GENERALIZED WEIGHTED COMPOSITION OPERATORS
FROM BERS–TYPE SPACES INTO BLOCH–TYPE SPACES

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Abstract. New criteria for the boundedness and the compactness of the generalized weighted composition operators from Bers-type spaces into Bloch-type spaces are given in this paper.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$. Let $\alpha > 0$. The Bers-type space, denoted by $H_{\alpha}^{\infty}$, is the space consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H_{\alpha}^{\infty}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha} |f(z)| < \infty.$$  

$H_{\alpha}^{\infty}$ is a Banach space under the norm $\|\cdot\|_{H_{\alpha}^{\infty}}$. The little Bers-type space, denoted by $H_{\alpha,0}^{\infty}$, is the subspace of $H_{\alpha}^{\infty}$ consisting of those $f \in H_{\alpha}^{\infty}$ such that

$$\lim_{|z| \to 1} (1 - |z|^2)^{\alpha} |f(z)| = 0.$$  

Let $\beta > 0$. The Bloch-type space $B_{\beta}$ is defined as the set of functions $f \in H(\mathbb{D})$ such that

$$B_{\beta}(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\beta} |f'(z)| < \infty.$$  

$B_{\beta}$ becomes a Banach space with the norm $\|f\|_{B_{\beta}} = |f(0)| + B_{\beta}(f)$. When $\beta = 1$, $B^1 = B$ is the classical Bloch space. For more information on Bloch-type spaces on the unit disk, see, e.g., [34].

In this paper, $\varphi$ always denotes a nonconstant analytic self-map of $\mathbb{D}$. The composition operator $C_{\varphi}$, induced by $\varphi$, is defined by

$$C_{\varphi}f = f \circ \varphi$$  


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for $f \in H(\mathbb{D})$. A fundamental and interesting problem concerning composition operators is to relate function theoretic properties of $\varphi$ to operator theoretic properties of $C_\varphi$ on various spaces (see, e.g., [3]).

Let $u \in H(\mathbb{D})$. The weighted composition operator $uC_\varphi$, induced by $\varphi$ and $u$, is defined by

$$(uC_\varphi f)(z) = u(z) \cdot f(\varphi(z)), \ f \in H(\mathbb{D}).$$

Let $D$ be the differentiation operator and $n$ be a nonnegative integer. Denote

$Df = f', \ D^n f = f^{(n)}, \ f \in H(\mathbb{D}).$

The generalized weighted composition operator $D^n_{\varphi, u}$, introduced by the author of this paper, is defined as follows (see [35, 36, 37]).

$$(D^n_{\varphi, u} f)(z) = u(z) \cdot f^{(n)}(\varphi(z)), \ f \in H(\mathbb{D}), \ z \in \mathbb{D}.$$

When $n = 0$, then $D^n_{\varphi, u} = uC_\varphi$. When $n = 0$ and $u(z) \equiv 1$, then we get the composition operator $C_\varphi$. When $n = 1$, $u(z) = \varphi'(z)$, then $D^n_{\varphi, u} = DC_\varphi$. When $n = 1$ and $u(z) = 1$, then $D^n_{\varphi, u} = C_\varphi D$. The operators $DC_\varphi$ and $C_\varphi D$ were studied, for example, in [7, 9, 12, 13, 19, 20, 24, 27, 31, 33].

Composition operators, weighted composition operators and generalized weighted composition operators between Bers-type spaces and some other spaces in one, as well as, in several complex variables were studied in [4, 5, 6, 17, 18, 19, 21, 22, 30, 32, 35, 38], while composition operators, weighted composition operators and generalized weighted composition operators between Bloch-type spaces and some other spaces in one and several complex variables were studied, for example, in [1, 2, 8, 10, 11, 14, 15, 16, 21, 22, 23, 25, 26, 28, 29, 31, 33, 32, 36, 39].

In this paper, motivated by [1, 2], we give a new criterion for the boundedness or compactness of the operator $D^n_{\varphi, u}$ from Bers-type spaces to Bloch-type spaces, namely we use two families of functions to characterize the generalized weighted composition operators $D^n_{\varphi, u} : \mathcal{H}_\alpha^\infty \to \mathcal{B}^\beta$.

Throughout the paper, $C$ denotes a positive constant which may differ from one occurrence to the other. The notation $A \asymp B$ means that there exists a positive constant $C$ such that $B/C \leq A \leq CB$.

2. Main results and proofs

In this section we give our main results and proofs. For this purpose, we need two lemmas as follows.

**Lemma 1.** Assume that $0 < \alpha < \infty$. Let $f \in \mathcal{H}_\alpha^\infty$. Then there is a positive constant $C$ independent of $f$ such that

$$|f^{(n)}(z)| \leq C \frac{\|f\|_{\mathcal{H}_\alpha^\infty}}{(1 - |z|^2)^{\alpha + n}}.$$
Proof. Using the fact
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\alpha+1} |f'(z)|,
\]
and the fact that for \( f \in \mathcal{B}_\beta \) (see [34]),
\[
\sup_{z \in \mathbb{D}} (1 - |z|^2)^\beta |f'(z)| \leq |f'(0)| + \cdots + |f^{(n-1)}(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)^{\beta+n-1} |f^{(n)}(z)|,
\]
we immediately get the desired result. \( \square \)

The following criterion follows from standard arguments similar to those outlined in Proposition 3.11 of [3].

**Lemma 2.** Let \( u \in H(\mathbb{D}) \), \( 0 < \alpha, \beta < \infty \), \( \varphi \) be an analytic self-map of \( \mathbb{D} \) and \( n \) be a nonnegative integer. The operator \( D^n_{\varphi,u} : H^\infty_\alpha \) (or \( H^\infty_{\alpha,0} \)) \( \to \mathcal{B}_\beta \) is compact if and only if \( D^n_{\varphi,u} : H^\infty_\alpha \) (or \( H^\infty_{\alpha,0} \)) \( \to \mathcal{B}_\beta \) is bounded and for any bounded sequence \( (f_k)_{k \in \mathbb{N}} \) in \( H^\infty_\alpha \) (or \( H^\infty_{\alpha,0} \)) which converges to zero uniformly on compact subsets of \( \mathbb{D} \), we have \( \|D^n_{\varphi,u}f_k\|_{\mathcal{B}_\beta} \to 0 \) as \( k \to \infty \).

For \( a \in \mathbb{D} \), set
\[
f_a(z) = \frac{(1 - |a|^2)}{(1 - \overline{a}z)^{\alpha+1}}, \quad \text{and} \quad g_a(z) = \left( \frac{1 - |a|^2}{1 - \overline{a}z} \right) f_a(z).
\]

Next, we will use these two families of functions to characterize the generalized weighted composition operators \( D^n_{\varphi,u} : H^\infty_\alpha \to \mathcal{B}_\beta \).

**Theorem 1.** Let \( u \in H(\mathbb{D}) \), \( 0 < \alpha, \beta < \infty \), \( \varphi \) be an analytic self-map of \( \mathbb{D} \) and \( n \) be a nonnegative integer. Then the following statements are equivalent:

(a) The operator \( D^n_{\varphi,u} : H^\infty_\alpha \to \mathcal{B}_\beta \) is bounded;

(b) The operator \( D^n_{\varphi,u} : H^\infty_{\alpha,0} \to \mathcal{B}_\beta \) is bounded;

(c) \( u\varphi \in \mathcal{B}_\beta \), \( u \in \mathcal{B}_\beta \),

\[
A := \sup_{w \in \mathbb{D}} \|D^n_{\varphi,u}f_{\varphi(w)}\|_{\mathcal{B}_\beta} < \infty \quad \text{and} \quad B := \sup_{w \in \mathbb{D}} \|D^n_{\varphi,u}g_{\varphi(w)}\|_{\mathcal{B}_\beta} < \infty;
\]

(d) \( M_1 := \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |u'(z)|}{(1 - |\varphi(z)|^2)^{\alpha+n}} < \infty \) \quad (2)

and \( M_2 := \sup_{z \in \mathbb{D}} \frac{(1 - |z|^2)^\beta |\varphi'(z)|u(z)}{(1 - |\varphi(z)|^2)^{\alpha+n+1}} < \infty \) \quad (3)
Proof. (d) ⇒ (a). Suppose that (d) holds. For arbitrary $z$ in $\mathbb{D}$ and $f \in H^\infty_\alpha$, by Lemma 1 we have

$$
(1 - |z|^2)^\beta |(D^n_{\varphi,u}f)'(z)|
\leq (1 - |z|^2)^\beta |u'(z)||f^{(n)}(\varphi(z))| + (1 - |z|^2)^\beta |f^{(n+1)}(\varphi(z))||u(z)||\varphi'(z)|
\leq C(1 - |z|^2)^\beta |u'(z)||f||_{H^\infty_\alpha} + C(1 - |z|^2)^\beta |u(z)||\varphi'(z)||f||_{H^\infty_\alpha}
\leq C(M_1 + M_2)||f||_{H^\infty_\alpha}.
$$

(4)

Taking the supremum in (4) over $\mathbb{D}$ and then using the condition in (d) we see that $D^n_{\varphi,u} : H^\infty_\alpha \to B^\beta$ is bounded.

(a) ⇒ (b). This implication is obvious.

(b) ⇒ (c). Assume $D^n_{\varphi,u} : H^\infty_{\alpha,0} \to B^\beta$ is bounded. Taking the functions $z^n$ and $z^{n+1}$ and using the boundedness of $D^n_{\varphi,u}$ we see that

$$
u \varphi \in B^\beta \quad \text{and} \quad u \in B^\beta.
$$

For each $a \in \mathbb{D}$, it is easy to check that $f_a, g_a \in H^\infty_{\alpha,0}$. Moreover $||f_a||_{H^\infty_\alpha}$ and $||g_a||_{H^\infty_\alpha}$ are bounded by constants independent of $a$. By the boundedness of $D^n_{\varphi,u} : H^\infty_{\alpha,0} \to B^\alpha$, we get

$$
\sup_{a \in \mathbb{D}} ||D^n_{\varphi,u}f(\varphi(a))||_{B^\beta} \leq ||D^n_{\varphi,u}|| \sup_{a \in \mathbb{D}} ||f(\varphi(a))||_{H^\infty_\alpha} \leq C||D^n_{\varphi,u}|| < \infty
$$

and

$$
\sup_{a \in \mathbb{D}} ||D^n_{\varphi,u}g(\varphi(a))||_{B^\beta} \leq ||D^n_{\varphi,u}|| \sup_{a \in \mathbb{D}} ||g(\varphi(a))||_{H^\infty_\alpha} \leq C||D^n_{\varphi,u}|| < \infty,
$$

as desired.

(c) ⇒ (d). Suppose that $u \varphi \in B^\beta$, $u \in B^\beta$, $A$ and $B$ are finite. A calculation shows that

$$
\begin{align*}
f_a^{(n)}(a) &= \prod_{j=1}^n (\alpha + j) \frac{\bar{a}^n}{(1 - |a|^2)^{\alpha+n}}, 
g_a^{(n)}(a) &= \prod_{j=2}^{n+1} (\alpha + j) \frac{\bar{a}^n}{(1 - |a|^2)^{\alpha+n}}. 
\end{align*}
$$

(5)

From (5), for $w \in \mathbb{D}$, we have

$$
(D^n_{\varphi,u}f(\varphi(w))' = \prod_{j=1}^n (\alpha + j) \frac{u'(w)\varphi'(w)^n}{(1 - |\varphi(w)|^2)^{\alpha+n}}
+ \prod_{j=1}^{n+1} (\alpha + j) \frac{u(w)\varphi'(w)\varphi(w)^{n+1}}{1 - |\varphi(w)|^2}^{1+\alpha+n}.
$$

(6)
Therefore
\[
\frac{(1 - |w|^2)^\beta |u'(w)||\varphi(w)|^n}{(1 - |\varphi(w)|^2)^{\alpha+n}} \leq \frac{1}{\prod_{j=1}^{n} (\alpha + j)} + \frac{\alpha + n + 1}{(1 - |\varphi(w)|^2)^{1+\alpha+n}} 
\]
\[
\leq \frac{\|D_{\alpha,u}\varphi(w)\|}{\prod_{j=1}^{n} (\alpha + j)} + \frac{\alpha + n + 1}{(1 - |\varphi(w)|^2)^{1+\alpha+n}} 
\]
\leq \frac{A}{\prod_{j=1}^{n} (\alpha + j)} + \frac{\alpha + n}{(1 - |\varphi(w)|^2)^{1+\alpha+n}}. \quad (7)
\]

In addition,
\[
(D_{\alpha,u}\varphi(w))^n(w) = \prod_{j=2}^{n+1} (\alpha + j) \frac{u'(w)|\varphi(w)|^n}{(1 - |\varphi(w)|^2)^{\alpha+n}} 
\]
\[
+ \prod_{j=2}^{n+1} (\alpha + j) \frac{u(w)|\varphi'(w)||\varphi(w)|^{n+1}}{(1 - |\varphi(w)|^2)^{1+\alpha+n}}. \quad (8)
\]

Therefore, by multiplying (6) by \(\alpha + n + 1\) and (8) by \(\alpha + 1\), then subtracting such obtained equalities and using the triangle inequality, we obtain
\[
\frac{|u(w)|\varphi'(w)||\varphi(w)|^{n+1}}{(1 - |\varphi(w)|^2)^{1+\alpha+n}} \leq \frac{\alpha + 1 + n}{\prod_{j=1}^{n+1} (\alpha + j)} \|D_{\alpha,u}\varphi(w)\|^n + \frac{\alpha + 1}{\prod_{j=1}^{n+1} (\alpha + j)} |(D_{\alpha,u}\varphi(w))^n'(w)|, \quad (9)
\]

which implies
\[
\frac{(1 - |w|^2)^\beta |u(w)|\varphi'(w)||\varphi(w)|^{n+1}}{(1 - |\varphi(w)|^2)^{1+\alpha+n}} \leq \frac{\alpha + 1 + n}{\prod_{j=1}^{n+1} (\alpha + j)} A + \frac{\alpha + 1}{\prod_{j=1}^{n+1} (\alpha + j)} B. \quad (10)
\]

From (7) and (10), we get
\[
\frac{(1 - |w|^2)^\beta |u'(w)||\varphi(w)|^n}{(1 - |\varphi(w)|^2)^{\alpha+n}} \leq \frac{\alpha + 2 + n}{\prod_{j=1}^{n} (\alpha + j)} A + \frac{\alpha + 1}{\prod_{j=1}^{n} (\alpha + j)} B. \quad (11)
\]

Fix \(r \in (0, 1)\). If \(|\varphi(w)| > r\), then from (10) we obtain
\[
\frac{(1 - |w|^2)^\beta |u(w)|\varphi'(w)|}{(1 - |\varphi(w)|^2)^{1+\alpha+n}} \leq \frac{1}{r^{n+1}} \left( \frac{\alpha + 1 + n}{\prod_{j=1}^{n+1} (\alpha + j)} A + \frac{\alpha + 1}{\prod_{j=1}^{n+1} (\alpha + j)} B \right). \quad (12)
\]

On the other hand, if \(|\varphi(w)| \leq r\), by the fact that
\[
(1 - |w|^2)^\beta |u(w)|\varphi'(w)| \leq \|u\|_{\tilde{\beta}} + \|u\|_{\tilde{\beta}},
\]

we get
\[
\frac{(1 - |w|^2)^\beta |u(w)\varphi'(w)|}{(1 - |\varphi(w)|^2)^{1+\alpha+n}} \leq \frac{1}{(1 - r^2)^{1+\alpha+n}} \left( \|u\varphi\|_{\mathcal{B}^\beta} + \|u\|_{\mathcal{B}^\beta} \right).
\] (13)

From (12) and (13) we see that $M_2$ is finite. Using similar arguments and (11) we can obtain that $M_1$ is finite as well. The proof of this theorem is finished. \[\square\]

**Theorem 2.** Let $u \in H(\mathbb{D})$, $0 < \alpha, \beta < \infty$, $\varphi$ be an analytic self-map of $\mathbb{D}$ and $n$ be a nonnegative integer. Suppose that the operator $D_{n}^{\varphi,u} : H_{\alpha}^{\infty} \to \mathcal{B}^{\beta}$ is bounded, then the following statements are equivalent:

(a) The operator $D_{n}^{\varphi,u} : H_{\alpha}^{\infty} \to \mathcal{B}^{\beta}$ is compact;

(b) The operator $D_{n}^{\varphi,u} : H_{\alpha,0}^{\infty} \to \mathcal{B}^{\beta}$ is compact;

(c) \[
\lim_{|\varphi(w)| \to 1} \|D_{n}^{\varphi,u}f_{\varphi(w)}\|_{\mathcal{B}^{\beta}} = 0 \quad \text{and} \quad \lim_{|\varphi(w)| \to 1} \|D_{n}^{\varphi,u}g_{\varphi(w)}\|_{\mathcal{B}^{\beta}} = 0;
\]

(d) \[
\lim_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)^\beta |u(z)|}{(1 - |\varphi(z)|^2)^{\alpha+n}} = 0 \quad \text{and} \quad \lim_{|\varphi(z)| \to 1} \frac{(1 - |z|^2)^\beta |u(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{1+\alpha+n}} = 0.
\]

**Proof.** (a) $\implies$ (b). This implication is clear.

(b) $\implies$ (c). Assume that $D_{n}^{\varphi,u} : H_{\alpha,0}^{\infty} \to \mathcal{B}^{\beta}$ is compact. Let $\{w_k\}_{k \in \mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $\lim_{k \to \infty} |\varphi(w_k)| = 1$ (if such a sequence does not exist then the limits in (c) automatically hold). Since the sequences $\{f_{\varphi(w_k)}\}$ and $\{g_{\varphi(w_k)}\}$ are bounded in $H_{\alpha,0}^{\infty}$ and converge to 0 uniformly on compact subsets of $\mathbb{D}$, by Lemma 2, we get
\[
\|D_{n}^{\varphi,u}f_{\varphi(w_k)}\|_{\mathcal{B}^{\beta}} \to 0 \quad \text{and} \quad \|D_{n}^{\varphi,u}g_{\varphi(w_k)}\|_{\mathcal{B}^{\beta}} \to 0
\] (14) as $k \to \infty$, which means that (c) holds.

(c) $\implies$ (d). Suppose that the limits in (c) are 0. Using the inequality (9), we get
\[
\frac{(1 - |w|^2)^\beta |u(w)\varphi'(w)|}{(1 - |\varphi(w)|^2)^{1+\alpha+n}} \leq \frac{(\alpha + 1+n)\|D_{n}^{\varphi,u}f_{\varphi(w)}\|_{\mathcal{B}^{\beta}} + (\alpha + 1)\|D_{n}^{\varphi,u}g_{\varphi(w)}\|_{\mathcal{B}^{\beta}}}{\prod_{j=1}^{n+1}(\alpha + j)|\varphi(w)|^{n+1}} \to 0
\] (15) as $|\varphi(w)| \to 1$. Moreover, using (7), we deduce
\[
\frac{(1 - |w|^2)^\beta |u'(w)|}{(1 - |\varphi(w)|^2)^{\alpha+n}} \leq \frac{\|D_{n}^{\varphi,u}f_{\varphi(w)}\|_{\mathcal{B}^{\beta}} + (\alpha + n + 1)(1 - |w|^2)^\beta |u(w)\varphi'(w)||\varphi(w)|}{\prod_{j=1}^{n}(\alpha + j)|\varphi(w)|^{n}} \to 0,
\] (16)
as $|\varphi(w)| \to 1$. The desired result follows.

(d) $\implies$ (a). Assume that (d) holds. By (d), we have that for any $\varepsilon > 0$, there is a constant $\delta$, $0 < \delta < 1$, such that

\[
\frac{(1 - |z|^2)^{\beta} |u'(z)|}{(1 - |\varphi(z)|^2)^{\alpha+n}} < \varepsilon \quad \text{and} \quad \frac{(1 - |z|^2)^{\beta} |u(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{\alpha+1+n}} < \varepsilon,
\]

whenever $\delta < |\varphi(z)| < 1$.

Let $(f_k)_{k \in \mathbb{N}}$ be a sequence in $H^\infty_\alpha$ with $\sup_{k \in \mathbb{N}} \|f_k\|_{H^\infty_\alpha} \leq M$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. In light of Lemma 2, we only need to show that $\|D_{\varphi,u}^n f_k\|_{\mathcal{B}_\beta} \to 0$ as $k \to \infty$. Using (17), for $|\varphi(w)| > r$, we have

\[
(1 - |w|^2)^{\beta} \|(D_{\varphi,u}^n f_k)'(w)\| \\
\leq (1 - |w|^2)^{\beta} |u'(w)f_k(n)\varphi(w)| + (1 - |w|^2)^{\beta} |u(w)f_k(n+1)\varphi(w)\varphi'(w)| \\
\leq C\|f_k\|_{H^\infty_\alpha} \left( \frac{(1 - |w|^2)^{\beta} |u'(w)|}{(1 - |\varphi(w)|^2)^{\alpha+n}} + \frac{(1 - |w|^2)^{\beta} |u(w)\varphi'(w)|}{(1 - |\varphi(w)|^2)^{1+\alpha+n}} \right) < 2MC\varepsilon.
\]

By Cauchy’s estimate, if $f_k$ is a sequence which converges on compact subset of $\mathbb{D}$ to zero, then the sequence $f_k(n)$ also converges on compact subset of $\mathbb{D}$ to zero as $k \to \infty$. Hence, for $|\varphi(w)| \leq r$, we have

\[
(1 - |w|^2)^{\beta} \|(D_{\varphi,u}^n f_k)'(w)\| < \varepsilon, \quad \text{as} \quad k \to \infty.
\]

Since $|u(0)f_k(n)(\varphi(0))| \to 0$ as $k \to \infty$, we obtain that

\[
\|D_{\varphi,u}^n f_k\|_{\mathcal{B}_\beta} = |u(0)f_k(n)(\varphi(0))| + \sup_{w \in \mathbb{D}} (1 - |w|^2)^{\beta} \|(D_{\varphi,u}^n f_k)'(w)\| \\
= \sup_{|\varphi(w)| > r} (1 - |w|^2)^{\beta} \|(D_{\varphi,u}^n f_k)'(w)\| + \sup_{|\varphi(w)| \leq r} (1 - |w|^2)^{\beta} \|(D_{\varphi,u}^n f_k)'(w)\| \to 0,
\]

as $k \to \infty$. Hence the operator $D_{\varphi,u}^n : H^\infty_\alpha \to \mathcal{B}_\beta$ is compact by Lemma 2. The proof of this theorem is finished.  

\[\square\]

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