Mathematical
Inequalities
& Applications
Volume 17, Number 4 (2014), 1579–1590 doi:10.7153/mia-17-116

REMARKS ON SHERMAN LIKE INEQUALITIES FOR (α,β)–CONVEX FUNCTIONS

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(*Communicated by I. Perić*)

Abstract. In this paper, Sherman's inequality is extended from convex functions to the class of (α,β)-convex functions including (k,h)-convex functions. Sherman's type results corresponding to Jensen-Steffensen, Mercer-Steffensen and Brunk inequalities are established. The obtained results are applied to mixed (α,β)-convex functions.

1. Introduction

A vector $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ is said to be *majorized* by a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, written as $\mathbf{y} \prec \mathbf{x}$, if

$$\sum_{i=1}^{j} y_{[i]} \leqslant \sum_{i=1}^{j} x_{[i]} \quad \text{for } j = 1, 2, \ldots, n$$

with equality for $j = n$ (see [10, p. 8]). Here $y_{[i]}$ and $x_{[i]}$ are the ith largest entry of $\mathbf{y}$ and $\mathbf{x}$, respectively.

An $m \times n$ real matrix $\mathbf{A} = (a_{ij})$ is said to be *row stochastic* if $a_{ij} \geqslant 0$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, and all row sums of $\mathbf{A}$ are equal to 1, i.e., $\sum_{j=1}^{n} a_{ij} = 1$ for $i = 1, 2, \ldots, m$. If in addition the transpose $\mathbf{A}^T = (a_{ji})$ of $\mathbf{A} = (a_{ij})$ is row stochastic, then $\mathbf{A}$ is called *doubly stochastic*. In other words, an $n \times n$ matrix $\mathbf{A}$ is doubly stochastic iff $\mathbf{A} \geqslant 0$ (entrywise) and $\mathbf{eA} = \mathbf{e} = \mathbf{eA}^T$, where $\mathbf{e} = (1, \ldots, 1)$ is the $1 \times n$ vector of ones.

It is well-known (see [10, pp. 33]) that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\mathbf{y} \prec \mathbf{x} \ \text{ if and only if } \ \mathbf{y} = \mathbf{xA} \ \text{ for some doubly stochastic } n \times n \text{ matrix } \mathbf{A}. \tag{1}$$

A real function F defined on a set $S \subset \mathbb{R}^n$ is said to be *Schur-convex* on S if for $\mathbf{x}, \mathbf{y} \in S$,

$$\mathbf{y} \prec \mathbf{x} \ \text{ implies } \ F(\mathbf{y}) \leqslant F(\mathbf{x}).$$

A relationship between Schur-convexity and standard convexity is included in the following *Majorization Theorem* (see [10, pp. 92-93]).

Mathematics subject classification (2010): 39B62, 26D15.

Keywords and phrases: Majorization, Schur-convex function, convex function, (α,β)-convex function, (k,h)-convex function, h-convex function, Toader's c-convex function, Breckner's s-convex function.

THEOREM A. *Assume that f is a real convex function defined on an interval $I \subset \mathbb{R}$.*

Then, for $\mathbf{x} = (x_1, x_2, \dots, x_n) \in I^n$ and $\mathbf{y} = (y_1, y_2, \dots, y_n) \in I^n$,

$$\mathbf{y} \prec \mathbf{x} \quad \textit{implies} \quad \sum_{i=1}^{n} f(y_i) \leqslant \sum_{i=1}^{n} f(x_i). \tag{2}$$

That is, the function

$$F(x_1, \dots, x_n) = \sum_{i=1}^{n} f(x_i) \quad \text{for } (x_1, x_2, \dots, x_n) \in I^n$$

is Schur-convex on I^n, whenever f is convex on I.

EXAMPLE 1.1. Petrović's inequality [6, p. 123] says that if f is a real convex function defined on interval $[0, \infty)$, then

$$f(x_1) + f(x_2) + \dots + f(x_n) \leqslant f(x_1 + x_2 + \dots + x_n) + (n-1)f(0) \tag{3}$$

for all $x_1, x_2, \dots, x_n \in [0, \infty)$.

Inequality (3) is a corollary to (2), because

$$(x_1, x_2, \dots, x_n) \prec (x_1 + x_2 + \dots + x_n, \underbrace{0, \dots, 0}_{n-1 \ \textit{times}})$$

for all $x_1, x_2, \dots, x_n \in [0, \infty)$.

Throughout, for a positive integer p, we denote $\mathbb{R}^p_+ = \{(x_1, \dots, x_p) \in \mathbb{R}^p : x_1 \geqslant 0, \dots, x_p \geqslant 0\}$.

A more general result is the following *Sherman Theorem* ([17], see also [3, 7]).

THEOREM B. *Assume that f is a real convex function defined on an interval $I \subset \mathbb{R}$.*

If $\mathbf{x} = (x_1, x_2, \dots, x_n) \in I^n$, $\mathbf{y} = (y_1, y_2, \dots, y_m) \in I^m$, $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n_+$, $\mathbf{b} = (b_1, b_2, \dots, b_m) \in \mathbb{R}^m_+$, and

$$\mathbf{y} = \mathbf{x}\mathbf{A}^T \quad \textit{and} \quad \mathbf{a} = \mathbf{b}\mathbf{A} \quad \textit{for some } m \times n \textit{ row stochastic matrix } \mathbf{A} = (a_{ij}), \tag{4}$$

then

$$\sum_{i=1}^{m} b_i f(y_i) \leqslant \sum_{j=1}^{n} a_j f(x_j). \tag{5}$$

If f is concave, then the inequality (5) is reversed.

REMARK 1.2. **(i)** Observe that Theorem B implies Theorem A by the substitution $m = n$ and $\mathbf{b} = \mathbf{e} = (1, \dots, 1) \in \mathbb{R}^n$, because $\mathbf{y} \prec \mathbf{x}$ gives $\mathbf{y} = \mathbf{x}\mathbf{A}^T$ with some doubly stochastic matrix $\mathbf{A}$ (see (1)) and $\mathbf{a} = \mathbf{b}\mathbf{A} = \mathbf{e}$.

(ii) In the case $m = 1$ and $\mathbf{b} = [1]$, Sherman's inequality (5) reduces to Jensen's inequality.

(iii) The Sherman theorem is usually given in the form of the equivalence of (4) and (5). See [3] for its proof.

To illustrate Sherman's inequality (5) we now provide an example.

EXAMPLE 1.3. Bougoffa [4, Theorem 1.2] showed that if f is a real convex function defined on an interval $I \subset \mathbb{R}$, then for $x_1, x_2, \dots, x_n \in I$,

$$\frac{n-1}{n}\left[f\left(\frac{x_1+x_2}{2}\right)+\ldots+f\left(\frac{x_{n-1}+x_n}{2}\right)+f\left(\frac{x_n+x_1}{2}\right)\right]+f\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)$$
$$\leqslant \sum_{i=1}^{n} f(x_i). \tag{6}$$

To prove (6) with the help of Theorem B, consider the matrices $\mathbf{A}$ and $\mathbf{A}^T$ of sizes $(n+1)\times n$ and $n\times(n+1)$, respectively, given by

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & \dots & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} \\ \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} & \frac{1}{n} \end{pmatrix} \quad \text{and} \quad \mathbf{A}^T = \begin{pmatrix} \frac{1}{2} & 0 & \dots & 0 & \frac{1}{2} & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{2} & \dots & 0 & 0 & \frac{1}{n} \\ 0 & \frac{1}{2} & \dots & 0 & 0 & \frac{1}{n} \\ 0 & 0 & \dots & 0 & 0 & \frac{1}{n} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{1}{2} & 0 & \frac{1}{n} \\ 0 & 0 & \dots & \frac{1}{2} & \frac{1}{2} & \frac{1}{n} \end{pmatrix}.$$

It is easily seen that for

$$\mathbf{x} = (x_1, x_2, \dots, x_n) \quad \text{and} \quad \mathbf{b} = \Bigg(\underbrace{1, 1, \dots, 1}_{n\ times}, \frac{n}{n-1}\Bigg),$$

we have

$$\mathbf{y} = \mathbf{x}\mathbf{A}^T = \left(\frac{x_1+x_2}{2}, \frac{x_2+x_3}{2}, \dots, \frac{x_{n-1}+x_n}{2}, \frac{x_n+x_1}{2}, \frac{x_1+x_2+\ldots+x_n}{n}\right)$$

and

$$\mathbf{a} = \mathbf{b}\mathbf{A} = \frac{n}{n-1}(1, 1, \dots, 1).$$

Now, inequality (6) is a direct consequence of (5).

REMARK 1.4. **(i)** The proof of Theorem B can be based on Jensen inequality.

(ii) On the other hand, it is not hard to check that Theorem B remains valid if convex functions are replaced by subadditive functions, and a row stochastic matrix is replaced by the all-ones matrix.

The first purpose of this paper is to extend Sherman's Theorem B from convex functions to so-called (α,β)-convex functions (see Section 2). The second purpose is to derive Sherman's type results corresponding to Jensen-Steffensen, Mercer-Steffensen and Brunk inequalities (see Section 3). Also, (k,h)-convex functions and Toader's c-convex functions are discussed in this context. The obtained results are applied to mixed (α,β)-convex functions in Section 4. This leads to a combination of Jensen and Bellman inequalities.

2. Sherman type results for (α,β)-convex functions

In this paper, for given vectors $\alpha,\beta \in \mathbb{R}^n$, we say that a function $f : I \to \mathbb{R}$ defined on an interval $I \subset \mathbb{R}$ is (α,β)*-convex* (resp. (α,β)*-concave*) on a set $S \subset I^n$ if the following inequality holds:

$$f(\langle \alpha, \mathbf{x} \rangle) \leqslant (\geqslant) \langle \beta, f(\mathbf{x}) \rangle \quad \text{for } \mathbf{x} \in S, \tag{7}$$

where $\langle \cdot,\cdot \rangle$ is the standard inner product on $\mathbb{R}^n$, and $f(\mathbf{x}) = (f(x_1), f(x_2), \ldots, f(x_n))$ for $\mathbf{x} = (x_1,x_2,\ldots,x_n) \in S$ (cf. [9]).

The class of functions satisfying inequality (7) (with $\alpha = \beta$ or $\alpha \neq \beta$) includes convex functions, subadditive functions, starshaped functions, Breckner's s-convex functions [6, p. 254], Godunova-Levin functions [18], P-functions [16, 18], h-convex functions [18], Toader's c-convex functions [8], (k,h)-convex functions [9, 12, 15], etc. For some details, see below.

(i) *Jensen's inequality* [1]: Let $f : I \to \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$. Then

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i), \tag{8}$$

where $\mathbf{x} = (x_1,x_2,\ldots,x_n) \in S = I^n$ and $p_i \geqslant 0$, $P_n = \sum\limits_{i=1}^{n} p_i > 0$.

(ii) *Jensen-Steffensen inequality* [1] asserts that if $f : I \to \mathbb{R}$ is a convex function on an interval $I \subset \mathbb{R}$, such that $[a,b] \subset I$ with $a < b$, and if $a \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant b$ and

$$0 \leqslant W_i \leqslant W_n\,, \quad W_n > 0 \quad \text{for } i = 1,\ldots,n, \tag{9}$$

where $W_i = \sum\limits_{j=1}^{i} w_j$, $i = 1,\ldots,n$, then

$$f\left(\frac{1}{W_n}\sum_{j=1}^{n} w_j x_j\right) \leqslant \frac{1}{W_n}\sum_{j=1}^{n} w_j f(x_j). \tag{10}$$

Statement (9) is called *Steffensen's condition*. Here $S = \{\mathbf{x} = (x_1,\ldots,x_n) \in I^n : a \leqslant x_1 \leqslant \ldots \leqslant x_n \leqslant b\}$.

(iii) *Brunk inequality* [5]: Let f be a real convex function defined on $[0,x_1]$ with $f(0) \leqslant 0$. Assume that $1 \geqslant h_1 \geqslant h_2 \geqslant \ldots \geqslant h_n \geqslant 0$ and $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant 0$. Then

$$f\left(\sum_{i=1}^{n}(-1)^{i-1}h_i x_i\right) \leqslant \sum_{i=1}^{n}(-1)^{i-1}h_i f(x_i). \tag{11}$$

(iv) *Mercer's inequality* [11, Theorem 1.2]: If f is a real convex function on an interval containing numbers x_i for $i = 1,\ldots,n$, and $0 < x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$, then

$$f(x_1 - \sum_{i=1}^{n} w_i x_i + x_n) \leqslant f(x_1) - \sum_{i=1}^{n} w_i f(x_i) + f(x_n), \tag{12}$$

where $\sum_{i=1}^{n} w_i = 1$ with $w_i > 0$.

(v) *Mercer-Steffensen inequality* [1, Theorem 2]: Let $f : I \to \mathbb{R}$ be a convex function, where I is an interval in $\mathbb{R}$, and let $[a,b] \subset I$ with $a < b$. Let $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{w} = (w_1,\ldots,w_n)$ be real n-tuples. If $a \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant b$ and

$$w_i \neq 0 \quad \text{and} \quad 0 \leqslant W_i \leqslant W_n\,, \quad W_n > 0 \quad \text{for } i = 1,\ldots,n, \tag{13}$$

where $W_i = \sum_{j=1}^{i} w_j$, $i = 1,\ldots,n$, then

$$f(a - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i + b) \leqslant f(a) - \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i) + f(b). \tag{14}$$

(vi) *Generalized Mercer type inequality* [13, Theorem 2.1]: Let $f : I \to \mathbb{R}$ be a continuous convex function on interval $I \subset \mathbb{R}$, $\mathbf{a} = (a_1,\ldots,a_m)$ with $a_j \in I$, and $\mathbf{X} = (x_{ij})$ be a real $n \times m$ matrix such that $x_{ij} \in I$ for all i,j.

If $\mathbf{a}$ majorizes each row of $\mathbf{X}$, i.e., $(x_{i1},\ldots,x_{im}) \prec (a_1,\ldots,a_m)$ for each $i = 1,\ldots,n$, then

$$f\left(\sum_{j=1}^{m} a_j - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i x_{ij}\right) \leqslant \sum_{j=1}^{m} f(a_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} w_i f(x_{ij}), \tag{15}$$

where $\sum_{i=1}^{n} w_i = 1$ with $w_i \geqslant 0$.

(vii) *Toader's c-convex functions* $f : I = [0,b] \to \mathbb{R}$ for $c \in [0,1]$ are defined by the following inequality (see [8]):

$$f(t_1x_1 + \ldots + t_{n-1}x_{n-1} + ct_nx_n) \leqslant t_1 f(x_1) + \ldots + t_{n-1} f(x_{n-1}) + ct_n f(x_n) \tag{16}$$

for all $(x_1,\ldots,x_n) \in S = [0,b]^n$ and $t_1,\ldots,t_n \in [0,1]$ with $t_1 + \ldots + t_n = 1$.

In particular, as noted in [8], (16) gives the standard convexity for $c = 1$, and the starshapedness for $n = 2$ and $c = 0$.

(viii) *Breckner's s-convex functions* $f : I \to \mathbb{R}_+$, $I \subset \mathbb{R}$, $s \in (0,1]$, are defined by the inequality (see [6, p. 254], cf. [18, p. 304]):

$$f(t_1x_1 + \ldots + t_nx_n) \leqslant t_1^s f(x_1) + \ldots + t_n^s f(x_n) \tag{17}$$

for all $(x_1,\ldots,x_n) \in S = I^n$ and $t_1,\ldots,t_n \in (0,1)$ with $t_1 + \ldots + t_n = 1$.

(ix) *P-functions* $f : I \to \mathbb{R}_+$, $I \subset \mathbb{R}$, are defined by the inequality (cf. [16],[18, p. 304]):

$$f(t_1x_1 + \ldots + t_nx_n) \leqslant f(x_1) + \ldots + f(x_n) \tag{18}$$

for all $(x_1,\ldots,x_n) \in S = I^n$ and $t_1,\ldots,t_n \in (0,1)$ with $t_1 + \ldots + t_n = 1$.

(x) *Godunova-Levin functions* $f : I \to \mathbb{R}_+$, $I \subset \mathbb{R}$, are defined by the inequality (cf. [18, p. 303]):

$$f(t_1x_1 + \ldots + t_nx_n) \leqslant \frac{1}{t_1} f(x_1) + \ldots + \frac{1}{t_n} f(x_n) \tag{19}$$

for all $(x_1,\ldots,x_n) \in S = I^n$ and $t_1,\ldots,t_n \in (0,1)$ with $t_1 + \ldots + t_n = 1$.

(xi) Let $h : (0,1) \to \mathbb{R}_+$ be a given function. An *h-convex function* $f : I \to \mathbb{R}_+$, $I \subset \mathbb{R}$, is defined by the inequality (cf. [18, p. 304]):

$$f(t_1x_1 + \ldots + t_nx_n) \leqslant h(t_1) f(x_1) + \ldots + h(t_n) f(x_n) \tag{20}$$

for all $(x_1,\ldots,x_n) \in S = I^n$ and $t_1,\ldots,t_n \in (0,1)$ with $t_1 + \ldots + t_n = 1$.

Now we are at position to state a result of Sherman type for (α,β)-convex functions.

THEOREM 2.1. *Let $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$ be $m \times n$-matrices with ith rows α_i and β_i, respectively, $i = 1,2,\ldots,m$. Let $f : I \to \mathbb{R}$ be an (α_i,β_i)-convex function on a set $\emptyset \neq S \subset I^n$ with an interval $I \subset \mathbb{R}$, i.e.,*

$$f(\langle \alpha_i, \mathbf{x} \rangle) \leqslant \langle \beta_i, f(\mathbf{x}) \rangle \quad \text{for } \mathbf{x} \in S, \quad i = 1,2,\ldots,m. \tag{21}$$

Let $\mathbf{x} = (x_1,x_2,\ldots,x_n) \in S$, $\mathbf{y} = (y_1,y_2,\ldots,y_m) \in I^m$, $\mathbf{a} = (a_1,a_2,\ldots,a_n) \in \mathbb{R}^n$, $\mathbf{b} = (b_1,b_2,\ldots,b_m) \in \mathbb{R}^m_+$.

If

$$\mathbf{y} = \mathbf{x}\mathbf{A}^T \quad \text{and} \quad \mathbf{a} = \mathbf{b}\mathbf{B} \tag{22}$$

then

$$\sum_{i=1}^m b_i f(y_i) \leqslant \sum_{j=1}^n a_j f(x_j). \tag{23}$$

If f is (α_i,β_i)-concave, $i = 1,2,\ldots,m$, then the inequality (23) is reversed.

Proof. By denoting $f(\mathbf{y}) = (f(y_1), f(y_2), \ldots, f(y_m))$ we have

$$\sum_{i=1}^{m} b_i f(y_i) = \langle \mathbf{b}, f(\mathbf{y}) \rangle. \tag{24}$$

Here $\langle \cdot, \cdot \rangle$ stands for the standard inner product on $\mathbb{R}^m$.

From $\mathbf{y} = \mathbf{x}\mathbf{A}^T$ we derive

$$y_i = \sum_{j=1}^{n} a_{ij} x_j, \quad i = 1, 2, \ldots, m.$$

Now, by using (21), we conclude that

$$f(y_i) = f\left(\sum_{j=1}^{n} a_{ij} x_j\right) \leqslant \sum_{j=1}^{n} b_{ij} f(x_j), \quad i = 1, 2, \ldots, m.$$

Hence

$$f(\mathbf{y}) \leqslant_C \left(\sum_{j=1}^{n} b_{1j} f(x_j), \sum_{j=1}^{n} b_{2j} f(x_j), \ldots, \sum_{j=1}^{n} b_{mj} f(x_j)\right) = f(\mathbf{x})\mathbf{B}^T,$$

where $f(\mathbf{x}) = (f(x_1), f(x_2), \ldots, f(x_n))$ and $\leqslant_C$ denotes the componentwise preorder on $\mathbb{R}^m$.

In consequence, from (24) and (22) and by the positivity of $\mathbf{b} \in \mathbb{R}_+^m$, we obtain

$$\sum_{i=1}^{m} b_i f(y_i) = \langle \mathbf{b}, f(\mathbf{y}) \rangle \leqslant \langle \mathbf{b}, f(\mathbf{x})\mathbf{B}^T \rangle = \langle \mathbf{bB}, f(\mathbf{x}) \rangle = \langle \mathbf{a}, f(\mathbf{x}) \rangle = \sum_{j=1}^{n} a_j f(x_j),$$

as required. □

Let $k : (0,1) \to \mathbb{R}$ be a given function and $n \geqslant 2$ be a given positive integer. Then a set $I \subset \mathbb{R}$ is said to be $(k;n)$*-convex* if

$$k(t_1)x_1 + \ldots + k(t_n)x_n \in I$$

for all $x_1, \ldots, x_n \in I$ and $t_1, \ldots, t_n \in (0,1)$ with $t_1 + \ldots + t_n = 1$. (Cf. [12, Def. 2.1], [15]).

Let $k, h : (0,1) \to \mathbb{R}$ be two given functions and $n \geqslant 2$ be a given positive integer. A function $f : I \to \mathbb{R}$ defined on a $(k;n)$-convex set $I \subset \mathbb{R}$ is said to be $(k,h;n)$*-convex* if

$$f(k(t_1)x_1 + \ldots + k(t_n)x_n) \leqslant h(t_1)f(x_1) + \ldots + h(t_n)f(x_n)$$

for all $x_1, \ldots, x_n \in I$ and $t_1, \ldots, t_n \in (0,1)$ with $t_1 + \ldots + t_n = 1$. (Cf. [12, Def. 2.4], [6, p. 254], [15]).

Given a function $g : (0,1) \to \mathbb{R}$, and an $m \times n$ matrix $\mathbf{T} = (t_{ij})$ such that $t_{ij} \in (0,1)$ for all i, j, we define $g(\mathbf{T})$ to be the matrix $(g(t_{ij}))$.

COROLLARY 2.2. *Let $k,h:(0,1)\to\mathbb{R}$ be two given functions and $n\geqslant 2$ be a given positive integer. Assume that f is a real $(k,h;n)$-convex function defined on a $(k;n)$-convex set $I\subset\mathbb{R}$.*

If $\mathbf{x}=(x_1,x_2,\dots,x_n)\in I^n$, $\mathbf{y}=(y_1,y_2,\dots,y_m)\in I^m$, $\mathbf{a}=(a_1,a_2,\dots,a_n)\in\mathbb{R}^n_+$, $\mathbf{b}=(b_1,b_2,\dots,b_m)\in\mathbb{R}^m_+$, and

$$\mathbf{y}=\mathbf{x}(k(\mathbf{T}))^T \quad \text{and} \quad \mathbf{a}=\mathbf{b}(h(\mathbf{T}))$$

for some $m\times n$ row stochastic matrix $\mathbf{T}$, then

$$\sum_{i=1}^{m} b_i f(y_i)\leqslant\sum_{j=1}^{n} a_j f(x_j). \tag{25}$$

Proof. Use Theorem 2.1 with $\mathbf{A}=k(\mathbf{T})$ and $\mathbf{B}=h(\mathbf{T})$. □

EXAMPLE 2.3. By setting $k(t)=t$ and $h(t)=t^s$, $t,s\in(0,1)$, we can apply Corollary 2.2 to Breckner's s-convex functions (see (17)).

Assume that $f:(0,\infty)\to\mathbb{R}_+$ is an s-convex function defined on $I=(0,\infty)$, $s\in(0,1)$.

If $\mathbf{x}=(x_1,x_2,\dots,x_n)\in I^n$, $\mathbf{y}=(y_1,y_2,\dots,y_m)\in I^m$, $\mathbf{a}=(a_1,a_2,\dots,a_n)\in\mathbb{R}^n_+$, $\mathbf{b}=(b_1,b_2,\dots,b_m)\in\mathbb{R}^m_+$, and

$$\mathbf{y}=\mathbf{x}\mathbf{T}^T \quad \text{and} \quad \mathbf{a}=\mathbf{b}\mathbf{T}^s \tag{26}$$

for some $m\times n$ row stochastic matrix $\mathbf{T}=(t_{ij})$, then (25) holds with (26).

3. The case $\mathbf{A}=\mathbf{B}$

Our aim in this section is to give some applications of Theorem 2.1 when $\mathbf{A}=\mathbf{B}$.

REMARK 3.1. Theorem A is a special case of Theorem 2.1 with $\mathbf{A}=\mathbf{B}$ connected with the classical Jensen's inequality (8).

The next result is related to Jensen-Steffensen inequality (10).

COROLLARY 3.2. *Let $f:I\to\mathbb{R}$ be a real convex function defined on an interval $I=[a,b]\subset\mathbb{R}$. Let $\mathbf{x}=(x_1,x_2,\dots,x_n)\in I^n$ with $a\leqslant x_1\leqslant x_2\leqslant\dots\leqslant x_n\leqslant b$, $\mathbf{y}=(y_1,y_2,\dots,y_m)\in I^m$, $\mathbf{a}=(a_1,a_2,\dots,a_n)\in\mathbb{R}^n$, and $\mathbf{b}=(b_1,b_2,\dots,b_m)\in\mathbb{R}^m_+$.*

If $\mathbf{y}=\mathbf{x}\mathbf{A}^T$ and $\mathbf{a}=\mathbf{b}\mathbf{A}$ for some $m\times n$ matrix $\mathbf{A}=(a_{ij})$ with rows satisfying Steffensen's condition:

$$0\leqslant\sum_{j=1}^{k} a_{ij}\leqslant 1=\sum_{j=1}^{n} a_{ij} \quad \text{for } i=1,2,\dots,m,\ k=1,2,\dots,n, \tag{27}$$

then (23) holds.

Proof. According to Jensen-Steffensen inequality (10) for convex f we get that f is (α_i,α_i)-convex, i.e.,

$$f(\langle \alpha_i, \mathbf{x}\rangle) \leqslant \langle \alpha_i, f(\mathbf{x})\rangle, \quad i = 1,2,\ldots,m,$$

where α_i is the ith row of $\mathbf{A}$. Now, it is enough to employ Theorem 2.1 with $\mathbf{A} = \mathbf{B}$. □

In the next example we present a generalization of the inequality of Brunk (11).

EXAMPLE 3.3. Suppose that f is a real convex function defined on interval $I = [0,x_1]$ with $f(0) \leqslant 0$. Let $\mathbf{x} = (x_1,x_2,\ldots,x_n) \in I^n$ with $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n > 0$, and $\mathbf{b} = (b_1,b_2,\ldots,b_m) \in \mathbb{R}_+^m$.

If $\mathbf{A} = (a_{ij}) = ((-1)^{j-1}h_{ij})$ is an $m\times n$ matrix with rows such that

$$1 \geqslant h_{i1} \geqslant h_{i2} \geqslant \ldots \geqslant h_{in} \geqslant 0 \quad \text{for } i = 1,2,\ldots,m,$$

then

$$\sum_{i=1}^{m} b_i f\left(\sum_{j=1}^{n} (-1)^{j-1} h_{ij} x_j\right) \leqslant \sum_{j=1}^{n} a_j f(x_j),$$

where $\mathbf{a} = \mathbf{bA}$, i.e., $a_j = (-1)^{j-1}\sum_{i=1}^{m} b_i h_{ij}$.

In the forthcoming example we use Toader's c-convexity (16) with $n = 2$.

EXAMPLE 3.4. Assume f is a real Toader's c-convex function defined on an interval $I = [0,b) \subset \mathbb{R}$. Let $\mathbf{x} = (x_1,x_2) \in I^2$ and $\mathbf{b} = (b_1,b_2,\ldots,b_m) \in \mathbb{R}_+^m$.

If $\mathbf{A}$ is an $m\times 2$ matrix with rows $\alpha_i = (a_{i1}, ca_{i2})$ such that

$$a_{i1} + a_{i2} = 1, \quad a_{i1}, a_{i2} \geqslant 0 \quad \text{for } i = 1,2,\ldots,m,$$

then

$$\sum_{i=1}^{m} b_i f(a_{i1}x_1 + ca_{i2}x_2) \leqslant a_1 f(x_1) + a_2 f(x_2),$$

where $a_1 = \sum_{i=1}^{m} b_i a_{i1}$ and $a_2 = c\sum_{i=1}^{m} b_i a_{i2}$.

In Corollary 3.5 we provide an extension of Mercer-Steffensen inequality (14).

COROLLARY 3.5. *Let $f : I \to \mathbb{R}$ be a real convex function defined on an interval $I = [a,b] \subset \mathbb{R}$. Let $\mathbf{x} = (x_1,x_2,\ldots,x_n) \in I^n$ with $a \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n \leqslant b$, and $\mathbf{b} = (b_1,b_2,\ldots,b_m) \in \mathbb{R}_+^m$.*

If $\mathbf{A} = (a_{ij})$ is an $m\times n$ matrix with rows satisfying Steffensen's condition (27), then

$$\sum_{i=1}^{m} b_i f\Big(a - \sum_{j=1}^{n} a_{ij}x_j + b\Big) \leqslant \sum_{i=1}^{m} b_i f(a) - \sum_{j=1}^{n} a_j f(x_j) + \sum_{i=1}^{m} b_i f(b), \tag{28}$$

where $a_j = \sum_{i=1}^{m} a_{ij} b_i$.

Proof. By $\mathbf{1}_m$ we denote the (column) m-tuple of ones.

By taking $\mathbf{y} = (y_1, y_2, \ldots, y_m) = \mathbf{x}\mathbf{A}^T \in I^m$, $\mathbf{a} = (a_1, a_2, \ldots, a_n) = \mathbf{b}\mathbf{A} \in \mathbb{R}^n$, $\widetilde{\mathbf{y}} = a\mathbf{1}^T - \mathbf{y} + b\mathbf{1}^T$, $\widetilde{\mathbf{b}} = \mathbf{b}$, and

$$\widetilde{\mathbf{A}} = (\mathbf{1}_m, -\mathbf{A}, \mathbf{1}_m) \quad \text{and} \quad \widetilde{\mathbf{x}} = (a, \mathbf{x}, b) = (a, x_1, x_2, \ldots, x_n, b) \in I^{n+2},$$

we derive $\widetilde{\mathbf{y}} = \widetilde{\mathbf{x}}\widetilde{\mathbf{A}}^T$ and $\widetilde{\mathbf{a}} = \widetilde{\mathbf{b}}\widetilde{\mathbf{A}}$.

Moreover, by Mercer-Steffensen inequality (14) we find that

$$f(a - \langle \alpha_i, \mathbf{x} \rangle + b) \leqslant f(a) - \langle \alpha_i, f(\mathbf{x}) \rangle + f(b), \quad i = 1, 2, \ldots, m,$$

where α_i denotes the ith row of the matrix $\mathbf{A}$. Thus we obtain

$$f(\langle \widetilde{\alpha}_i, \widetilde{\mathbf{x}} \rangle) \leqslant \langle \widetilde{\alpha}_i, f(\widetilde{\mathbf{x}}) \rangle, \quad i = 1, 2, \ldots, m,$$

where $\widetilde{\alpha}_i$ denotes the ith row of the matrix $\widetilde{\mathbf{A}}$. Consequently, f is $(\widetilde{\alpha}_i, \widetilde{\alpha}_i)$-convex.

So, by employing Theorem 2.1 we infer that

$$\sum_{i=1}^{m} \widetilde{b}_i f(\widetilde{y}_i) \leqslant \sum_{j=1}^{n+2} \widetilde{a}_j f(\widetilde{x}_j). \tag{29}$$

Hence, for $a_j = \sum_{i=1}^{m} a_{ij} b_i$,

$$\sum_{i=1}^{m} b_i f(\widetilde{y}_i) \leqslant \left(\sum_{i=1}^{m} b_i \right) f(a) - \sum_{j=1}^{n} a_j f(x_j) + \left(\sum_{i=1}^{m} b_i \right) f(b), \tag{30}$$

since

$$\widetilde{y}_i = a - y_i + b = a - \sum_{j=1}^{n} a_{ij} x_j + b,$$

$$\widetilde{a}_1 = \widetilde{a}_{n+2} = \sum_{i=1}^{m} b_i, \quad \widetilde{a}_{j+1} = -\sum_{i=1}^{m} a_{ij} b_i = -a_j, \quad j = 1, 2, \ldots, n.$$

For this reason inequality (28) follows from (29)-(30), completing the proof. □

We now interpret the last corollary in the context of Mercer's inequality (12).

EXAMPLE 3.6. Let $f : I \to \mathbb{R}$ be a real convex function defined on an interval $I = [a, b] \subset \mathbb{R}$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n$ with $x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n$, and $\mathbf{b} = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m_+$.

If $\mathbf{A} = (a_{ij})$ is an $m \times n$ row stochastic matrix, then

$$\sum_{i=1}^{m} b_i f(x_1 - \sum_{j=1}^{n} a_{ij} x_j + x_n) \leqslant \sum_{i=1}^{m} b_i f(x_1) - \sum_{j=1}^{n} a_j f(x_j) + \sum_{i=1}^{m} b_i f(x_n),$$

where $a_j = \sum_{i=1}^{m} a_{ij} b_i$.

4. Combining (α,β)-convexity and (γ,δ)-convexity

In this section we focus on convex combinations of (α,β)-convexity and (γ,δ)-convexity.

LEMMA 4.1. *Let $f: I \to \mathbb{R}$ be convex on an interval $I \subset \mathbb{R}$. Let α, β, γ and δ be given vectors in $\mathbb{R}^n$. Assume that f is (α,β)-convex and (γ,δ)-convex on a set $S \subset I^n$, i.e.,*

$$f(\langle\alpha,\mathbf{x}\rangle) \leqslant \langle\beta, f(\mathbf{x})\rangle \quad \text{and} \quad f(\langle\gamma,\mathbf{x}\rangle) \leqslant \langle\delta, f(\mathbf{x})\rangle, \quad \mathbf{x}\in S. \tag{31}$$

Then for any $t \in [0,1]$ function f is $(t\alpha+(1-t)\gamma, t\beta+(1-t)\delta)$-convex on S, i.e.,

$$f(\langle t\alpha+(1-t)\gamma,\mathbf{x}\rangle) \leqslant \langle t\beta+(1-t)\gamma, f(\mathbf{x})\rangle, \quad \mathbf{x}\in S.$$

Proof. Fix arbitrarily $t \in [0,1]$ and $\mathbf{x}\in S$. It follows that

$$f(\langle t\alpha+(1-t)\gamma,\mathbf{x}\rangle) = f(t\langle\alpha,\mathbf{x}\rangle+(1-t)\langle\gamma,\mathbf{x}\rangle) \leqslant tf(\langle\alpha,\mathbf{x}\rangle)+(1-t)f(\langle\gamma,\mathbf{x}\rangle)$$
$$\leqslant t\langle\beta,f(\mathbf{x})\rangle+(1-t)\langle\delta,f(\mathbf{x})\rangle = \langle t\beta+(1-t)\delta, f(\mathbf{x})\rangle,$$

as claimed. The former inequality is a consequence of the (standard) convexity of f, and the latter follows from (31). This finishes the proof. □

EXAMPLE 4.2. Suppose that $f: I \to \mathbb{R}$ is convex on an interval $I=[0,x_1]$ and $f(0)\leqslant 0$. By Bellman's inequality (see [2, p. 462]), for $x_1 \geqslant x_2 \geqslant \ldots \geqslant x_n \geqslant 0$, it holds that

$$f(x_1-x_2+x_3-x_4+\ldots+(-1)^{n-1}x_n) \leqslant f(x_1)-f(x_2)+f(x_3)-f(x_4)+\ldots+(-1)^{n-1}f(x_n).$$

In other words, by putting $\alpha=\beta=(1,-1,1,-1,\ldots,(-1)^{n-1})$, we see that f is (α,β)-convex. Furthermore, by Jensen's inequality, f is also (γ,δ)-convex, where $\gamma=\delta=(w_1,w_2,\ldots,w_n)\in[0,1]^n$ with $w_1+w_2+\ldots+w_n=1$.

In summary, in light of Lemma 4.1, for $t\in[0,1]$ the function f is $(t\alpha+(1-t)\gamma, t\beta+(1-t)\delta)$-convex on $S=\{\mathbf{x}\in I^n : x_1\geqslant\ldots\geqslant x_n\}$, i.e., for $\mathbf{x}\in S$,

$$f\left(\sum_{j=1}^{n}((-1)^{j-1}t+(1-t)w_j)x_j\right) \leqslant \sum_{j=1}^{n}((-1)^{j-1}t+(1-t)w_j)f(x_j). \tag{32}$$

It is evident that (32) becomes Jensen's inequality for $t=0$, and Bellman's inequality for $t=1$.

Finally, we demonstrate a Sherman type result based on inequality (32).

COROLLARY 4.3. *Let $f: I\to\mathbb{R}$ be a real convex function defined on an interval $I=[0,x_1]\subset\mathbb{R}$ and $f(0)\leqslant 0$. Let $\mathbf{x}=(x_1,x_2,\ldots,x_n)\in I^n$ with $x_1\geqslant x_2\geqslant\ldots\geqslant x_n\geqslant 0$, and $\mathbf{b}=(b_1,b_2,\ldots,b_m)\in\mathbb{R}^m_+$.*

If $\mathbf{A} = (a_{ij})$ *is an* $m \times n$ *matrix with rows* α_i *of the form*

$$\alpha_i = t(1,-1,1,-1,\ldots,(-1)^{n-1}) + (1-t)(w_{i1},w_{i2},\ldots,w_{in}) \text{ for } i = 1,2,\ldots,m,$$

with $t \in [0,1]$, $w_{i1} + w_{i2} + \ldots + w_{in} = 1$, $w_{ij} \geqslant 0$, *then*

$$\sum_{i=1}^{m} b_i f\left(\sum_{j=1}^{n}((-1)^{j-1}t + (1-t)w_{ij})x_j\right) \leqslant \sum_{j=1}^{n} a_j f(x_j),$$

where $a_j = \sum_{i=1}^{m}\left((-1)^{j-1}t + (1-t)w_{ij}\right)b_i$.

Proof. It is sufficient to apply Theorem 2.1. □

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(Received September 12, 2013)

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