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A NOTE ON THE RELATIONS OF CLASSES OF NUMERICAL SEQUENCES

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Abstract. We investigate the interrelations of three classes of numerical sequences that have connection with the L^1-convergence of cosine series.

1. Introduction

Several authors have studied the question of L^1-convergence of the following cosine series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx. \tag{1.1}$$

They defined numerous classes of numerical sequences such that if the coefficients of the series (1.1) belong to their class then the series (1.1) is a Fourier series of some $f \in L^1(0,\pi)$ and the condition

$$a_n \log n = o(1), \quad n \to \infty, \tag{1.2}$$

is a necessary and sufficient condition for the convergence of the partial sums of (1.1) in $L^1(0,\pi)$-norm.

Now we define only the classes to be considered in our note, to see more classes we refer to [2].

Referee kindly recommended appending three further papers to the reference list. They are added in the supplementary references.

1. A null-sequence $\mathbf{a} := \{a_n\}$ $(a_n \to 0)$ belongs to the class S if there exists a monotonically decreasing sequence $\{A_n\}$ such that

$$\sum_{n=1}^{\infty} A_n < \infty \quad \text{and} \quad |\Delta a_n| \leqslant A_n \tag{1.3}$$

for all n.

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2. A null-sequence $\boldsymbol{a}$ belongs to the class $\boldsymbol{F}_p$ if for some $p \geqslant 1$

$$\sum_{n=1}^{\infty} n^{-1/p} \left(\sum_{k=n}^{\infty} |\Delta a_k|^p \right)^{1/p} < \infty. \tag{1.4}$$

3. A null-sequence $\boldsymbol{a}$ belongs to the class $\boldsymbol{F}_p^*$ if for some $p \geqslant 1$

$$\sum_{m=1}^{\infty} 2^{m\left(1-\frac{1}{p}\right)} \left\{ \sum_{n=2^m+1}^{2^{m+1}} |\Delta a_n|^p \right\}^{1/p} < \infty. \tag{1.5}$$

4. A null-sequence $\boldsymbol{a}$ belongs to the class $\boldsymbol{S}_p$ $(p \geqslant 0)$ if there exists a monotonically decreasing sequence $\{A_n\}$ such that

$$\sum_{n=1}^{\infty} A_n < \infty \quad \text{and} \quad \sum_{k=1}^{n} \frac{|\Delta a_k|^p}{A_k^p} = O(n). \tag{1.6}$$

The definition **1** is due to Telyakovskiĭ [4], the definitions **2** and **3** for $p > 1$ were given by Fomin [1], and the definition **4** is due to C. V. Stanojevič and V. B. Stanojevič [3].

In [2] (see also Fomin [1]) we verified the following embedding relations: If $p > 1$ then

$$\boldsymbol{F}_p \subseteq \boldsymbol{S}_p \subseteq \boldsymbol{F}_p^* \subseteq \boldsymbol{F}_p \tag{1.7}$$

and

$$\boldsymbol{S} \subsetneqq \boldsymbol{F}_p, \tag{1.8}$$

that is, the classes appearing in (1.7) are equivalent, furthermore the embedding $\boldsymbol{S} \subset \boldsymbol{F}_p$ is a strict one, namely there exists a null-sequence $\boldsymbol{a}$ such that $\boldsymbol{a} \in \boldsymbol{F}_p$ but $\boldsymbol{a} \notin \boldsymbol{S}$.

Rereading these results my first optimism was that similar statements will be valid for the case $p = 1$, too; naturally not expecting the convergence of the partial sums of (1.1).

Unfortunately we discovered immediately that this is not the case, namely if $p = 1$, then

$$\boldsymbol{a} \in \boldsymbol{F}_1 \quad \text{if} \quad \sum_{n=1}^{\infty} |\Delta a_n| \log n < \infty \tag{1.9}$$

and

$$\boldsymbol{a} \in \boldsymbol{F}_p^* \quad \text{if} \quad \sum_{n=1}^{\infty} |\Delta a_n| < \infty,$$

and these conditions are not equivalent.

Consequently we decreased our aim to analyzing of the relationships of the classes $\boldsymbol{S}, \boldsymbol{F}_1$ and $\boldsymbol{F}_p,\ p > 1$.

Before presenting the results we introduce the notion $A \nsim B$ to denote that the classes of sequences A and B are not comparable.

2. Results

Next we show the following class-relations:

$$\boldsymbol{F}_1 \not\sim \boldsymbol{F}_p \quad (p>1) \tag{2.1}$$

and

$$\boldsymbol{F}_1 \not\sim \boldsymbol{S}. \tag{2.2}$$

3. Proofs

Proof of (2.1). First we give a sequence $\boldsymbol{a}$ which belongs to $\boldsymbol{F}_p$, but $\boldsymbol{a} \notin \boldsymbol{F}_1$. Let

$$a_k := \frac{1}{\log k}.$$

Then

$$\Delta a_k \leqslant \frac{1}{k \log^2 k}.$$

By (1.7) it is enough to verify that $\boldsymbol{a} \in \boldsymbol{F}_p^*$ and $\boldsymbol{a} \notin \boldsymbol{F}_1$. Then

$$\begin{aligned}\sum_{m=1}^{\infty} 2^{m\left(1-\frac{1}{p}\right)} \left\{ \sum_{k=2^m+1}^{2^{m+1}} (k \log^2 k)^{-p} \right\}^{1/p} &\leqslant \sum_{m=1}^{\infty} 2^{m\left(1-\frac{1}{p}\right)} \{2^m (2^m m^2)^{-p}\}^{1/p} \\ &\leqslant \sum_{m=1}^{\infty} m^{-2} < \infty,\end{aligned}$$

but (1.9) clearly does not hold, that is,

$$\boldsymbol{a} \in \boldsymbol{F}_p \quad \text{and} \quad \boldsymbol{a} \notin \boldsymbol{F}_1$$

are verified. This shows that

$$\boldsymbol{F}_p \not\subset \boldsymbol{F}_1. \tag{3.1}$$

In the following calculation we use the notation $b \approx c$ if there exist two positive constant K_1 and K_2 such that

$$K_1 c \leqslant b \leqslant K_2 c$$

holds.

The following sequence has the opposite properties. Let

$$a_n := (2^{2^m})^{\frac{1}{p}-1} \quad \text{if} \quad 2^{2^m} < n \leqslant 2^{2^{m+1}}, \quad m \geqslant 0.$$

Then

$$|\Delta a_n| = 0 \quad \text{if} \quad n \neq 2^{2^m} \quad \text{and} \quad |\Delta a_{2^{2^{m+1}}}| \approx (2^{2^m})^{\frac{1}{p}-1}.$$

By (1.9) $\boldsymbol{a}\in \boldsymbol{F}_1$ if

$$\sum_{n=1}^{\infty}|\Delta a_n|\log n \approx \sum_{m=0}^{\infty}(2^{2^m})^{\frac{1}{p}-1}2^m<\infty,$$

now this clearly holds.

On the other hand

$$\sum_{m=1}^{\infty}\sum_{n=2^{2^m}+1}^{2^{2^{m+1}}} n^{-1/p}\left\{\sum_{k=n}^{\infty}|\Delta a_k|^p\right\}^{1/p} \geqslant \sum_{m=1}^{\infty}(2^{2^m})^{1-\frac{1}{p}}(2^{2^m})^{\frac{1}{p}-1}=\infty,$$

thus $\boldsymbol{a}\notin \boldsymbol{F}_p$, consequently

$$\boldsymbol{F}_1 \not\subset \boldsymbol{F}_p. \tag{3.2}$$

Due to (3.1) and (3.2), the classes $\boldsymbol{F}_1$ and $\boldsymbol{F}_p$ are not comparable, herewith (2.1) is proved. □

Proof of (2.2). Let us define a sequence $\boldsymbol{a}$ as follows. Let

$$a_1 := 1 \quad \text{and} \quad a_n := 2^{-m} \quad \text{if} \quad 2^{m-1}<n\leqslant 2^m, \quad m\geqslant 1. \tag{3.3}$$

Then

$$\sum_{m=1}^{\infty}\sum_{n=2^m+1}^{2^{m+1}}\frac{1}{n}\sum_{k=n}^{\infty}|\Delta a_k| \leqslant \sum_{m=1}^{\infty}2\cdot 2^{-m}<\infty,$$

thus $\boldsymbol{a}\in \boldsymbol{F}_1$.

On the other hand, if $\boldsymbol{a}\in \boldsymbol{S}$, then the suitable monotone decreasing sequence $\{A_n\}$ appearing in the definition of $\boldsymbol{S}$, would have the properties:

$$|\Delta a_n|\leqslant A_n \quad \text{and} \quad \sum_{m=1}^{\infty}2^m A_{2^m}<\infty. \tag{3.4}$$

Since, in the case of the sequence given in (3.3)

$$|\Delta a_{2^m}| = 2^{-m-1},$$

there exists no monotone sequence $\{A_n\}$ which satisfies (3.4), thus our sequence does not belongs to $\boldsymbol{S}$.

Since

$$\boldsymbol{a}\in \boldsymbol{F}_1 \quad \text{but} \quad \boldsymbol{a}\notin \boldsymbol{S},$$

thus

$$\boldsymbol{F}_1 \not\subset \boldsymbol{S}. \tag{3.5}$$

The sequence

$$a_k := \frac{1}{\log k},$$

as we have seen in the proof of (2.1), $\mathbf{a} \notin \mathbf{F}_1$, but due to

$$|\Delta a_k| \leqslant \frac{1}{k \log^2 k} =: A_k,$$

$\mathbf{a} \in \mathbf{S}$ clearly follows, therefore

$$\mathbf{S} \not\subset \mathbf{F}_1. \tag{3.6}$$

The results (3.5) and (3.6) verify that the classes $\mathbf{F}_1$ and $\mathbf{S}$ are not comparable, consequently (2.2) is verified. □

Next we recall one of the sharpest results, proved by Fomin [1], mentioned implicitly at the forepart of the Introduction.

THEOREM. *If $1 < p \leqslant 2$ and $\mathbf{a} \in \mathbf{F}_p$, then series* (1.1) *is a Fourier series of an integrable function f and*

$$\|S_n - f\| = o(1), \quad n \to \infty, \tag{3.7}$$

if and only if (1.2) *holds, where S_n are the partial sums of* (1.1) *and $\|\cdot\|$ denotes the L^1-norm.*

Our results (2.1) and (2.2) show that the class $\mathbf{F}_1$, generally, does not belong to the classes that imply the equivalence of (1.2) and (3.7).

This raises the following problem: What modification is required on the definition of $\mathbf{F}_1$ in order that if $\mathbf{a}$ belongs to the modified class, then (1.2) and (3.7) should be equivalent?

We show that if in the definition of $\mathbf{F}_1$ we put $n^{\varepsilon-1}$ in place of n^{-1}, $\varepsilon > 0$ can be arbitrarily small, then the new class $\mathbf{F}_{1,\varepsilon}$ defined by the inequality

$$\sum_{n=1}^{\infty} n^{\varepsilon-1} \sum_{k=n}^{\infty} |\Delta a_k| < \infty,$$

has the property, that if $\mathbf{a} \in \mathbf{F}_{1,\varepsilon}$, then (1.2) and (3.7) are equivalent.

More precisely we can prove the following

COROLLARY. *If $\varepsilon > 0$ and $\mathbf{a} \in \mathbf{F}_{1,\varepsilon}$, then series* (1.1) *is a Fourier series of an integrable function f and the statements* (1.2) *and* (3.7) *are equivalent.*

Proof of Corollary. We show that if $1 < p \leqslant \min\left(2, \dfrac{1}{1-\varepsilon}\right)$ then the class $\mathbf{F}_{1,\varepsilon}$ is a subclass of $\mathbf{F}_p$, thus the cited Theorem of Fomin conveys the statement of Corollary.

In the following calculation we shall use the well-known inequality

$$\left(\sum b_n\right)^{\alpha} \leqslant \sum b_n^{\alpha}, \quad b_n \geqslant 0, \quad 0 < \alpha \leqslant 1,$$

with $\alpha = \dfrac{1}{p}$ and $b_n = |\Delta a_n|^p$. Then, due to $\boldsymbol{a} \in \boldsymbol{F}_{1,\varepsilon}$ and $p \leqslant 1/(1-\varepsilon)$, we get

$$\sum_{n=1}^{\infty} n^{-1/p} \left(\sum_{k=n}^{\infty} |\Delta a_k|^p \right)^{1/p} \leqslant \sum_{n=1}^{\infty} n^{-1/p} \sum_{k=n}^{\infty} |\Delta a_k| \leqslant \sum_{n=1}^{\infty} n^{\varepsilon-1} \sum_{k=n}^{\infty} |\Delta a_k| < \infty,$$

consequently, $\boldsymbol{F}_{1,\varepsilon} \subset \boldsymbol{F}_p$ holds, herewith the proof is complete. □

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