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ON A SYSTEM OF REGULARIZED NONCONVEX VARIATIONAL INEQUALITIES

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Abstract. In the present paper, we point out that the basic result, which is the main tool, in [D. J. Wen, Projection methods for a generalized system of nonconvex variational inequalities with different nonlinear operators, Nonlinear Anal. 73 (2010) 2292–2297] has some fatal errors. Therefore, the results and algorithms in the above mentioned paper are no longer valid. To overcome with the problems in the above mentioned paper, we introduce a system of regularized nonconvex variational inequalities (SRNVI) and establish an equivalence between this system and a fixed point problem. By using this equivalence, we suggest a projection iterative algorithm for solving SRNVI. Furthermore, we also prove the existence and uniqueness of a solution of SRNVI. The convergence analysis of the suggested iterative algorithm is studied. As a consequence, we derive the correct version of the algorithms and results presented in the above mentioned paper.

1. Introduction

In the last two decades, the system of variational inequalities is used as a tool to study the Nash equilibrium problem [1, 2]; See, for example, [3, 4, 5, 6, 7] and the references therein. In 1985, Pang [7] showed that the system of variational inequalities is the model of several equilibrium problems, namely, the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem, and the general equilibrium programming problem, etc. Several existence results for the solutions of the systems of variational inequalities with their applications to Nash equilibrium problem are investigated in [3, 4, 5] and the references therein. In the recent past, several authors considered different kinds of systems of variational inequalities and suggested iterative algorithms to find the approximate solutions of such systems; See, for example, [6, 7, 8, 9, 10, 12, 13] and the references therein. We note that almost all the results regarding the existence of solutions and iterative schemes for solving system of variational inequalities and related problems are being considered in the setting of convex sets. Consequently, the techniques are based on the properties of the projection operator over convex sets, which may not hold in general, when the sets are nonconvex. It is known that the uniformly prox-regular sets are nonconvex and include the convex sets as special cases; See, for example, [14, 15, 16, 17, 18] and the references therein.

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Very recently, Wen [13] introduced and considered a system of nonconvex variational inequalities (SNVI) with different nonlinear operators and asserted that this system is equivalent to the fixed point problem. He used this equivalence formulation to suggest an iterative algorithm for solving the SNVI. The convergence analysis of the proposed iterative algorithm under some certain conditions is also studied.

In this paper, we point out that the equivalence formulation used by Wen [13] is not correct. Therefore, all the results and algorithms in [13] are not correct. To overcome with the problems in [13], we introduce a system of regularized nonconvex variational inequalities (SRNVI) and establish an equivalence between this system and a fixed point problem. By using this equivalence, we suggest a projection iterative algorithm for solving SRNVI. Furthermore, we also prove the existence and uniqueness of a solution of SRNVI. The convergence analysis of the proposed iterative algorithm is studied. As a consequence, we derive the correct version of the algorithms and results presented in [13].

2. Preliminaries and basic results

Throughout the paper, unless otherwise specified, we shall use the following notations, terminology and assumptions. Let $\mathscr{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let K be a nonempty closed subset of $\mathscr{H}$. We denote by $d_K(.)$ or $d(.,K)$ the usual distance function from a point to a set K, that is, $d_K(u) = \inf_{v\in K} \|u-v\|$.

DEFINITION 2.1. Let $u \in \mathscr{H}$ be a point not lying in K. A point $v \in K$ is called *a closest point or a projection of* u *onto* K if $d_K(u) = \|u-v\|$. The set of all such closest points is denoted by $P_K(u)$, that is,

$$P_K(u) := \{v \in K : d_K(u) = \|u-v\|\}.$$

DEFINITION 2.2. The *proximal normal cone* of K at a point $u \in K$ is given by

$$N_K^P(u) := \{\xi \in \mathscr{H} : u \in P_K(u+\alpha\xi)\},$$

where $\alpha > 0$ is a constant.

It can be easily seen $N_K^P(\cdot)$ is a closed set-valued map.

The following lemmas give the characterization of the proximal normal cone.

LEMMA 2.1. [15, Proposition 1.1.5] *Let* K *be a nonempty closed subset of* $\mathscr{H}$. *Then* $\xi \in N_K^P(u)$ *if and only if there exists a constant* $\alpha = \alpha(\xi,u) > 0$ *such that* $\langle \xi, v-u\rangle \leqslant \alpha\|v-u\|^2$ *for all* $v \in K$.

LEMMA 2.2. [15, Proposition 1.1.10] *Let* K *be a nonempty, closed and convex subset of* $\mathscr{H}$. *Then* $\xi \in N_K^P(u)$ *if and only if* $\langle \xi, v-u\rangle \leqslant 0$ *for all* $v \in K$.

DEFINITION 2.3. [19] Let $f:\mathscr{H}\to\mathbb{R}$ be locally Lipschitz near a point x. The *Clarke's directional derivative* of f at x in the direction v, denoted by $f^{\circ}(x;v)$, is defined by

$$f^{\circ}(x;v)=\limsup_{\substack{y\to x\\ t\downarrow 0}}\frac{f(y+tv)-f(y)}{t},$$

where y is a vector in $\mathscr{H}$ and t is a positive scalar.

The *tangent cone* to K at a point $x\in K$, denoted by $T_K(x)$, is defined by

$$T_K(x):=\{v\in\mathscr{H}:d_K^{\circ}(x;v)=0\}.$$

The *normal cone* to K at $x\in K$, denoted by $N_K(x)$, is defined by

$$N_K(x):=\{\xi\in\mathscr{H}:\langle\xi,v\rangle\leqslant 0 \text{ for all } v\in T_K(x)\}.$$

The *Clarke normal cone*, denoted by $N_K^C(x)$, is defined by $N_K^C(x)=\overline{co}[N_K^P(x)]$, where $\overline{co}[S]$ denotes the closure of the convex hull of S.

Clearly, $N_K^P(x)\subseteq N_K^C(x)$. Note that $N_K^C(x)$ is a closed and convex cone, whereas $N_K^P(x)$ is convex, but may not be closed. For further details on this topic, we refer to [19, 15, 18] and the references therein.

In 1995, Clarke et al. [16] introduced a nonconvex set, called *proximally smooth set*. Subsequently, it has been investigated by Poliquin et al. [18] but under the name of uniformly prox-regular set. Such kind of sets are used in many nonconvex applications in optimization, economic models, dynamical systems, differential inclusions, etc. For further details and applications, we refer to [20, 21, 22] and the references therein. This class of nonconvex sets seems particularly well suited to overcome the difficulties which arise due to the nonconvexity assumption.

DEFINITION 2.4. [16] For a given $r\in(0,+\infty]$, a subset K_r of $\mathscr{H}$ is said to be *normalized uniformly prox-regular* (or *uniformly r-prox-regular*) if every nonzero proximal normal to K_r can be realized by an r-ball. This means that for all $\bar{x}\in K_r$ and all $0\neq\xi\in N_{K_r}^P(\bar{x})$,

$$\left\langle\frac{\xi}{\|\xi\|},x-\bar{x}\right\rangle\leqslant\frac{1}{2r}\|x-\bar{x}\|^2,\quad \text{for all } x\in K_r.$$

It is evident that for all $\bar{x}\in K_r$ and all $0\neq\xi\in N_{K_r}^P(\bar{x})$ with $\|\xi\|=1$, we have

$$\langle\xi,x-\bar{x}\rangle\leqslant\frac{1}{2r}\|x-\bar{x}\|^2,\quad \text{for all } x\in K_r.$$

The class of normalized uniformly prox-regular sets includes the class of convex sets, p-convex sets [23], $C^{1,1}$ submanifolds (possibly with boundary) of $\mathscr{H}$, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets [16].

LEMMA 2.3. [16] *A closed set $K\subseteq\mathscr{H}$ is convex if and only if it is uniformly r-prox-regular for every $r>0$.*

If $r = +\infty$, then in view of Definition 2.4 and Lemma 2.3, the uniform r-prox-regularity of K_r is equivalent to the convexity of K_r. That is, for $r = +\infty$, we set $K_r = K$.

The following proposition summarizes some important consequences of the uniform prox-regularity needed in the sequel.

PROPOSITION 2.1. [16, 18] *Let $r > 0$ and K_r be a nonempty closed and uniformly r-prox-regular subset of $\mathscr{H}$. Let $U(r) = \{u \in \mathscr{H} : 0 < d_{K_r}(u) < r\}$. Then the following statements hold:*

(a) For all $x \in U(r)$, $P_{K_r}(x) \neq \emptyset$;

(b) For all $r' \in (0, r)$, P_{K_r} is Lipschitz continuous with constant $\frac{r}{r-r'}$ on $U(r') = \{u \in \mathscr{H} : 0 < d_{K_r}(u) < r'\}$.

Since $N_K^P(\cdot)$ is a closed set-valued map, we have $N_{K_r}^C(x) = N_{K_r}^P(x)$. Therefore, we define $N_{K_r}(x) := N_{K_r}^C(x) = N_{K_r}^P(x)$.

The union of two disjoint intervals $[a,b]$ and $[c,d]$ is uniformly r-prox-regular with $r = \frac{c-b}{2}$ [14, 15, 18]. The finite union of disjoint intervals is also uniformly r-prox-regular and r depends on the distances between the intervals.

3. Formulations and some basic comments

Let K_r be an uniformly r-prox-regular (nonconvex) set, and $g : K_r \to K_r$ be a given mapping. For given two different mappings $T_1, T_2 : K_r \to K_r$, Wen [13] considered the problem of finding $(x^*, y^*) \in K_r \times K_r$ such that

$$\langle \rho T_1(y^*, x^*) + g(x^*) - g(y^*), x - g(x^*) \rangle \geqslant 0, \quad \text{for all } x \in K_r,\ \rho > 0, \tag{3.1}$$

$$\langle \eta T_2(x^*, y^*) + g(y^*) - g(x^*), x - g(y^*) \rangle \geqslant 0, \quad \text{for all } x \in K_r,\ \eta > 0. \tag{3.2}$$

He also considered several special cases of the above system. If $T_1 = T_2 = T : K_r \to K_r$ is an univariate nonlinear operator, $g \equiv I$ (the identity operator) and $x^* = y^* = u$, then the system (3.1)–(3.2) reduces to the following *classical variational inequality* (VI) defined on the nonconvex set K_r: Find $u \in K_r$ such that

$$\langle Tu, v - u \rangle \geqslant 0, \quad \text{for all } v \in K_r. \tag{3.3}$$

Wen [13] has also mentioned that the VI is equivalent to find $u \in K_r$ such that

$$0 \in Tu + N_{K_r}^P(u), \tag{3.4}$$

where $N_{K_r}^P(u)$ denotes the normal cone of K_r at u in the sense of nonconvex analysis. An iterative algorithm is proposed in [13] by utilizing the following lemma.

LEMMA 3.1. [13, Lemma 3.1] *$(x^*, y^*) \in K_r \times K_r$ is a solution of problems (3.1)–(3.2) if and only if*

$$g(x^*) = P_{K_r}[g(y^*) - \rho T_1(y^*, x^*)], \tag{3.5}$$

$$g(y^*) = P_{K_r}[g(x^*) - \eta T_2(x^*, y^*)], \tag{3.6}$$

where P_{K_r} *is the projection of* $\mathscr{H}$ *onto the uniformly* r*-prox-regular set* K_r*.*

By a careful reading, we found that there are two fatal errors in the proof of this lemma. Firstly, in view of Proposition 2.1, it should be pointed that for any $r' \in (0, r)$, the projection of points in the tube $U(r') = \{u \in \mathscr{H} : 0 < d_{K_r}(u) < r'\}$ onto the set K_r exists and unique, that is, for any $x \in U(r')$, the set $P_{K_r}(x)$ is nonempty and singleton. From the equations (3.5)–(3.6) and Proposition 2.1, it follows that two points $g(y^*) - \rho T_1(y^*, x^*)$ and $g(x^*) - \eta T_2(x^*, y^*)$ should be belonged to $U(r')$ for some $r' \in (0, r)$. Unfortunately, it is not necessarily true. Indeed, the equations (3.5)–(3.6) are not necessarily well defined. If $\rho < \frac{r'}{1+\|T_1(y^*,x^*)\|}$ and $\eta < \frac{r'}{1+\|T_2(x^*,y^*)\|}$, for some $r' \in (0, r)$, then we have

$$\begin{aligned} d_{K_r}(g(y^*) - \rho T_1(y^*, x^*)) &\leqslant d_{K_r}(g(y^*)) + \rho \|T_1(y^*, x^*)\| \\ &< \frac{r'\|T_1(y^*, x^*)\|}{1 + \|T_1(y^*, x^*)\|} < r', \end{aligned}$$

because $g(y^*) \in K_r$. Therefore, $g(y^*) - \rho T_1(y^*, x^*) \in U(r')$. Similarly, one can deduce that $g(x^*) - \eta T_2(x^*, y^*) \in U(r')$. Hence, if $\rho < \frac{r'}{1+\|T_1(y^*,x^*)\|}$ and $\eta < \frac{r'}{1+\|T_2(x^*,y^*)\|}$, for some $r' \in (0, r)$, then the equations (3.5)–(3.6) are well defined.

Secondly, we note that Wen [13] used the following system of nonconvex variational inclusions as an equivalence formulation of the system (3.1)–(3.2):

$$\begin{cases} 0 \in \rho T_1(y^*, x^*) + g(x^*) - g(y^*) + \rho N^P_{K_r}(g(x^*)), & \rho > 0 \\ 0 \in \eta T_2(x^*, y^*) + g(y^*) - g(x^*) + \eta N^P_{K_r}(g(y^*)), & \eta > 0. \end{cases} \tag{3.7}$$

Since $N^P_{K_r}(g(x^*))$ and $N^P_{K_r}(g(y^*))$ are cone, the system (3.7) is equivalent to the following system:

$$\begin{cases} 0 \in \rho T_1(y^*, x^*) + g(x^*) - g(y^*) + N^P_{K_r}(g(x^*)), & \rho > 0 \\ 0 \in \eta T_2(x^*, y^*) + g(y^*) - g(x^*) + N^P_{K_r}(g(y^*)), & \eta > 0. \end{cases} \tag{3.8}$$

Therefore, according to the proof of [13, Lemma 3.1], the system (3.1)–(3.2) is equivalent to the system (3.8). Unfortunately, it is not true.

REMARK 3.1. Every solution of the system (3.1)–(3.2) is a solution of the system (3.8), but the converse is not necessarily true.

Proof. Let $(x^*, y^*) \in K_r \times K_r$ be a solution of the system (3.1)–(3.2). Then,

$$\begin{cases} \langle \rho T_1(y^*, x^*) + g(x^*) - g(y^*), x - g(x^*)\rangle \geqslant 0, & \text{for all } x \in K_r,\ \rho > 0 \\ \langle \eta T_2(x^*, y^*) + g(y^*) - g(x^*), x - g(y^*)\rangle \geqslant 0, & \text{for all } x \in K_r,\ \eta > 0. \end{cases} \tag{3.9}$$

It follows from the inequalities (3.9) that for all $\alpha > 0$,

$$\begin{cases} \langle \rho T_1(y^*, x^*) + g(x^*) - g(y^*), x - g(x^*)\rangle + \alpha\|x - g(x^*)\|^2 \geqslant 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for all } x \in K_r,\ \rho > 0 \\ \langle \eta T_2(x^*, y^*) + g(y^*) - g(x^*), x - g(y^*)\rangle + \alpha\|x - g(y^*)\|^2 \geqslant 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for all } x \in K_r,\ \eta > 0. \end{cases} \tag{3.10}$$

The inequalities (3.10) and Lemma 2.1 imply that

$$\begin{cases} -(\rho T_1(y^*,x^*)+g(x^*)-g(y^*)) \in N^P_{K_r}(g(x^*)), & \text{for all } \rho>0,\\ -(\eta T_2(x^*,y^*)+g(y^*)-g(x^*)) \in N^P_{K_r}(g(y^*)), & \text{for all } \eta>0,\end{cases}$$

and therefore,

$$\begin{cases} 0 \in \rho T_1(y^*,x^*)+g(x^*)-g(y^*)+N^P_{K_r}(g(x^*)), & \text{for all } \rho>0,\\ 0 \in \eta T_2(x^*,y^*)+g(y^*)-g(x^*)+N^P_{K_r}(g(y^*)), & \text{for all } \eta>0.\end{cases} \tag{3.11}$$

We see that the converse is not true in general. Indeed, suppose that the inclusions (3.11) holds for some $(x^*,y^*) \in K_r \times K_r$. Then, Lemma 2.1 implies that the system (3.10) hold for some $\alpha>0$. However, by using the system (3.10), we cannot deduce the system (3.9).

The following example illustrates that the system (3.10) does not imply the system (3.9).

EXAMPLE 3.1. Let $\mathscr{H}=\mathbb{R}$ and $K_r=[0,\beta]\cup[\gamma,\delta]$ be the union of two disjoint intervals $[0,\beta]$ and $[\gamma,\delta]$ where $0<\beta<\gamma<\delta$. Then K_r is an uniformly r-prox-regular set with $r=\frac{\gamma-\beta}{2}$. For each $i=1,2$, define $T_i: K_r\times K_r \to K_r$ and $g: K_r \to K_r$ by

$$T_i(x,y)=\theta_i e^{s_i xy} \quad \text{and} \quad g(x)=kx^l, \quad \text{for all } x,y\in K_r,$$

where for each $i=1,2$, $s_i,l\in\mathbb{R}$, $\theta_i<0$ and $\beta^{1-l}\leqslant k<\frac{\gamma}{\beta^l}$ are arbitrary but fixed. Take $x^*=y^*=\beta$, and let $\rho,\eta>0$ and $\alpha \geqslant \max\left\{-\frac{\rho\theta_1 e^{s_1\beta^2}}{\gamma-k\beta^l}, -\frac{\eta\theta_2 e^{s_2\beta^2}}{\gamma-k\beta^l}\right\}$ be arbitrary and fixed. Then, we have

$$\begin{aligned} &\langle \rho T_1(y^*,x^*)+g(x^*)-g(y^*), w-g(x^*)\rangle + \alpha\|w-g(x^*)\|^2\\ &=\rho\theta_1 e^{s_1\beta^2}(w-k\beta^l)+\alpha(w-k\beta^l)^2\\ &=(w-k\beta^l)\left(\alpha(w-k\beta^l)+\rho\theta_1 e^{s_1\beta^2}\right), \quad \text{for all } w\in K_r.\end{aligned} \tag{3.12}$$

If $w\in[0,\beta]$, then $-k\beta^l\leqslant w-k\beta^l\leqslant \beta-k\beta^l=\beta(1-k\beta^{l-1})$ and

$$-k\alpha\beta^l+\rho\theta_1 e^{s_1\beta^2}\leqslant \alpha\left(w-k\beta^l\right)+\rho\theta_1 e^{s_1\beta^2}\leqslant \rho\theta_1 e^{s_1\beta^2}+\alpha\beta\left(1-k\beta^{l-1}\right).$$

For $w\in[\gamma,\delta]$, we have $\gamma-k\beta^l\leqslant w-k\beta^l\leqslant\delta-k\beta^l$ and

$$\alpha\left(\gamma-k\beta^l\right)+\rho\theta_1 e^{s_1\beta^2}\leqslant \alpha\left(w-k\beta^l\right)+\rho\theta_1 e^{s_1\beta^2}\leqslant \alpha\left(\delta-k\beta^l\right)+\rho\theta_1 e^{s_1\beta^2},$$

and therefore,

$$\left(w-k\beta^l\right)\left(\alpha\left(w-k\beta^l\right)+\rho\theta_1 e^{s_1\beta^2}\right)\geqslant 0, \quad \text{for all } w\in K_r. \tag{3.13}$$

By (3.12) and (3.13), it follows that

$$\langle \rho T_1(y^*,x^*)+g(x^*)-g(y^*), w-g(x^*)\rangle+\alpha\|w-g(x^*)\|^2\geqslant 0, \quad \text{for all } w\in K_r.$$

However, it is obvious that $\rho\theta_1 e^{s_1\beta^2}(w-k\beta^l)<0$ for all $w\in[\gamma,\delta]$, that is,

$$\langle \rho T_1(y^*,x^*)+g(x^*)-g(y^*),w-g(x^*)\rangle < 0, \quad \text{for all } w\in[\gamma,\delta].$$

Hence, the inequality

$$\langle \rho T_1(y^*,x^*)+g(x^*)-g(y^*),w-g(x^*)\rangle \geqslant 0,$$

cannot hold for all $w\in K_r$.

Similarly, we have

$$\langle \eta T_2(x^*,y^*)+g(y^*)-g(x^*),w-g(y^*)\rangle+\alpha\|w-g(y^*)\|^2 \geqslant 0, \quad \text{for all } w\in K_r,$$

while the inequality

$$\langle \eta T_2(x^*,y^*)+g(y^*)-g(x^*),w-g(y^*)\rangle \geqslant 0$$

cannot hold for all $w\in K_r$.

Similarly, we can see that every solution of (3.3) is a solution of (3.4), but the converse is not true in general.

In view of the above remark, the results in [13] and in the papers where the same technique and method are used, are no longer valid.

Instead of the system (3.1)–(3.2) of [13], in this paper, we consider another system of nonconvex variational inequalities called *system of regularized nonconvex variational inequalities* (SRNVI). We prove the equivalence between SRNVI and the system of nonconvex variational inclusions (3.8) as well as the fixed point problems (3.5)–(3.6).

For given nonlinear mappings $T_1,T_2:\mathscr{H}\times\mathscr{H}\to\mathscr{H}$ and $g:\mathscr{H}\to\mathscr{H}$, and the constants $\rho,\eta>0$, we consider the following *system of regularized nonconvex variational inequalities* (SRNVI): Find $(x^*,y^*)\in\mathscr{H}\times\mathscr{H}$ such that $(g(x^*),g(y^*))\in K_r\times K_r$ and

$$\begin{cases} \langle \rho T_1(y^*,x^*)+g(x^*)-g(y^*),x-g(x^*)\rangle+\frac{\|\rho T_1(y^*,x^*)+x^*-y^*\|}{2r}\|x-g(x^*)\|^2 \geqslant 0, \\ \qquad\qquad \text{for all } x\in K_r, \\ \langle \eta T_2(x^*,y^*)+g(y^*)-g(x^*),x-g(y^*)\rangle+\frac{\|\eta T_2(x^*,y^*)+g(y^*)-g(x^*)\|}{2r}\|x-g(y^*)\|^2 \geqslant 0, \\ \qquad\qquad \text{for all } x\in K_r. \end{cases} \tag{3.14}$$

If $g\equiv I$, then the system (3.14) collapses to the following system of finding $(x^*,y^*)\in K_r\times K_r$ such that

$$\begin{cases} \langle \rho T_1(y^*,x^*)+x^*-y^*,x-x^*\rangle+\frac{\|\rho T_1(y^*,x^*)+x^*-y^*\|}{2r}\|x-x^*\|^2 \geqslant 0, \quad \text{for all } x\in K_r, \\ \langle \eta T_2(x^*,y^*)+y^*-x^*,x-y^*\rangle+\frac{\|\eta T_2(x^*,y^*)+g(y^*)-g(x^*)\|}{2r}\|x-y^*\|^2 \geqslant 0, \quad \text{for all } x\in K_r, \end{cases} \tag{3.15}$$

which appears to be a new system of regularized nonconvex variational inequalities.

If $r=\infty$, that is, $K_r=K$, the convex set in $\mathscr{H}$, then the system (3.15) reduces to the following system of finding $(x^*,y^*)\in K\times K$ such that

$$\begin{cases} \langle \rho T_1(y^*,x^*)+x^*-y^*, x-x^*\rangle \geqslant 0, & \text{for all } x\in K,\\ \langle \eta T_2(x^*,y^*)+y^*-x^*, x-y^*\rangle \geqslant 0, & \text{for all } x\in K, \end{cases} \tag{3.16}$$

which is considered and studied in [9].

If $r=\infty$ and $T_1=T_2=T$, then the system (3.15) considered and studied by Chang et al. [8] and Verma [10].

If $r=\infty$ and $T_1=T_2=T:\mathscr{H}\to\mathscr{H}$ is an univariate nonlinear operator, then the system (3.15) reduces to the system of variational inequalities considered in [12].

If $T_1=T_2=T:\mathscr{H}\to\mathscr{H}$ is an univariate nonlinear operator, $x^*=y^*$ and $\rho=\eta$, then the system (3.15) becomes the problem of finding $x^*\in K_r$ such that

$$\langle \rho T(x^*), x-x^*\rangle + \frac{\|\rho T(x^*)\|}{2r}\|x-x^*\|^2 \geqslant 0, \quad \text{for all } x\in K_r, \tag{3.17}$$

which appears to be a new problem of nonlinear regularized nonconvex variational inequality.

If $r=\infty$, then the problem (3.17) reduces to the classical variational inequality.

In the next proposition, the equivalence between the system of regularized nonconvex variational inequalities (3.14) and the system of nonconvex variational inclusions (3.8) is established.

PROPOSITION 3.1. *Let K_r be an uniformly r-prox-regular set. The system (3.14) is equivalent to the system (3.8).*

Proof. Let $(x^*,y^*)\in\mathscr{H}\times\mathscr{H}$ with $(g(x^*),g(y^*))\in K_r\times K_r$ be a solution of the system (3.14). If $\rho T_1(y^*,x^*)+g(x^*)-g(y^*)=0$, then $0\in\rho T_1(y^*,x^*)+g(x^*)-g(y^*)+N^P_{K_r}(g(x^*))$ because the vector zero always belongs to any normal cone. If $\rho T_1(y^*,x^*)+g(x^*)-g(y^*)\neq 0$, then for all $x\in K_r$, we have

$$\langle -(\rho T_1(y^*,x^*)+g(x^*)-g(y^*)), x-g(x^*)\rangle \leqslant \frac{\|\rho T_1(y^*,x^*)+g(x^*)-g(y^*)\|}{2r}\|x-g(x^*)\|^2.$$

By Lemma 2.1, we have

$$-(\rho T_1(y^*,x^*)+g(x^*)-g(y^*))\in N^P_{K_r}(g(x^*)),$$

which implies that

$$0\in\rho T_1(y^*,x^*)+g(x^*)-g(y^*)+N^P_{K_r}(g(x^*)).$$

Similarly, we obtain

$$0\in\eta T_2(x^*,y^*)+g(y^*)-g(x^*)+N^P_{K_r}(g(y^*)).$$

Conversely, if $(x^*,y^*)\in\mathscr{H}\times\mathscr{H}$ with $(g(x^*),g(y^*))\in K_r\times K_r$ is a solution of the system (3.8), then it follows from Definition 2.4 that $(x^*,y^*)\in\mathscr{H}\times\mathscr{H}$ with $(g(x^*),g(y^*))\in K_r\times K_r$ is a solution of the system (3.14). □

By using the projection operator technique, we establish the equivalence between the system (3.14) and the fixed point problems (3.5)–(3.6).

LEMMA 3.2. *Let* T_1, T_2, g, ρ *and* η *be the same as in the system* (3.14)*. Then* $(x^*,y^*)\in\mathscr{H}\times\mathscr{H}$ *with* $(g(x^*),g(y^*))\in K_r\times K_r$ *is a solution of the system* (3.14) *if and only if* (x^*,y^*) *satisfies the system* (3.5)–(3.6) *provided that* $\rho<\frac{r'}{1+\|T_1(y^*,x^*)\|}$ *and* $\eta<\frac{r'}{1+\|T_2(x^*,y^*)\|}$ *for some* $r'\in(0,r)$*.*

Proof. Let $(x^*,y^*)\in\mathscr{H}\times\mathscr{H}$ with $(g(x^*),g(y^*))\in K_r\times K_r$ be a solution of the system (3.14). Since $g(x^*),g(y^*)\in K_r$, $\rho<\frac{r'}{1+\|T_1(y^*,x^*)\|}$ and $\eta<\frac{r'}{1+\|T_2(x^*,y^*)\|}$, it follows that the equations (3.5)–(3.6) are well defined. By using the well-known fact that $P_{K_r}=(I+N^P_{K_r})^{-1}$ and Proposition 3.1, we obtain

$$\begin{aligned}
&0\in\rho T_1(y^*,x^*)+g(x^*)-g(y^*)+N^P_{K_r}(g(x^*))\\
&\Leftrightarrow g(y^*)-\rho T_1(y^*,x^*)\in g(x^*)+N^P_{K_r}(g(x^*))\\
&\Leftrightarrow g(y^*)-\rho T_1(y^*,x^*)\in(I+N^P_{K_r})(g(x^*))\\
&\Leftrightarrow g(x^*)=P_{K_r}(g(y^*)-\rho T_1(y^*,x^*)),
\end{aligned}$$

where I is the identity operator. Similarly, we deduce

$$\begin{aligned}
&0\in\eta T_2(x^*,y^*)+g(y^*)-g(x^*)+N^P_{K_r}(g(y^*))\\
&\Leftrightarrow g(y^*)=P_{K_r}(g(x^*)-\eta T_2(x^*,y^*)).
\end{aligned}$$

This completes the proof. □

4. Existence and uniqueness of a solution and convergence analysis

DEFINITION 4.1. A mapping $T:\mathscr{H}\to\mathscr{H}$ is said to be

(a) *monotone in the first variable* if for all $x,y\in\mathscr{H}$,

$$\langle T(x,u)-T(y,v),x-y\rangle\geqslant 0,\quad\text{for all } u,v\in\mathscr{H};$$

(b) r*-strongly monotone in the first variable* if there exists a constant $r>0$ such that for all $x,y\in\mathscr{H}$,

$$\langle T(x,u)-T(y,v),x-y\rangle\geqslant r\|x-y\|^2,\quad\text{for all } u,v\in\mathscr{H};$$

(c) (ξ,ρ)*-relaxed cocoercive in the first variable* if there exist two constants $\xi,\rho>0$ such that for all $x,y\in\mathscr{H}$,

$$\langle T(x,u)-T(y,v),x-y\rangle\geqslant-\xi\|T(x,u)-T(y,v)\|^2+\rho\|x-y\|^2,\quad\text{for all } u,v\in\mathscr{H};$$

(d) μ*-Lipschitz continuous in the first variable* if there exists a constant $\mu>0$ such that for all $x,y\in\mathscr{H}$,

$$\|T(x,u)-T(y,v)\|\leqslant\mu\|x-y\|,\quad\text{for all } u,v\in\mathscr{H}.$$

DEFINITION 4.2. A mapping $g : \mathscr{H} \to \mathscr{H}$ is said to be

(a) κ-*strongly monotone* if there exists a constant $\kappa > 0$ such that

$$\langle g(x) - g(y), x - y\rangle \geqslant \kappa \|x - y\|^2, \quad \text{for all } x, y \in \mathscr{H};$$

(b) γ-*Lipschitz continuous* if there exists a constant $\gamma > 0$ such that

$$\|g(x) - g(y)\| \leqslant \gamma \|x - y\|, \quad \text{for all } x, y \in \mathscr{H}.$$

Now, we prove the existence of a unique solution of the system of regularized nonconvex variational inequalities (3.14).

THEOREM 4.1. *Let the mappings T_i, g $(i = 1,2)$ and the constants ρ and η be the same as in the system (3.14) such that $g(\mathscr{H}) \subseteq K_r$. For each $i = 1,2$, let T_i be ξ_i-strongly monotone and μ_i-Lipschitz continuous in the first variable, and let g be ξ_3-strongly monotone and μ_3-Lipschitz continuous. If the constants ρ and η satisfy the following conditions*

$$\rho < \frac{r'}{1 + \|T_1(y,x)\|} \quad \text{and} \quad \eta < \frac{r'}{1 + \|T_2(x,y)\|}, \quad \text{for some } r' \in (0,r) \text{ and } \forall x,y \in \mathscr{H} \tag{4.18}$$

and

$$\begin{cases} \left|\rho - \frac{\xi_1}{\mu_1^2}\right| < \dfrac{\sqrt{\delta^2\xi_1^2 - \mu_1^2(\delta^2 - (1-(1+\delta)k)^2)}}{\delta\mu_1^2}, \\ \left|\eta - \frac{\xi_2}{\mu_2^2}\right| < \dfrac{\sqrt{\delta^2\xi_2^2 - \mu_2^2(\delta^2 - (1-(1+\delta)k)^2)}}{\delta\mu_2^2}, \\ \xi_i > \dfrac{\sqrt{\delta^2 - (1-(1+\delta)k)^2}}{\delta}\mu_i, \quad i = 1,2, \\ k = \sqrt{1 - 2\xi_3 + \mu_3^2} < 1, \quad 2\xi_3 \leqslant 1 + \mu_3^2, \end{cases} \tag{4.19}$$

$\delta = \frac{r}{r-r'}$*, then the system (3.14) admits a unique solution.*

Proof. Define $\psi, \phi : \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ by

$$\begin{aligned} \psi(x,y) &= x - g(x) + P_{K_r}(g(y) - \rho T_1(y,x)), \\ \phi(x,y) &= y - g(y) + P_{K_r}(g(x) - \eta T_2(x,y)), \end{aligned} \tag{4.20}$$

for all $x, y \in \mathscr{H}$. Define $\|.\|_*$ on $\mathscr{H} \times \mathscr{H}$ by

$$\|(x,y)\|_* = \|x\| + \|y\|, \quad \text{for all } (x,y) \in \mathscr{H} \times \mathscr{H}.$$

It is obvious that $(\mathscr{H} \times \mathscr{H}, \|.\|_*)$ is a Hilbert space. Also, define $F : \mathscr{H} \times \mathscr{H} \to \mathscr{H} \times \mathscr{H}$ by

$$F(x,y) = (\psi(x,y), \phi(x,y)), \quad \text{for all } (x,y) \in \mathscr{H} \times \mathscr{H}. \tag{4.21}$$

We claim that F is a contraction mapping. Indeed, let $(x,y),(\hat{x},\hat{y}) \in \mathscr{H} \times \mathscr{H}$ be given. Since $g(y) \in K_r$ and $\rho < \frac{r'}{1+\|T_1(y,x)\|}$, for some $r' \in (0,r)$, it follows that $g(y) - \rho T_1(y,x) \in U(r')$ and the r-prox-regularity of K_r implies that $P_{K_r}(g(y) - \rho T_1(y,x)$ exists and unique. Similarly, we can deduce that $P_{K_r}(g(x) - \rho T_2(x,y)$ exists and unique. By using Proposition 2.1, we have

$$\begin{aligned} &\|\psi(x,y) - \psi(\hat{x},\hat{y})\| \\ &= \|x - g(x) + P_{K_r}(g(y) - \rho T_1(y,x)) - (\hat{x} - g(\hat{x}) + P_{K_r}(g(\hat{y}) - \rho T_1(\hat{y},\hat{x})))\| \\ &\leqslant \|x - \hat{x} - (g(x) - g(\hat{x}))\| + \delta\|g(y) - g(\hat{y}) - \rho(T_1(y,x) - T_1(\hat{y},\hat{x}))\| \\ &\leqslant \|x - \hat{x} - (g(x) - g(\hat{x}))\| \\ &\quad + \delta\left(\|y - \hat{y} - (g(y) - g(\hat{y}))\| + \|y - \hat{y} - \rho(T_1(y,x) - T_1(\hat{y},\hat{x}))\|\right), \end{aligned} \tag{4.22}$$

where $\delta = \frac{r}{r-r'}$. By ξ_3-strongly monotonicity and μ_3-Lipschitz continuity of g, we have

$$\begin{aligned} \|x - \hat{x} - (g(x) - g(\hat{x}))\|^2 &= \|x - \hat{x}\|^2 - 2\langle g(x) - g(\hat{x}), x - \hat{x}\rangle + \|g(x) - g(\hat{x})\|^2 \\ &\leqslant (1 - 2\xi_3 + \mu_3^2)\|x - \hat{x}\|^2, \end{aligned}$$

which leads to

$$\|x - \hat{x} - (g(x) - g(\hat{x}))\| \leqslant \sqrt{1 - 2\xi_3 + \mu_3^2}\|x - \hat{x}\|. \tag{4.23}$$

Similarly, we obtain

$$\|y - \hat{y} - (g(y) - g(\hat{y}))\| \leqslant \sqrt{1 - 2\xi_3 + \mu_3^2}\|y - \hat{y}\|. \tag{4.24}$$

Since T_1 is ξ_1-strongly monotone and μ_1-Lipschitz continuous in the first variable, we get

$$\begin{aligned} &\|y - \hat{y} - \rho(T_1(y,x) - T_1(\hat{y},\hat{x}))\|^2 \\ &= \|y - \hat{y}\|^2 - 2\rho\langle T_1(y,x) - T_1(\hat{y},\hat{x}), y - \hat{y}\rangle + \rho^2\|T_1(y,x) - T_1(\hat{y},\hat{x})\|^2 \\ &\leqslant (1 - 2\rho\xi_1 + \rho^2\mu_1^2)\|y - \hat{y}\|^2. \end{aligned} \tag{4.25}$$

By (4.22)–(4.25), we have

$$\|\psi(x,y) - \psi(\hat{x},\hat{y})\| \leqslant k\|x - \hat{x}\| + \theta_1\|y - \hat{y}\|, \tag{4.26}$$

where

$$k = \sqrt{1 - 2\xi_3 + \mu_3^2}, \qquad \text{and} \qquad \theta_1 = \delta\left(k + \sqrt{1 - 2\rho\xi_1 + \rho^2\mu_1^2}\right).$$

Since T_2 is ξ_2-strongly monotone and μ_2-Lipschitz continuous in the first variable, and g is ξ_3-strongly monotone and μ_3-Lipschitz continuous, in a similar way, we have

$$\|\phi(x,y) - \phi(\hat{x},\hat{y})\| \leqslant \theta_2\|x - \hat{x}\| + k\|y - \hat{y}\|, \tag{4.27}$$

where

$$\theta_2=\delta\left(k+\sqrt{1-2\eta\xi_2+\eta^2\mu_2^2}\right).$$

It follows from (4.21), (4.26) and (4.27) that

$$\begin{aligned}\|F(x,y)-F(\hat{x},\hat{y})\|_*&=\|\psi(x,y)-\psi(\hat{x},\hat{y})\|+\|\phi(x,y)-\phi(\hat{x},\hat{y})\|\\&\leqslant(k+\theta_2)\|x-\hat{x}\|+(k+\theta_1)\|y-\hat{y}\|\\&\leqslant\vartheta\|(x,y)-(\hat{x},\hat{y})\|_*,\end{aligned}\tag{4.28}$$

where $\vartheta=\max\{k+\theta_i:i=1,2\}$. The condition (4.19) implies that $0\leqslant\vartheta<1$, and so, (4.28) guarantees that F is a contraction mapping.

By Banach fixed point theorem, there exists a unique point $(x^*,y^*)\in\mathscr{H}\times\mathscr{H}$ such that $F(x^*,y^*)=(x^*,y^*)$. From (4.20) and (4.21), we conclude that

$$g(x^*)=P_{K_r}(g(y^*)-\rho T_1(y^*,x^*))$$

and

$$g(y^*)=P_{K_r}(g(x^*)-\eta T_2(x^*,y^*)),$$

because of the choice of the constants ρ and η, in similar way, deduce that the above equations are well defined. Now, Lemma 3.2 guarantees that $(x^*,y^*)\in\mathscr{H}\times\mathscr{H}$ with $(g(x^*),g(y^*))\in K_r\times K_r$ is a solution of the system (3.14). This completes the proof. □

COROLLARY 4.1. *Suppose that the mappings T_i $(i=1,2)$ and the constants ρ and η are the same as in the system (3.15) and assume that for each $i=1,2$, the mapping T_i is ξ_i-strongly monotone and μ_i-Lipschitz continuous in the first variable. If the constants ρ and η satisfy the condition (4.18) and the following conditions*

$$\begin{cases}\left|\rho-\frac{\xi_1}{\mu_1^2}\right|<\frac{\sqrt{\delta^2\xi_1^2-\mu_1^2(\delta^2-1)}}{\delta\mu_1^2},\\\left|\eta-\frac{\xi_2}{\mu_2^2}\right|<\frac{\sqrt{\delta^2\xi_2^2-\mu_2^2(\delta^2-1)}}{\delta\mu_2^2},\\\xi_i>\frac{\sqrt{\delta^2-1}}{\delta}\mu_i,\quad i=1,2,\end{cases}\tag{4.29}$$

where $\delta=\frac{r}{r-r'}$, then the system (3.15) admits a unique solution.

COROLLARY 4.2. *Let the mappings T_i $(i=1,2)$ and the constants ρ and η be the same as in the system (3.16) such that for each $i=1,2$, the mapping T_i is ξ_i-strongly monotone and μ_i-Lipschitz continuous in the first variable. If the constants ρ and η satisfy the following conditions*

$$\left|\rho-\frac{\xi_1}{\mu_1^2}\right|<\frac{\xi_1}{\mu_1^2}\quad\text{and}\quad\left|\eta-\frac{\xi_2}{\mu_2^2}\right|<\frac{\xi_2}{\mu_2^2},\tag{4.30}$$

then the system (3.16) admits a unique solution.

COROLLARY 4.3. *Assume that the mapping T and the constant ρ are the same as in the problem (3.17) and suppose that the mapping T is ξ-strongly monotone and μ-Lipschitz continuous. If the constant ρ satisfies the following conditions*

$$\rho < \frac{r'}{1+\|T(x)\|}, \qquad \textit{for some } r' \in (0,r) \textit{ and } \forall x \in \mathscr{H}, \tag{4.31}$$

and

$$\left|\rho - \frac{\xi}{\mu^2}\right| < \frac{\sqrt{\delta^2\xi^2 - \mu^2(\delta^2-1)}}{\delta\mu^2} \quad \text{and} \quad \xi > \frac{\sqrt{\delta^2-1}}{\delta}\mu, \tag{4.32}$$

where $\delta = \frac{r}{r-r'}$, *then the problem* (3.17) *admits a unique solution.*

By utilizing Lemma 3.2, we suggest and analyze the following explicit projection iterative method for solving the system of regularized nonconvex variational inequalities (3.14).

ALGORITHM 4.1. Let the mappings T_i $(i=1,2)$, g and the constants ρ and η be the same as in the system (3.14) such that $g(\mathscr{H}) \subseteq K_r$. For arbitrary initial points $x_0, y_0 \in \mathscr{H}$, compute the sequences $\{x_n\}$ and $\{y_n\}$ in $\mathscr{H}$ in the following way:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n[x_n - g(x_n) + P_{K_r}(g(y_n) - \rho T_1(y_n,x_n))], \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n[y_n - g(y_n) + P_{K_r}(g(x_n) - \eta T_2(x_n,y_n))], \end{cases} \tag{4.33}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$.

ALGORITHM 4.2. Let the mappings T_i $(i=1,2)$ and the constants ρ and η be the same as in the system (3.15). For arbitrary initial points $x_0, y_0 \in K_r$, compute the sequences $\{x_n\}$ and $\{y_n\}$ in K_r in the following way:

$$\begin{cases} x_{n+1} = P_{K_r}(y_{n+1} - \rho T_1(y_{n+1},x_n)), \\ y_{n+1} = P_{K_r}(x_n - \eta T_2(x_n,y_n)), \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$.

ALGORITHM 4.3. Let the mappings T_i $(i=1,2)$ and the constants ρ and η be the same as in the system (3.16). For arbitrary initial points $x_0, y_0 \in K$, compute the sequences $\{x_n\}$ and $\{y_n\}$ in the following way:

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_K(y_n - \rho T_1(y_n,x_n)), \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n P_K(x_n - \eta T_2(x_n,y_n)), \end{cases}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$.

ALGORITHM 4.4. Let the mapping T and the constant ρ be the same as in the problem (3.17). For an arbitrary initial point $x_0 \in K_r$, compute the sequence $\{x_n\}$ in K_r by the following iterative process:

$$x_{n+1} = P_{K_r}(x_n - \rho T(x_n)),$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$.

Now, we prove the strong convergence of the sequences generated by Algorithm 4.1 to a unique solution of the system (3.14).

THEOREM 4.2. *Let the mappings $T_i,\ g\ (i=1,2)$ and the constants ρ and η be the same as in Theorem 4.1, and let all the conditions of Theorem 4.1 hold. If the constants ρ and η satisfy the conditions (4.18) and (4.19) and $\sum_{n=0}^{\infty}\alpha_n=\infty$, then the iterative sequence $\{(x_n,y_n)\}$ generated by Algorithm 4.1 converges strongly to a unique solution (x^*,y^*) of the system (3.14).*

Proof. Theorem 4.1 guarantees the existence of a unique solution (x^*,y^*) in $\mathscr{H}\times\mathscr{H}$ with $(g(x^*),g(y^*))$ in $K_r\times K_r$ for the system (3.14). Since $\rho<\frac{r'}{1+\|T_1(y^*,x^*)\|}$ and $\eta<\frac{r'}{1+\|T_2(x^*,y^*)\|}$, for some $r'\in(0,r)$, it follows from Lemma 3.2 that (x^*,y^*) satisfies the system (3.5)–(3.6). Then for each $n\geqslant 0$, we have

$$\begin{cases} x^*=(1-\alpha_n)x^*+\alpha_n[x^*-g(x^*)+P_{K_r}(g(y^*)-\rho T_1(y^*,x^*))],\\ y^*=(1-\alpha_n)y^*+\alpha_n[y^*-g(y^*)+P_{K_r}(g(x^*)-\eta T_2(x^*,y^*))],\end{cases} \tag{4.34}$$

where the sequence $\{\alpha_n\}$ is the same as in Algorithm 4.1. Since for each $n\in\mathbb{N}$, $g(y^*),g(y_n)\in K_r$, $\rho<\frac{r'}{1+\|T_1(y^*,x^*)\|}$ and $\rho<\frac{r'}{1+\|T_1(y_n,x_n)\|}$, it is easy to check that for each $n\in\mathbb{N}$, $g(y^*)-\rho T_1(y^*,x^*),g(y_n)-\rho T_1(y_n,x_n)\in U(r')$. From (4.33), (4.34) and Proposition 2.1, we obtain

$$\begin{aligned} &\|x_{n+1}-x^*\|\\ &\leqslant(1-\alpha_n)\|x_n-x^*\|+\alpha_n\Big(\|x_n-x^*-(g(x_n)-g(x^*))\|\\ &\quad+\|P_{K_r}(g(y_n)-\rho T_1(y_n,x_n))-P_{K_r}(g(y^*)-\rho T_1(y^*,x^*))\|\Big)\\ &\leqslant(1-\alpha_n)\|x_n-x^*\|+\alpha_n\Big(\|x_n-x^*-(g(x_n)-g(x^*))\|\\ &\quad+\delta\|g(y_n)-g(y^*)-\rho(T_1(y_n,x_n)-T_1(y^*,x^*))\|\Big)\\ &\leqslant(1-\alpha_n)\|x_n-x^*\|+\alpha_n\Big(\|x_n-x^*-(g(x_n)-g(x^*))\|\\ &\quad+\delta(\|y_n-y^*-(g(y_n)-g(y^*))\|+\|y_n-y^*-\rho(T_1(y_n,x_n)-T_1(y^*,x^*))\|\Big).\end{aligned} \tag{4.35}$$

Since T_1 is ξ_1-strongly monotone and μ_1-Lipschitz continuous in the first variable, and g is ξ_3-strongly monotone and μ_3-Lipschitz continuous, in a similar way, we have

$$\|x_n-x^*-(g(x_n)-g(x^*))\|\leqslant\sqrt{1-2\xi_3+\mu_3^2}\|x_n-x^*\|, \tag{4.36}$$

$$\|y_n-y^*-(g(y_n)-g(y^*))\|\leqslant\sqrt{1-2\xi_3+\mu_3^2}\|y_n-y^*\| \tag{4.37}$$

and

$$\|y_n-y^*-\rho(T_1(y_n,x_n)-T_1(y^*,x^*))\|\leqslant\sqrt{1-2\rho\xi_1+\rho^2\mu_1^2}\|y_n-y^*\|. \tag{4.38}$$

Combining (4.35)–(4.38), we obtain

$$\|x_{n+1}-x^*\| \leqslant (1-\alpha_n)\|x_n-x^*\| + \alpha_n\left(k\|x_n-x^*\| + \theta_1\|y_n-y^*\|\right), \tag{4.39}$$

where k and θ_1 are the same as in (4.26). By $g(x^*), g(x_n) \in K_r$ $(n \in \mathbb{N})$, $\eta < \frac{r'}{1+\|T_2(x^*,y^*)\|}$ and $\eta < \frac{r'}{1+\|T_2(x_n,y_n)\|}$, it follows that for each $n \in \mathbb{N}$, $g(x^*) - \eta T_2(x^*,y^*)$, $g(x_n) - \eta T_2(x_n,y_n) \in U(r')$. Since T_2 is ξ_2-strongly monotone and μ_2-Lipschitz continuous in the first variable, and g is ξ_3-strongly monotone and μ_3-Lipschitz continuous, in a similar way as that of proof (4.35)–(4.39), we get

$$\|y_{n+1}-y^*\| \leqslant (1-\alpha_n)\|y_n-y^*\| + \alpha_n\left(k\|y_n-y^*\| + \theta_2\|x_n-x^*\|\right), \tag{4.40}$$

where θ_2 is the same as in (4.27). By (4.39) and (4.40), we have

$$\begin{aligned}
&\|(x_{n+1},y_{n+1})-(x^*,y^*)\|_* \\
&\leqslant (1-\alpha_n)\|(x_n,y_n)-(x^*,y^*)\|_* + \alpha_n\left((k+\theta_2)\|x_n-x^*\| + (k+\theta_1)\|y_n-y^*\|\right) \\
&\leqslant (1-\alpha_n)\|(x_n,y_n)-(x^*,y^*)\|_* + \alpha_n\vartheta\|(x_n,y_n)-(x^*,y^*)\|_* \\
&= (1-(1-\vartheta)\alpha_n)\|(x_n,y_n)-(x^*,y^*)\|_* \\
&\leqslant \prod_{i=0}^{n}(1-(1-\vartheta)\alpha_i)\|(x_0,y_0)-(x^*,y^*)\|_*,
\end{aligned} \tag{4.41}$$

where ϑ is the same as in (4.28). The condition (4.19) guarantees that $\vartheta \in (0,1)$. Since $\sum_{n=0}^{\infty}\alpha_n = \infty$, we get

$$\lim_{n\to\infty}\prod_{i=0}^{n}(1-(1-\vartheta)\alpha_i) = 0. \tag{4.42}$$

It follows from (4.41) and (4.42) that $\|(x_n,y_n)-(x^*,y^*)\|_* \to 0$, as $n \to \infty$, and so the sequence $\{(x_n,y_n)\}$ generated by Algorithm 4.1, converges strongly to a unique solution (x^*,y^*) of the system (3.14). This completes the proof. □

Similarly, we can prove the convergence of iterative sequences generated by Algorithm 4.2.

COROLLARY 4.4. *Assume that the mappings T_i $(i=1,2)$ and the constants ρ and η are the same as in Corollary 4.1, and let all the conditions of Corollary 4.1 hold. If the constants ρ and η satisfy the conditions (4.18) and (4.29), then the iterative sequence $\{(x_n,y_n)\}$ generated by Algorithm 4.2 converges strongly to a unique solution (x^*,y^*) of the system (3.15).*

COROLLARY 4.5. *Assume that the mappings T_i $(i=1,2)$ and the constants ρ and η are the same as in Corollary 4.2, and let all the conditions of Corollary 4.2 hold. If the constants ρ and η satisfy the condition (4.30) and $\sum_{n=0}^{\infty}\alpha_n = \infty$, then the iterative sequence $\{(x_n,y_n)\}$ generated by Algorithm 4.3 converges strongly to a unique solution (x^*,y^*) of the system (3.16).*

Proof. Since $r = \infty$, that is, $K_r = K$, we have $\delta = 1$. On the other hand, since $g \equiv I$, it follows that g is ξ_3-strongly monotone and μ_3-Lipschitz continuous with $\xi_3 = \mu_3 = 1$, and we have $k = \sqrt{1 - 2\xi_3 + \mu_3^2} = 0$. Taking $\delta = 1$ and $k = 0$ in the condition (4.19) of Theorem 4.1, the condition (4.19) reduces to the condition (4.30) of Corollary 4.2, and result follows from Corollary 4.2. □

COROLLARY 4.6. *Let the mapping T and the constant ρ be the same as in Corollary 4.3 and suppose that all the conditions of Corollary 4.3 hold. If the constant ρ satisfies the conditions (4.31) and (4.32), then the iterative sequence $\{x_n\}$ generated by Algorithm 4.4 converges strongly to a unique solution x^* of the problem (3.17).*

5. Conclusions

In this paper, we have investigated and analyzed, so called, generalized system of nonconvex variational inequalities (2.1a) and (2.1b) from [13] and verified that this system of inequalities is not equivalent to the fixed point problems (3.1a) and (3.1b) from [13]. That is, the [13, Lemma 3.1] is incorrect. Lemma 3.1 in [13] is the main key to suggest the algorithm and to prove the strong convergence of the sequences generated by the proposed algorithm. Since [13, Lemma 3.1] is no longer valid, the algorithms and results in [13] are also no longer valid. To overcome with the problems in [13], we introduced a system of regularized nonconvex variational inequalities (SRNVI). By using the projection operator technique, we have verified the equivalence between the SRNVI and the fixed point problems. By using this equivalence, we suggested and analyzed an explicit projection iterative method for solving the SRNVI. The existence of a unique solution of the SRNVI is proved and the convergence analysis of the suggested iterative algorithm under some suitable conditions is studied.

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