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APPLICATIONS OF FAN'S MATCHING THEOREM IN MKKM SPACES

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Abstract. Using some concepts in MKKM space, we prove the generalized forms of open and closed versions of Fan's matching theorem. The S-weakly KKM multimap is applied to obtain a type of KKM theorem. As an application, a Fan type minimax theorem is proved. Moreover, a minimax inequality is given in these new settings.

1. Introduction

It is known that Fan-KKM principle is an important part in the study of nonlinear analysis which has become one of the most applicable tools in mathematical economics, game theory and control theory. As an application of Fan-KKM principle, Fan's minimax inequality [10] has been studied by many authors and several extensions have been obtained in spaces with convex structure [8, 9, 11]. In 1993, Park and Kim [24] introduced the concept of generalized convex space as a general model of many well known spaces with convex structure. Since then, many minimax inequalities of Ky Fan type have been improved and generalized in generalized convex spaces [5, 9, 15].

Motivated by [16], M. Balaj [6] introduced the concept of weakly G-KKM mapping for G-convex spaces and obtained Fan type and Sion type minimax inequalities. Recently by introducing the new concept of abstract convex space with certain broad classes $\mathfrak{KC}$ and $\mathfrak{KO}$ of multimaps (having the KKM property) by Park [23], KKM type mappings were used to obtain matching theorems, coincidence theorems, fixed point theorems and minimax inequalities. In [20], the author motivated by [5] achieved the basic results for weakly KKM mappings and minimax theorems on the KKM spaces [22] as genuine generalization of G-convex spaces and Φ_A-spaces $(X,D;\{\Phi_A\}_{A\in\langle D\rangle})$.

2. Preliminaries

The concepts of minimal structure and minimal space, as generalizations of topology and topological spaces were introduced in [18]. Further results about minimal spaces can be found in [2, 3, 4, 17] and [25]. For easy understanding of the material

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incorporated in this paper we recall some basic definitions and results. Also some new concepts are introduced in minimal spaces.

A family $\mathscr{M} \subseteq \mathscr{P}(X)$ is said to be a *minimal structure* on X if $\emptyset$, $X \in \mathscr{M}$. In this case $(X, \mathscr{M})$ is called a *minimal space*. For example, let (X, τ) be a topological space, then τ, $SO(X)$, $PO(X)$, $\alpha O(X)$ and $\beta O(X)$ are minimal structures on X [17]. In a minimal space $(X, \mathscr{M})$, $A \in \mathscr{P}(X)$ is said to be an *m-open set* if $A \in \mathscr{M}$ and also $B \in \mathscr{P}(X)$ is an *m-closed set* if $B^c \in \mathscr{M}$. For any set $A \subseteq X$, set $m\text{-Int}(A) = \bigcup\{U : U \subseteq A, U \in \mathscr{M}\}$ and $m\text{-Cl}(A) = \bigcap\{F : A \subseteq F,\ F^c \in \mathscr{M}\}$. Note that for $A \subseteq X$, $m\text{-Cl}(A)$ (resp. $m\text{-Int}(A)$) is not necessarily m-closed (resp. m-open).

DEFINITION 2.1. [25] Let $(X, \mathscr{M})$ and $(Y, \mathscr{N})$ be two minimal spaces. A function $f : (X, \mathscr{M}) \longrightarrow (Y, \mathscr{N})$ is called *minimal continuous* (briefly m-continuous) if $f^{-1}(U) \in \mathscr{M}$ for any $U \in \mathscr{N}$.

DEFINITION 2.2. [2] Consider a minimal space $(X, \mathscr{M})$ and a nonempty subset Y of X. The family $\mathscr{M}|_Y = \{U \cap Y : U \in \mathscr{M}\}$ is called *the minimal structure induced* by $\mathscr{M}$ on Y. $(Y, \mathscr{M}|_Y)$ is called *minimal subspace* of $(X, \mathscr{M})$. Also for any subset A of X, $m\text{-Int}_Y(A) = \bigcup\{V :\ V \in \mathscr{M}|_Y \text{ and } V \subseteq A\}$ and $m\text{-Cl}_Y(A) = \bigcap\{F :\ F^c \in \mathscr{M}|_Y \text{ and } A \subseteq F\}$.

THEOREM 2.3. [3] Suppose $(X, \mathscr{M})$ and $(Y, \mathscr{N})$ are two minimal spaces and $f : (X, \mathscr{M}) \longrightarrow (Y, \mathscr{N})$ is a function. Consider a subset Z of Y with $f(X) \subseteq Z$. Then f is m-continuous if and only if $g : (X, \mathscr{M}) \longrightarrow (Z, \mathscr{N}|_Z)$ defined by $g(x) = f(x)$ for all $x \in X$ is m-continuous.

3. MKKM Space and Fan's matching Theorem

A *multimap* $F : X \multimap Y$ is a function from a set X into the power set of Y. Given $A \subseteq X$, set $F(A) := \bigcup_{x \in A} F(x)$.

DEFINITION 3.1. [23] Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D. An *abstract convex space* (X, D, Γ), consists of two nonempty sets X, D and a multimap $\Gamma : \langle D \rangle \multimap X$. In case to emphasize $X \supseteq D$, (X, D, Γ) will be denoted by $(X \supseteq D, \Gamma)$; and if $X = D$, then (X, Γ) is denoted for $(X \supseteq X; \Gamma)$. When X is a minimal (resp. topological) space, (X, D, Γ) is called *abstract convex minimal* (*resp. topological*) *space*. If $D \subseteq X$ and $E \subseteq X$, then E is called *abstract convex* if $\Gamma(A) \subseteq E$ for each $A \in \langle D \cap E \rangle$.

In [21] and some references cited therein we can see many examples of abstract convex spaces. For instance, any G-convex space or equivalently any Φ_A-space [22] and all of their subclasses are particular forms of abstract convex spaces.

The concept of abstract convex subset of an abstract convex space (X, D, Γ), given in Definition 3.1 has a generalized form as the following.

DEFINITION 3.2. Suppose that (X,D,Γ) is an abstract convex space, Y is a nonempty set and $f : X \to Y$ is a mapping. A subset $C \subseteq Y$ is called *f-abstract convex with respect to the set $B \subseteq D$*, if

$$f(\Gamma(A)) \subseteq C \text{ for any } A \in \langle B \rangle.$$

It should be noticed that any abstract convex subset $C \subseteq X$ defined in Definition 3.1, is I_X-abstract convex with respect to the set $C \cap D$, where $D \subseteq X$ and I_X is the identity map on X.

EXAMPLE 3.3. Consider sets $X = \{1,2,3,4\}$, $D = \{a,b\}$ and $Y = \{a_1,a_2,a_3,a_4\}$. Define the multimap $\Gamma : \langle D \rangle \multimap X$ by $\Gamma(\{a\}) = \{1,2\}$, $\Gamma(\{b\}) = \{2,3\}$, $\Gamma(\{a,b\}) = \{1,3\}$. Also define the function $f : X \to Y$ as $f(i) = a_i$ for $i = 1,2,3,4$. (X,D,Γ) is an abstract convex space. Since for any finite subset A of $\{a,b\}$, $f(\Gamma(A)) \subseteq \{a_1,a_2,a_3\}$, then we can say that the set $\{a_1,a_2,a_3\}$ is f-abstract convex with respect to the set $\{a,b\}$.

DEFINITION 3.4. [22] Let (X,D,Γ) be an abstract convex space and Y be a nonempty set. A multimap $F : D \multimap X$ is said to be KKM map if for any $A \in \langle D \rangle$, $\Gamma(A) \subseteq F(A)$.

For a multimap $T : X \multimap Y$ with nonempty values, if a multimap $F : D \multimap Y$ satisfies

$$T(\Gamma(A)) \subseteq F(A) \text{ for all } A \in \langle D \rangle,$$

then F is called a KKM *map with respect to T*. It is clear that a KKM map, is a KKM map with respect to the identity map I_X.

DEFINITION 3.5. [19, 22] Suppose that (X,D,Γ) is an abstract convex space and Y is a minimal space. The multimap $T : X \multimap Y$ has m-KKMC (*resp.* m-KKMO) *property* if for any m-closed (resp. m-open) valued multimap $F : D \multimap Y$ which is KKM map with respect to T, the family $\{F(x) : x \in D\}$ has the finite intersection property; i.e., $\bigcap_{x \in A} F(x) \neq \emptyset$ for any $A \in \langle D \rangle$. Denote by

(a) m-KKMC$(X,Y) = \{T : X \multimap Y : T$ has m-KKMC property$\}$,

(b) m-KKMO$(X,Y) = \{T : X \multimap Y : T$ has m-KKMO property$\}$.

A abstract convex minimal space (X,D,Γ) is called

(a) MKKMC *space* if $I_X \in m$-KKMC(X,X).

(b) MKKMO *space* if $I_X \in m$-KKMO(X,X).

(c) MKKM *space* if $I_X \in m$-KKMC$(X,X) \cap m$-KKMO(X,X).

The classes m-KKMC(X,Y) and m-KKMO(X,Y) in abstract convex minimal spaces are generalizations of classes $\mathfrak{KC}$ and $\mathfrak{KO}$ in abstract convex topological spaces. A abstract convex topological space (X,D,Γ) is called KKM *space* if $I_X \in \mathfrak{KC} \cap \mathfrak{KO}$ (see [19, 22]). According to the Definition 3.5, any KKM space is an MKKM space whereas the converse may not be true as it is shown in the following example.

EXAMPLE 3.6. Some examples of the MKKM spaces are the following.

(1) Suppose that $D=\{1,2\}$ and $X=\{1,2,3,\}$ and $\mathscr{M}=\{\emptyset,\{1,2\},\{1,3\},\{2,3\},X\}$ is a minimal structure on X. Consider multimap $\Gamma:\langle D\rangle\multimap X$ defined by $\Gamma(\{1\})=\{1\}$, $\Gamma(\{2\})=\{2,3\}$, $\Gamma(\{1,2\})=\{1,3\}$. It is not hard to see that for any KKM multimap $F:D\multimap X$ with m-closed (resp. m-open) values, the set $\{F(x):x\in D\}$ has the finite intersection property which implies that $I_X\in m$-KKMC$(X,X)\cap m$-KKMO(X,X). Then (X,D,Γ) is an MKKM space. Now consider $\tau=\{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},X\}$ as the smallest topology on X including $\mathscr{M}$ (the topology generated by the family $\mathscr{M}$). Define open (closed) valued multimap $F:D\multimap X$ such that $F(1)=\{1\}$, $F(2)=\{2,3\}$. We can see that F is a KKM multimap without finite intersection property on the set of its values. Then $I_X\notin\mathfrak{KC}(X,X)\cap\mathfrak{KO}(X,X)$ which implies that (X,D,Γ) is not a KKM space.

(2) In [3], the authors introduced the concept of *minimal generalized convex space* (briefly MG-convex space) (X,D,Γ) and proved that $I_X\in m$-KKMC$(X,X)\cap m$-KKMO(X,X), which implies that any MG-convex space (X,D,Γ) is an MKKM space.

(3) Suppose $(X,\mathscr{M})$ is a minimal vector space which is not a topological vector space (see example 2.16 in [3]). Consider the multimap $\Gamma:\langle X\rangle\multimap X$ defined by $\Gamma(\{a_0,a_1,\ldots,a_n\})=\{\sum_{i=0}^{n}\alpha_i a_i:0\leqslant\alpha_i\leqslant 1,\sum_{i=0}^{n}\alpha_i=1\}$. One can deduce that (X,Γ) is a minimal generalized convex space whereas it is not generalized convex space. Then every G-convex space is an MG-convex space but the converse is not true. It follows that any G-convex space is an MKKM space. Also since $\mathscr{M}$ is not a topology on X, so $I_X\notin\mathfrak{KC}(X,X)\cap\mathfrak{KO}(X,X)$. Observe that there is an MKKM space, which is not a KKM space. So it is clear that any space contained in G-convex space is an example of MKKM spaces such as L-space [7], H-spaces [12], convex space [14] and convex subset of a topological vector space.

A unified generalization of the open (resp. closed) versions of Fan's matching theorem in MKKM space is given in the following.

THEOREM 3.7. *Suppose that* (X,D,Γ) *is an* MKKMC (*resp.* MKKMO) *space and* Y *is a minimal space. Consider a function* $S:X\to Y$ *and a multimap* $G:D\multimap S(X)$ *satisfying*

(a) S is m-continuous,

(b) for any $x\in D$, $G(x)$ is m-open (resp. m-closed) in $S(X)$,

(c) $S(X)=\bigcup_{x\in A}G(x)$, for some $A\in\langle D\rangle$.

Then there exists $B\in\langle D\rangle$ such that $S(\Gamma(B))\cap\bigcap_{x\in B}G(x)\neq\emptyset$.

Proof. On the contrary suppose that for any $B\in\langle D\rangle$,

$$S(\Gamma(B))\subseteq\bigcup_{x\in B}S(X)\setminus G(x).$$

So for any $B\in\langle D\rangle$, $\Gamma(B)\subseteq\bigcup_{x\in B}S^{-1}(S(X)\setminus G(x))$. Define the multimap $H:D\multimap X$ with $H(x)=S^{-1}(S(X)\setminus G(x))$, for each $x\in D$. Since S is an m-continuous function, Theorem 2.3 implies that $H(x)$ is m-closed (resp. m-open) for any $x \in D$. Also for any $B \in \langle D \rangle$, $\Gamma(B) \subseteq \bigcup_{x\in B} H(x) = H(B)$. According to the Definition 3.5, the set $\{H(x) : x \in D\}$ has the finite intersection property and so $\bigcap_{x\in A} H(x) \neq \emptyset$. This means that $\bigcap_{x\in A} S^{-1}(S(X) \setminus G(x)) \neq \emptyset$ or equivalently $\bigcap_{x\in A} S(X) \setminus G(x) \neq \emptyset$. Then $S(X) \neq \bigcup_{x\in A} G(x)$ which contradicts assertion (b). $\square$

REMARK 3.8. If (X,D,Γ) is a KKM space and $S = I_X$, then Theorem 3.7 goes back to Theorem 3.3 and Corollary 3.4 in [20]. When (X,D,Γ) is a G-convex space and $S = I_X$, Theorem 3.7 reduces to Lemma 1 in [6]. Also Theorem 3.7 is a generalization of Theorem 3.2 in [27] for FC-space.

4. S-weakly KKM Theorems

DEFINITION 4.1. Suppose that (X,D,Γ) is an abstract convex space, Y and Z are nonempty sets, $S : X \to Y$ is a function and $F : Y \multimap Z$ is a multimap. The multimap $G : D \multimap Z$ is called S-*weakly* KKM *with respect to* F if $F(y) \cap G(A) \neq \emptyset$, for any $A \in \langle D \rangle$ and all $y \in S(\Gamma(A))$.

It is obvious that any weakly KKM map with respect to F, discussed in [6, 20, 27] is I_X-weakly KKM map with respect to F.

DEFINITION 4.2. [25] For a minimal space $(X,\mathscr{M})$

(a) a family $\mathscr{A} = \{A_j : j \in J\}$ of m-open sets in X is called an m-*open cover* of K if $K \subseteq \bigcup_{j\in J} A_j$. Any subfamily of $\mathscr{A}$ which is also an m-open cover of K is called a *subcover* of $\mathscr{A}$ for K;

(b) a subset K of X is m-*compact* if any given m-open cover of K has a finite subcover.

LEMMA 4.3. [25] *Suppose that X and Y are two minimal spaces and $f : X \to Y$ is an m-continuous function. For any m-compact subset $K \subseteq X$, $f(K)$ is m-compact in Y.*

THEOREM 4.4. *Suppose that (X,D,Γ) is an m-compact* MKKMC *space, Y is a minimal space and Z is a nonempty set. Consider function $S : X \to Y$ and multimaps $F : S(X) \multimap Z$ and $G : D \multimap Z$ satisfying*

(a) S is m-continuous,

(b) G is S-weakly KKM with respect to F,

(c) the set $\{y \in S(X) : F(y) \cap G(x) \neq \emptyset\}$ is m-closed in $S(X)$ for each $x \in D$.

Then there exists $y_0 \in S(X)$ such that $F(y_0) \cap G(x) \neq \emptyset$, for any $x \in D$.

Proof. Define the multimap $H : D \multimap S(X)$ such that $H(x) = \{y \in S(X) : F(y) \cap G(x) = \emptyset\}$, for each $x \in D$. The multimap H is m-open valued in $S(X)$. Now suppose that the conclusion does not hold and so for any $y \in S(X)$, $F(y) \cap G(x) = \emptyset$, for some $x \in D$. This means that

$$S(X) \subseteq \bigcup_{x \in D} \{y \in Y : F(y) \cap G(x) = \emptyset\} = \bigcup_{x \in D} H(x) \subseteq S(X).$$

According to Lemma 4.3, $S(X)$ is m-compact. So there exists $A \in \langle D \rangle$ such that $S(X) = \bigcup_{x \in A} H(x)$. Now Theorem 3.7 implies that there exists $B \in \langle D \rangle$ such that $S(\Gamma(B)) \cap \bigcap_{x \in B} H(x) \neq \emptyset$. Choose $\bar{y} \in S(\Gamma(B)) \cap \bigcap_{x \in B} H(x)$. Since G is S-weakly KKM with respect to F, then $F(\bar{y}) \cap G(B) \neq \emptyset$. On the other hand from $\bar{y} \in \bigcap_{x \in B} H(x)$ we have $F(\bar{y}) \cap G(x) = \emptyset$ for each $x \in B$ or $F(\bar{y}) \cap G(B) = \emptyset$, which is a contradiction. □

REMARK 4.5. Theorem 4.4 is an extended version of Theorem 2 in [16], Theorem 4.3 in [20] and Theorem 3.3 in [27].

5. Application to minimax inequality

DEFINITION 5.1. Suppose that X and Y are minimal spaces. A multimap $T : X \multimap Y$ is said to be *minimal upper semicontinuous* (m-usc), if for any m-closed subset $B \subseteq Y$, the set $\{x \in X : T(x) \cap B \neq \emptyset\}$ is m-closed. Also a real valued bifunction $f : X \times Y \to \mathbb{R}$ is called *m-usc in the second variable* if for any $x \in X$ and $\alpha \in \mathbb{R}$, the set $\{y \in Y : f(x,y) < \alpha\}$ is m-open. More details can be found in [1].

THEOREM 5.2. *Suppose that (X, D, Γ) is an m-compact* MKKMC *space, Y and Z are minimal spaces. Consider a function $S : X \to Y$, an m-usc multimap $F : S(X) \multimap Z$, two real valued bifunctions $\varphi : S(X) \times Z \to \mathbb{R}$ and $\psi : D \times Z \to \mathbb{R}$ satisfying*

(a) *S is m-continuous,*

(b) *ψ is m-usc in the second variable,*

(c) *for any $\alpha < \inf_{y \in S(X)} \sup_{z \in F(y)} \varphi(y,z)$ and $z \in F(y)$, the set $\{y \in S(X) : \varphi(y,z) < \alpha\}$ is S-abstract convex with respect to the set $\{x \in D : \psi(x,z) < \alpha\}$.*

Then

$$\inf_{y \in S(X)} \sup_{z \in F(y)} \varphi(y,z) \leqslant \sup_{y \in S(X)} \inf_{x \in D} \sup_{z \in F(y)} \psi(x,z).$$

Proof. Let $\beta = \inf_{y \in S(X)} \sup_{z \in F(y)} \varphi(y,z)$. Choose a fixed $\alpha < \beta$ and define $G : D \multimap Z$ by

$$G(x) = \{z \in Z : \psi(x,z) \geqslant \alpha\} \text{ for each } x \in D.$$

By (b), $G(x)$ is m-closed for each $x \in D$. We claim that G is S-weakly KKM with respect to F. On the contrary suppose that there is $A \in \langle D \rangle$ and $\bar{y} \in S(\Gamma(A))$ such that $F(\bar{y}) \cap G(A) = \emptyset$. So $z \notin G(A)$ for any $z \in F(\bar{y})$. Hence $A \subseteq \{x \in D : \psi(x,z) < \alpha\}$ for all $z \in F(\bar{y})$. It follows from (c),

$$\bar{y} \in S(\Gamma(A)) \subseteq \{y \in S(X) : \varphi(y,z) < \alpha\} \text{ for all } z \in F(\bar{y}).$$

Therefore $\sup_{z\in F(\bar{y})} \varphi(\bar{y},z) \leqslant \alpha$, which contradicts $\alpha < \beta$.

Since F is an m-usc multimap and G is m-closed valued, then the set $\{y \in S(X) : F(y)\cap G(x) \neq \emptyset\}$ is m-closed in $S(X)$, for each $x \in D$. According to the Theorem 4.4, there exists $y_0 \in S(X)$ such that $F(y_0)\cap G(x) \neq \emptyset$ or $\alpha \leqslant \psi(x,z_0)$ for some $z_0 \in F(y_0)$ and all $x \in D$. That is $\alpha \leqslant \inf_{x\in D} \sup_{z\in F(y_0)} \psi(x,z)$, for some $y_0 \in S(X)$. Hence $\alpha \leqslant \sup_{y\in S(X)} \inf_{x\in D} \sup_{z\in F(y)} \psi(x,z)$, which gives the inequality. □

REMARK 5.3. The origin of Theorem 5.2 goes back to Ky Fan's minimax inequality [10]. Theorem 5.2 improves Theorem 4.11 in [20], part (a) of Theorem 4 in [6] and part (a) of Theorem 4.1 in [27]. Also it is a generalization of Fan type minimax inequalities in [11, 13, 16, 22].

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