1t_1)^{1/2}}{(R_1T_1)^{1/2}} \right)^{-1} \left(\frac{R_1T_1}{r_1t_1} + \frac{r_1t_1}{R_1T_1} \right)^{-1}, \right. \\ & \left. \frac{\phi_2(nb)}{m\phi_2(0) + n\phi_2(b)} \left(\frac{(R_2T_2)^{1/2}}{(r_2t_2)^{1/2}} + \frac{(r_2t_2)^{1/2}}{(R_2T_2)^{1/2}} \right)^{-1} \left(\frac{R_2T_2}{r_2t_2} + \frac{r_2t_2}{R_2T_2} \right)^{-1} \right\}, \end{aligned}$$

$0 \leq m, n \leq 1, m + n = 1$, and

$$L(x_1, x_2) = \min \left\{ x_2 \left(\int_0^{x_1} \left(\frac{\phi_1(P_1(s_1))}{P_1(s_1)} \right)^2 ds_1 \right)^{1/2}, x_1 \left(\int_0^{x_2} \left(\frac{\phi_2(P_2(s_2))}{P_2(s_2)} \right)^2 ds_2 \right)^{1/2} \right\}.$$

REMARK 1.2. In Theorem C we deal with sub-multiplicative function, while the reverse inequality in Theorem 1.2 is achieved for super-multiplicative function. In addition, the molecule of the integrand of the left side of inequality (1.4) becomes $\phi_1(F_1(s_1)) + \phi_2(F_2(s_2))$, but inequality (1.3) in Theorem C is $\phi_1(F_1(s_1))\phi_2(F_2(s_2))$.

2. Main results

In this section, we start with two auxiliary results (Lemmas 2.1 and 2.2), which will be the base of our further study. Our main results are given in the following Theorems (Theorems 2.1 and 2.2).

LEMMA 2.1. [19] (Pólya-Szegő's integral inequality) Let $f(x), g(x)$ be non-negative continuous functions on $[a, b]$. Let $p, q > 0, \frac{1}{p} + \frac{1}{q} = 1$ and $f^{1/p}(x)g^{1/q}(x)$ be integrable on $[a, b]$. If $0 < m_1 \leq f(x) \leq M_1$ and $0 < m_2 \leq g(x) \leq M_2$, then

$$\left(\int_a^b f(x) dx \right)^{1/p} \left(\int_a^b g(x) dx \right)^{1/q} \leq \Gamma_{p,q} \left(\frac{m_1 m_2}{M_1 M_2} \right) \int_a^b f^{1/p}(x) g^{1/q}(x) dx, \tag{2.1}$$

where

$$\Gamma_{p,q}(\xi) = (\sqrt[p]{p} \cdot \sqrt[q]{q})^{-1} \frac{1 - \xi}{(1 - \xi^{1/p})^{1/p} (1 - \xi^{1/q})^{1/q}} \cdot \xi^{-1/pq}. \tag{2.2}$$

Putting $p = q = 2$ in (2.1), (2.1) reduces an integral form of the Pólya-Szegő's inequality [1].

LEMMA 2.2. (Reverse Jensen's inequality) *Let $p(x)$ be a positive continuous function and $f(x)$ be convex continuous function on $I := [a, b]$. For $a \leq x \leq b$, there exist m, n and $0 \leq m, n \leq 1$ and $m + n = 1$ such that $x = ma + nb$. Then*

$$f\left(\frac{\int_a^b p(x)xdx}{\int_a^b p(x)dx}\right) \geq E_f(a,b) \frac{\int_a^b p(x)f(x)dx}{\int_a^b p(x)dx}, \tag{2.3}$$

where

$$E_f(a,b) = \frac{f(ma + nb)}{mf(a) + nf(b)}. \tag{2.4}$$

Proof. Since $a \leq x \leq b$, there is t ($0 \leq t \leq 1$) such that $x = ta + (1 - t)b$. Therefore

$$\begin{aligned} & \frac{\int_a^b p(x)f(x)dx}{\int_a^b p(x)dx} = \frac{\int_0^1 p(ta + (1 - t)b)f(ta + (1 - t)b)dt}{\int_0^1 p(ta + (1 - t)b)dt} \\ f\left(\frac{\int_a^b p(x)xdx}{\int_a^b p(x)dx}\right) &= f\left(\frac{\int_0^1 p(ta + (1 - t)b)(ta + (1 - t)b)dt}{\int_0^1 p(ta + (1 - t)b)dt}\right) \\ &= \frac{\int_0^1 p(ta + (1 - t)b)(tf(a) + (1 - t)f(b))dt}{\int_0^1 p(ta + (1 - t)b)dt} \\ &\leq \frac{f\left(\frac{a \int_0^1 tp(t + (1 - t)b)dt + b \int_0^1 (1 - t)p(t + (1 - t)b)dt}{\int_0^1 p(ta + (1 - t)b)dt}\right)}{f\left(\frac{a \int_0^1 tp(ta + (1 - t)b)dt}{\int_0^1 p(ta + (1 - t)b)dt} + f(b) \left(1 - \frac{\int_0^1 tp(ta + (1 - t)b)dt}{\int_0^1 p(ta + (1 - t)b)dt}\right)\right)} \\ &= \frac{f\left(\frac{a \int_0^1 tp(ta + (1 - t)b)dt}{\int_0^1 p(ta + (1 - t)b)dt} + b \left(1 - \frac{\int_0^1 tp(ta + (1 - t)b)dt}{\int_0^1 p(ta + (1 - t)b)dt}\right)\right)}{f\left(\frac{a \int_0^1 tp(ta + (1 - t)b)dt}{\int_0^1 p(ta + (1 - t)b)dt} + b \left(1 - \frac{\int_0^1 tp(ta + (1 - t)b)dt}{\int_0^1 p(ta + (1 - t)b)dt}\right)\right)}. \end{aligned}$$

On the other hand, let

$$m = \frac{\int_0^1 tp(ta + (1 - t)b)dt}{\int_0^1 p(ta + (1 - t)b)dt}$$

and

$$n = 1 - \frac{\int_0^1 tp(ta + (1 - t)b)dt}{\int_0^1 p(ta + (1 - t)b)dt}.$$

It is clear that $0 \leq m, n \leq 1$ and $m + n = 1$. Hence, it easy follows the inequality in Lemma 2.2. \square

THEOREM 2.1. For $i = 1, 2$, let $h_i \geq 1$ and $0 < r_i \leq f_i(\sigma_i) \leq R_i$ for $\sigma_i \in (0, x_i)$, where x_i are positive real numbers and define $F_i(s_i) = \int_0^{s_i} f_i(\sigma_i) d\sigma_i$, for $s_i \in (0, x_i)$. If $\alpha > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then

$$\begin{aligned} & \int_0^{x_1} \int_0^{x_2} \frac{F_1^{h_1}(s_1) + F_2^{h_2}(s_2)}{\min\{s_1^{1/\alpha}, s_2^{1/\beta}\}} ds_1 ds_2 \\ & \geq M_{\alpha,\beta} (h_1 h_2)^{1/2} x_1^{(1+\alpha)/(2\alpha)} x_2^{(1+\beta)/(2\beta)} \left(\int_0^{x_1} (x_1 - s_1) \left(F_1^{h_1-1}(s_1) f_1(s_1) \right)^\beta ds_1 \right)^{1/(2\beta)} \\ & \quad \times \left(\int_0^{x_2} (x_2 - s_2) \left(F_2^{h_2-1}(s_2) f_2(s_2) \right)^\alpha ds_2 \right)^{1/(2\alpha)}, \end{aligned} \tag{2.5}$$

where

$$M_{\alpha,\beta} = 2 \min \left\{ \Gamma_{\alpha,\beta}^{-2} \left(\frac{r_1^{\beta h_1}}{R_1^{\beta h_1}} \right), \Gamma_{\alpha,\beta}^{-2} \left(\frac{r_2^{\alpha h_2}}{R_2^{\alpha h_2}} \right) \right\},$$

and $\Gamma_{\alpha,\beta}$ is as in (2.2).

Proof. From the hypotheses, we have

$$F_i^{h_i}(s_i) = h_i \int_0^{s_i} F_i^{h_i-1}(\sigma_i) f_i(\sigma_i) d\sigma_i, \quad i = 1, 2.$$

Hence

$$\begin{aligned} F_1^{h_1}(s_1) + F_2^{h_2}(s_2) &= h_1 \int_0^{s_1} 1^{1/\alpha} \cdot \left(F_1^{\beta(h_1-1)}(\sigma_1) f_1^\beta(\sigma_1) \right)^{1/\beta} d\sigma_1 \\ & \quad + h_2 \int_0^{s_2} \left(F_2^{\alpha(h_2-1)}(\sigma_2) f_2^\alpha(\sigma_2) \right)^{1/\alpha} \cdot 1^{1/\beta} d\sigma_2. \end{aligned}$$

On the other hand

$$R_1^{\beta h_1} (s_1)^{\beta(h_1-1)} \geq F_1^{\beta(h_1-1)}(\sigma_1) f_1^\beta(\sigma_1) \geq r_1^{\beta h_1} s_1^{\beta(h_1-1)}$$

and

$$R_2^{\alpha h_2} (s_2)^{\alpha(h_2-1)} \geq F_2^{\alpha(h_2-1)}(\sigma_2) f_2^\alpha(\sigma_2) \geq r_2^{\alpha h_2} s_2^{\alpha(h_2-1)}.$$

By applying Pólya-Szegő's integral inequality in Lemma 2.1, we have

$$\begin{aligned} F_1^{h_1}(s_1) + F_2^{h_2}(s_2) & \geq \Gamma_{\alpha,\beta}^{-1} \left(\frac{r_1^{\beta h_1}}{R_1^{\beta h_1}} \right) \cdot h_1 s_1^{1/\alpha} \left(\int_0^{s_1} \left(F_1^{h_1-1}(\sigma_1) f_1(\sigma_1) \right)^\beta d\sigma_1 \right)^{1/\beta} \\ & \quad + \Gamma_{\alpha,\beta}^{-1} \left(\frac{r_2^{\alpha h_2}}{R_2^{\alpha h_2}} \right) \cdot h_2 s_2^{1/\beta} \left(\int_0^{s_2} \left(F_2^{h_2-1}(\sigma_2) f_2(\sigma_2) \right)^\alpha d\sigma_2 \right)^{1/\alpha}. \end{aligned}$$

Hence

$$\begin{aligned} & \frac{F_1^{h_1}(s_1) + F_2^{h_2}(s_2)}{\min\{s_1^{1/\alpha}, s_2^{1/\beta}\}} \\ & \geq \Gamma_{\alpha,\beta}^{-1} \left(\frac{r_1^{\beta h_1}}{R_1^{\beta h_1}} \right) h_1 \left(\int_0^{s_1} \left(F_1^{h_1-1}(\sigma_1) f_1(\sigma_1) \right)^\beta d\sigma_1 \right)^{1/\beta} \\ & + \Gamma_{\alpha,\beta}^{-1} \left(\frac{r_2^{\alpha h_2}}{R_2^{\alpha h_2}} \right) \cdot h_2 \left(\int_0^{s_2} \left(F_2^{h_2-1}(\sigma_2) f_2(\sigma_2) \right)^\alpha d\sigma_2 \right)^{1/\alpha}. \end{aligned} \tag{2.6}$$

Integrating both sides of (2.6) over s_i from 0 to x_i ($i = 1, 2$), we obtain

$$\begin{aligned} & \int_0^{x_1} \int_0^{x_2} \frac{F_1^{h_1}(s_1) + F_2^{h_2}(s_2)}{\min\{s_1^{1/\alpha}, s_2^{1/\beta}\}} ds_1 ds_2 \\ & \geq \Gamma_{\alpha,\beta}^{-1} \left(\frac{r_1^{\beta h_1}}{R_1^{\beta h_1}} \right) \cdot x_2 h_1 \int_0^{x_1} \left(\int_0^{s_1} \left(F_1^{h_1-1}(\sigma_1) f_1(\sigma_1) \right)^\beta d\sigma_1 \right)^{1/\beta} dx_1 \\ & + \Gamma_{\alpha,\beta}^{-1} \left(\frac{r_2^{\alpha h_2}}{R_2^{\alpha h_2}} \right) \cdot x_1 h_2 \int_0^{x_2} \left(\int_0^{s_2} \left(F_2^{h_2-1}(\sigma_2) f_2(\sigma_2) \right)^\alpha d\sigma_2 \right)^{1/\alpha} dx_2. \end{aligned}$$

Notice that

$$r_1^{\beta h_1} s_1^{1+\beta(h_1-1)} \leq \int_0^{s_1} \left(F_1^{h_1-1}(\sigma_1) f_1(\sigma_1) \right)^\beta d\sigma_1 \leq R_1^{\beta h_1} s_1^{1+\beta(h_1-1)}$$

and

$$r_2^{\alpha h_2} s_2^{1+\alpha(h_2-1)} \leq \int_0^{s_2} \left(F_2^{h_2-1}(\sigma_2) f_2(\sigma_2) \right)^\alpha d\sigma_2 \leq R_2^{\alpha h_2} s_2^{1+\alpha(h_2-1)}.$$

By using Pólya-Szegő's integral inequality again,

$$\begin{aligned} & \int_0^{x_1} \int_0^{x_2} \frac{F_1^{h_1}(s_1) + F_2^{h_2}(s_2)}{\min\{s_1^{1/\alpha}, s_2^{1/\beta}\}} ds_1 ds_2 \\ & \geq \Gamma_{\alpha,\beta}^{-2} \left(\frac{r_1^{\beta h_1}}{R_1^{\beta h_1}} \right) h_1 x_2 x_1^{1/\alpha} \left(\int_0^{x_1} \left(\int_0^{s_1} \left(F_1^{h_1-1}(\sigma_1) f_1(\sigma_1) \right)^\beta d\sigma_1 \right) ds_1 \right)^{1/\beta} \\ & + \Gamma_{\alpha,\beta}^{-2} \left(\frac{r_2^{\alpha h_2}}{R_2^{\alpha h_2}} \right) h_2 x_1 x_2^{1/\beta} \left(\int_0^{x_2} \left(\int_0^{s_2} \left(F_2^{h_2-1}(\sigma_2) f_2(\sigma_2) \right)^\alpha d\sigma_2 \right) ds_2 \right)^{1/\alpha} \\ & = \frac{1}{2} M_{\alpha,\beta} \left[h_1 x_2 x_1^{1/\alpha} \left(\int_0^{x_1} (x_1 - s_1) \left(F_1^{h_1-1}(s_1) f_1(s_1) \right)^\beta ds_1 \right)^{1/\beta} \right. \\ & \left. + h_2 x_1 x_2^{1/\beta} \left(\int_0^{x_2} (x_2 - s_2) \left(F_2^{h_2-1}(s_2) f_2(s_2) \right)^\alpha ds_2 \right)^{1/\alpha} \right], \end{aligned} \tag{2.7}$$

where

$$M_{\alpha,\beta} = 2 \min \left\{ \Gamma_{\alpha,\beta}^{-2} \left(\frac{r_1^{\beta h_1}}{R_1^{\beta h_1}} \right), \Gamma_{\alpha,\beta}^{-2} \left(\frac{r_2^{\alpha h_2}}{R_2^{\alpha h_2}} \right) \right\}.$$

By using the arithmetic-geometric mean inequality $c + d \geq 2c^{1/2}d^{1/2}$ (for c, d nonnegative reals) on the right side of (2.7), the inequality in Theorem 2.1 follows easy. The proof is complete. \square

REMARK 2.1. Putting $\alpha = \beta = 2$ in (2.5), (2.5) reduces to (1.2) stated in the introduction.

THEOREM 2.2. For $i = 1, 2$, let $f_i(\sigma_i) \geq 0$ on $(0, x_i)$, where x_i are positive real numbers and define $F_i(s_i) = \int_0^{s_i} f_i(\sigma_i) d\sigma_i$, for $s_i \in (0, x_i)$. Let $0 < r_i \leq p_i(\sigma_i) \leq R_i$ and define $P_i(s_i) = \int_0^{s_i} p_i(\sigma_i) d\sigma_i$. Let ϕ_i be real-valued nonnegative, convex, and supermultiplicative functions defined on $[0, b]$ and $0 \leq t_i \leq \phi_i \leq T_i$. If $\alpha > 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then

$$\begin{aligned} & \int_0^{x_1} \int_0^{x_2} \frac{\phi_1(F_1(s_1)) + \phi_2(F_2(s_2))}{\min\{s_1^{1/\alpha}, s_2^{1/\beta}\}} ds_1 ds_2 \\ & \geq N_{\alpha,\beta} \cdot L_{\alpha,\beta}(x_1, x_2) \left(\int_0^{x_1} (x_1 - s_1) \left(p_1(s_1) \phi_1 \left(\frac{f_1(s_1)}{p_1(s_1)} \right) \right)^\beta ds_1 \right)^{1/(2\beta)} \\ & \quad \times \left(\int_0^{x_2} (x_2 - s_2) \left(p_2(s_2) \phi_2 \left(\frac{f_2(s_2)}{p_2(s_2)} \right) \right)^\alpha ds_2 \right)^{1/(2\alpha)}, \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} N_{\alpha,\beta} &= 2 \min \left\{ E_{\phi_1}(0, b) \Gamma_{\alpha,\beta}^{-1} \left(\frac{(r_1 t_1)^\beta}{(R_1 T_1)^\beta} \right) \Gamma_{\alpha,\beta}^{-1} \left(\frac{(r_1 t_1)^{\alpha+\beta}}{(R_1 T_1)^{\alpha+\beta}} \right), \right. \\ & \left. E_{\phi_2}(0, b) \Gamma_{\alpha,\beta}^{-1} \left(\frac{(r_2 t_2)^\alpha}{(R_2 T_2)^\alpha} \right) \Gamma_{\alpha,\beta}^{-1} \left(\frac{(r_2 t_2)^{\alpha+\beta}}{(R_2 T_2)^{\alpha+\beta}} \right) \right\}, \end{aligned}$$

and

$$L_{\alpha,\beta}(x_1, x_2) = \min \left\{ x_2 \left(\int_0^{x_1} \left(\frac{\phi_1(P_1(s_1))}{P_1(s_1)} \right)^\alpha ds_1 \right)^{1/\alpha}, x_1 \left(\int_0^{x_2} \left(\frac{\phi_2(P_2(s_2))}{P_2(s_2)} \right)^\beta ds_2 \right)^{1/\beta} \right\}.$$

Proof. From the hypotheses and applying reverse Jensen's inequality in Lemma

2.2, we have for $i = 1, 2$

$$\begin{aligned} \phi_i(F_i(s_i)) &= \phi_i \left(\frac{P_i(s_i) \int_0^{s_i} p_i(\sigma_i) \frac{f_i(\sigma_i)}{p_i(\sigma_i)} d\sigma_i}{\int_0^{s_i} p_i(\sigma_i) d\sigma_i} \right) \\ &\geq \phi_i(P_i(s_i)) \phi_i \left(\frac{\int_0^{s_i} p_i(\sigma_i) \frac{f_i(\sigma_i)}{p_i(\sigma_i)} d\sigma_i}{\int_0^{s_i} p_i(\sigma_i) d\sigma_i} \right) \\ &\geq E_{\phi_i}(0, b) \frac{\phi_i(P_i(s_i))}{P_i(s_i)} \int_0^{s_i} p_i(\sigma_i) \phi_i \left(\frac{f_i(\sigma_i)}{p_i(\sigma_i)} \right) d\sigma_i. \end{aligned}$$

Hence

$$\begin{aligned} \phi_1(F_1(s_1)) + \phi_2(F_2(s_2)) &\geq E_{\phi_1}(0, b) \frac{\phi_1(P_1(s_1))}{P_1(s_1)} \int_0^{s_1} p_1(\sigma_1) \phi_1 \left(\frac{f_1(\sigma_1)}{p_1(\sigma_1)} \right) d\sigma_1 \\ &\quad + E_{\phi_2}(0, b) \frac{\phi_2(P_2(s_2))}{P_2(s_2)} \int_0^{s_2} p_2(\sigma_2) \phi_2 \left(\frac{f_2(\sigma_2)}{p_2(\sigma_2)} \right) d\sigma_2. \end{aligned}$$

On the other hand, notice that

$$(r_1 t_1)^\beta \leq \left(p_1(\sigma_1) \phi_1 \left(\frac{f_1(\sigma_1)}{p_1(\sigma_1)} \right) \right)^\beta \leq (R_1 T_1)^\beta,$$

and

$$(r_2 t_2)^\alpha \leq \left(p_2(\sigma_2) \phi_2 \left(\frac{f_2(\sigma_2)}{p_2(\sigma_2)} \right) \right)^\alpha \leq (R_2 T_2)^\alpha.$$

By applying Pólya-Szegő's integral inequality in Lemma 2.1, we obtain

$$\begin{aligned} &\phi_1(F_1(s_1)) + \phi_2(F_2(s_2)) \\ &\geq E_{\phi_1}(0, b) \Gamma_{\alpha, \beta}^{-1} \left(\frac{(r_1 t_1)^\beta}{(R_1 T_1)^\beta} \right) s_1^{1/\alpha} \frac{\phi_1(P_1(s_1))}{P_1(s_1)} \left(\int_0^{s_1} \left(p_1(\sigma_1) \phi_1 \left(\frac{f_1(\sigma_1)}{p_1(\sigma_1)} \right) \right)^\beta d\sigma_1 \right)^{1/\beta} \\ &\quad + E_{\phi_2}(0, b) \Gamma_{\alpha, \beta}^{-1} \left(\frac{(r_2 t_2)^\alpha}{(R_2 T_2)^\alpha} \right) s_2^{1/\beta} \frac{\phi_2(P_2(s_2))}{P_2(s_2)} \left(\int_0^{s_2} \left(p_2(\sigma_2) \phi_2 \left(\frac{f_2(\sigma_2)}{p_2(\sigma_2)} \right) \right)^\alpha d\sigma_2 \right)^{1/\alpha}. \end{aligned}$$

Hence

$$\begin{aligned} &\frac{\phi_1(F_1(s_1)) + \phi_2(F_2(s_2))}{\min\{s_1^{1/\alpha}, s_2^{1/\beta}\}} \\ &\geq E_{\phi_1}(0, b) \Gamma_{\alpha, \beta}^{-1} \left(\frac{(r_1 t_1)^\beta}{(R_1 T_1)^\beta} \right) \frac{\phi_1(P_1(s_1))}{P_1(s_1)} \left(\int_0^{s_1} \left(p_1(\sigma_1) \phi_1 \left(\frac{f_1(\sigma_1)}{p_1(\sigma_1)} \right) \right)^\beta d\sigma_1 \right)^{1/\beta} \end{aligned}$$

$$+E_{\phi_2}(0, b)\Gamma_{\alpha, \beta}^{-1} \left(\frac{(r_2 t_2)^\alpha}{(R_2 T_2)^\alpha} \right) \frac{\phi_2(P_2(s_2))}{P_2(s_2)} \left(\int_0^{s_2} \left(p_2(\sigma_2) \phi_2 \left(\frac{f_2(\sigma_2)}{p_2(\sigma_2)} \right) \right)^\alpha d\sigma_2 \right)^{1/\alpha} \tag{2.9}$$

Integrating two sides of (2.9) over s_i from 0 to x_i ($i = 1, 2$), we have

$$\begin{aligned} & \int_0^{x_1} \int_0^{x_2} \frac{\phi_1(F_1(s_1)) + \phi_2(F_2(s_2))}{\min\{s_1^{1/\alpha}, s_2^{1/\beta}\}} ds_1 ds_2 \\ & \geq x_2 E_{\phi_1}(0, b) \Gamma_{\alpha, \beta}^{-1} \left(\frac{(r_1 t_1)^\beta}{(R_1 T_1)^\beta} \right) \int_0^{x_1} \frac{\phi_1(P_1(s_1))}{P_1(s_1)} \\ & \times \left(\int_0^{s_1} \left(p_1(\sigma_1) \phi_1 \left(\frac{f_1(\sigma_1)}{p_1(\sigma_1)} \right) \right)^\beta d\sigma_1 \right)^{1/\beta} ds_1 \\ & + x_1 E_{\phi_2}(0, b) \Gamma_{\alpha, \beta}^{-1} \left(\frac{(r_2 t_2)^\alpha}{(R_2 T_2)^\alpha} \right) \int_0^{x_2} \frac{\phi_2(P_2(s_2))}{P_2(s_2)} \\ & \times \left(\int_0^{s_2} \left(p_2(\sigma_2) \phi_2 \left(\frac{f_2(\sigma_2)}{p_2(\sigma_2)} \right) \right)^\alpha d\sigma_2 \right)^{1/\alpha} ds_2. \end{aligned}$$

Moreover, notice that

$$\begin{aligned} & \left(\frac{t_1}{R_1} \right)^\alpha \leq \left(\frac{\phi_1(P_1(s_1))}{P_1(s_1)} \right)^\alpha \leq \left(\frac{T_1}{r_1} \right)^\alpha, \\ & s_1 (r_1 t_1)^\beta \leq \int_0^{s_1} \left(p_1(\sigma_1) \phi_1 \left(\frac{f_1(\sigma_1)}{p_1(\sigma_1)} \right) \right)^\beta d\sigma_1 \leq s_1 (R_1 T_1)^\beta, \end{aligned}$$

and

$$\begin{aligned} & \left(\frac{t_2}{R_2} \right)^\beta \leq \left(\frac{\phi_2(P_2(s_2))}{P_2(s_2)} \right)^\beta \leq \left(\frac{T_2}{r_2} \right)^\beta, \\ & s_2 (r_2 t_2)^\alpha \leq \int_0^{s_2} \left(p_2(\sigma_2) \phi_2 \left(\frac{f_2(\sigma_2)}{p_2(\sigma_2)} \right) \right)^\alpha d\sigma_2 \leq s_2 (R_2 T_2)^\alpha. \end{aligned}$$

By applying Pólya-Szegő's integral inequality again, we obtain

$$\begin{aligned} & \int_0^{x_1} \int_0^{x_2} \frac{\phi_1(F_1(s_1)) + \phi_2(F_2(s_2))}{\min\{s_1^{1/\alpha}, s_2^{1/\beta}\}} ds_1 ds_2 \\ & \geq x_2 E_{\phi_1}(0, b) \Gamma_{\alpha, \beta}^{-1} \left(\frac{(r_1 t_1)^\beta}{(R_1 T_1)^\beta} \right) \Gamma_{\alpha, \beta}^{-1} \left(\frac{(r_1 t_1)^{\alpha+\beta}}{(R_1 T_1)^{\alpha+\beta}} \right) \\ & \times \left(\int_0^{x_1} \left(\frac{\phi_1(P_1(s_1))}{P_1(s_1)} \right)^\alpha ds_1 \right)^{1/\alpha} \left(\int_0^{x_1} \int_0^{s_1} \left(p_1(\sigma_1) \phi_1 \left(\frac{f_1(\sigma_1)}{p_1(\sigma_1)} \right) \right)^\beta d\sigma_1 ds_1 \right)^{1/\beta} \end{aligned}$$

$$\begin{aligned}
 & +x_1 E_{\phi_2}(0, b) \Gamma_{\alpha, \beta}^{-1} \left(\frac{(r_2 t_2)^\alpha}{(R_2 T_2)^\alpha} \right) \Gamma_{\alpha, \beta}^{-1} \left(\frac{(r_2 t_2)^{\alpha+\beta}}{(R_2 T_2)^{\alpha+\beta}} \right) \\
 & \times \left(\int_0^{x_2} \left(\frac{\phi_2(P_2(s_2))}{P_2(s_2)} \right)^\beta ds_2 \right)^{1/\beta} \left(\int_0^{x_2} \int_0^{s_2} \left(p_2(\sigma_2) \phi_2 \left(\frac{f_2(\sigma_2)}{p_2(\sigma_2)} \right) \right)^\alpha d\sigma_2 ds_2 \right)^{1/\alpha} \\
 & \geq \frac{1}{2} N_{\alpha, \beta} L_{\alpha, \beta}(x_1, x_2) \left\{ \left(\int_0^{x_1} \int_0^{s_1} \left(p_1(\sigma_1) \phi_1 \left(\frac{f_1(\sigma_1)}{p_1(\sigma_1)} \right) \right)^\beta d\sigma_1 ds_1 \right)^{1/\beta} \right. \\
 & \quad \left. + \left(\int_0^{x_2} \int_0^{s_2} \left(p_2(\sigma_2) \phi_2 \left(\frac{f_2(\sigma_2)}{p_2(\sigma_2)} \right) \right)^\alpha d\sigma_2 ds_2 \right)^{1/\alpha} \right\} \\
 & = \frac{1}{2} N_{\alpha, \beta} L_{\alpha, \beta}(x_1, x_2) \left\{ \left(\int_0^{x_1} (x_1 - s_1) \left(p_1(s_1) \phi_1 \left(\frac{f_1(s_1)}{p_1(s_1)} \right) \right)^\beta ds_1 \right)^{1/\beta} \right. \\
 & \quad \left. + \left(\int_0^{x_2} (x_2 - s_2) \left(p_2(s_2) \phi_2 \left(\frac{f_2(s_2)}{p_2(s_2)} \right) \right)^\alpha ds_2 \right)^{1/\alpha} \right\}. \tag{2.9}
 \end{aligned}$$

By using the the arithmetic-geometric mean inequality $c + d \geq 2c^{1/2}d^{1/2}$ (for c, d non-negative reals) on the right side of (2.9), (2.9) becomes the inequality in Theorem 2.2. The proof is complete. \square

REMARK 2.2. Putting $\alpha = \beta = 2$ in (2.8), (2.8) reduces to (1.4) stated in the introduction.

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(Received May 8, 2013)

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