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DISCRETE BERNSTEIN INEQUALITIES FOR POLYNOMIALS

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Abstract. We study discrete versions of some classical inequalities of Berstein for algebraic and trigonometric polynomials.

1. Introduction and statement of the results

Let $\mathscr{P}_n$ denote the class of polynomials $p(z) = \sum_{k=0}^{n} a_k z^k$ of degree at most n with complex coefficients and $\mathscr{T}_n$ the class of trigonometric polynomials $t(\theta) = \sum_{k=-n}^{n} a_k e^{ik\theta}$. According to the famous inequalities of Bernstein and Markov we have for any $p \in \mathscr{P}_n$

$$|p'|_{\mathbb{D}} \leqslant n|p|_{\mathbb{D}} \tag{1}$$

and

$$|p'|_{[-1,1]} \leqslant n^2 |p|_{[-1,1]}. \tag{2}$$

Here $|q|_{\mathbb{D}}$ (resp. $|q|_{[-1,1]}$) means the maximum modulus of the analytic function q over the unit disc $\mathbb{D} = \{z \mid |z| < 1\}$ (resp. the unit real interval $[-1,1]$). We refer the reader to the book of Rahman and Schmeisser [10] concerning the ubiquity of these inequalities in modern approximation theory. The following discrete versions of (1) and (2) are also available:

$$|p'|_{\mathbb{D}} \leqslant n \max_{1 \leqslant j \leqslant 2n} |p(e^{ij\pi/n})| \tag{3}$$

$$|p'|_{[-1,1]} \leqslant n^2 \max_{0 \leqslant j \leqslant n} \left|p\big(\cos(j\pi/n)\big)\right|. \tag{4}$$

The inequality (4) has been obtained by Duffin and Schaeffer [5] in 1947. Surprisingly, the inequality (3) was only obtained in 1985 by Frappier, Rahman and Ruscheweyh [8]. All of the above inequalities are sharp and the extremal polynomials are known (see [4] concerning in this respect the statement (3)); the number $2n$ of polynomial values appearing in (3) cannot be replaced, as it is explained in [8], by a smaller one and any closed subset E of $[-1,1]$ such that

$$|p'|_{[-1,1]} \leqslant n^2 \max_{\zeta \in E} |p(\zeta)|, \quad p \in \mathscr{P}_n,$$

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must contain the point set $\{\cos(j\pi/n) \mid 0 \leqslant j \leqslant n\}$. Dryanov [2] has more recently obtained a refinement of (4).

There are of course other types of Bernstein–Markov inequalities. We mention for example

$$|t'|_{[0,2\pi]} \leqslant n|t|_{[0,2\pi]}, \quad t \in \mathscr{T}_n \tag{5}$$

or else

$$|\sqrt{1-x^2}p'(x)| \leqslant n|p|_{[-1,1]}, \quad p \in \mathscr{P}_n, \quad -1 \leqslant x \leqslant 1. \tag{6}$$

Both of (5) and (6) are due to Bernstein; concerning their discrete counterparts, we make the two following observations:

OBSERVATION 1. Given two positive integers n and N and a set of distinct nodes $\{\theta_j\}_{j=0}^N \subset [0,2\pi]$, there exists $t \in \mathscr{T}_n$ such that $|t'|_{[0,2\pi]} > n\max_{0\leqslant j\leqslant N}|t(\theta_j)|$. It suffices to consider a polynomial of the type $t(\theta) = A\mathrm{e}^{in\theta} + B\mathrm{e}^{-in\theta}$ with properly chosen complex coefficients A and B.

OBSERVATION 2. Given a positive integer n and a set $\{x_j\}_{j=0}^n \subset [-1,1]$ of distinct nodes, there exists $p\in\mathscr{P}_n$ such that $\max_{-1\leqslant x\leqslant 1}|\sqrt{1-x^2}p'(x)| > n\max_{0\leqslant j\leqslant n}|p(x_j)|$.

About the second observation, we assume first that $x_j = \cos(\theta_j)$ with $0 \leqslant \theta_0 < \theta_1 \cdots < \theta_n \leqslant \pi$ and $\{\cos(\theta_j)_{j=0}^n \neq \{\cos(j\pi/n)\}_{j=0}^n$. By the Lagrange interpolation formula we have, for any $p \in \mathscr{P}_n$,

$$p'(z) = \sum_{j=0}^n \frac{\mathrm{d}}{\mathrm{d}z}\frac{w(z)}{(z-\cos\theta_j)w'(\cos\theta_j)}p(\cos\theta_j) \tag{7}$$

where $w(z) = \prod_{j=0}^n(z-\cos\theta_j)$. We claim that

$$\max_{x\in[-1,1]}\sum_{j=0}^n\left|\frac{\mathrm{d}}{\mathrm{d}x}\frac{w(x)}{(x-\cos\theta_j)w'(\cos\theta_j)}\right| > n^2. \tag{8}$$

If this was not the case, we would obtain from (7)

$$\max_{-1\leqslant x\leqslant 1}|p'(x)| \leqslant n^2\max_{0\leqslant j\leqslant n}|p(\cos\theta_j)|, \quad p \in \mathscr{P}_n,$$

and this is impossible, if $\{\theta_j\}_{j=0}^n \neq \{j\pi/n\}_{j=0}^n$, by the unicity result of Duffin and Schaeffer [5] (see also [10, p. 574] for a detailed proof). We also claim the existence of $\widetilde{\theta} \in [0,2\pi)$ with

$$\sum_{j=0}^n\left|\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{w(\cos\theta)}{(\cos\theta-\cos\theta_j)w'(\cos\theta_j)}\right|_{\theta=\widetilde{\theta}} > n \tag{9}$$

because otherwise

$$\sqrt{1-x^2}\left|\frac{\mathrm{d}}{\mathrm{d}x}\sum_{j=0}^n \pm\frac{w(x)}{(x-\cos\theta_j)w'(\cos\theta_j)}\right| \leqslant n, \quad x \in [-1,1],$$

for any choice of the $\pm$ sign structure and by the classical result of Bernstein [10, p. 567]

$$\left|\frac{\mathrm{d}}{\mathrm{d}x}\sum_{j=0}^{n}\pm\frac{w(x)}{(x-\cos\theta_j)w'(\cos\theta_j)}\right|\leqslant n^2, \quad x\in[-1,1],$$

which of course contradicts (8). We now consider a polynomial p defined as

$$p(x)=\sum_{j=0}^{n}\pm\frac{w(x)}{(x-\cos\theta_j)w'(\cos\theta_j)}$$

where the sign structure $\pm$ is to be determined such that

$$\begin{aligned}\sin(\widetilde{\theta})p'\big(\cos(\widetilde{\theta})\big) &= -\frac{\mathrm{d}}{\mathrm{d}\theta}p(\cos(\theta))\big|_{\theta=\widetilde{\theta}}\\ &=\sum_{j=0}^{n}\mp\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{w(\cos(\theta))}{(\cos(\theta)-\cos(\theta_j))w'(\cos(\theta_j))}\Bigg|_{\theta=\widetilde{\theta}}\\ &=\sum_{j=0}^{n}\left|\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{w(\cos(\theta))}{(\cos(\theta)-\cos(\theta_j))w'(\cos(\theta_j))}\right|_{\theta=\widetilde{\theta}}\\ &> n = n\max_{0\leqslant j\leqslant n}|p(\cos(\theta_j))|.\end{aligned}$$

This settles our claim except in the situation where $\{\cos(\theta_j\}_{j=0}^{n}=\{\cos(j\pi/n)\}_{j=0}^{n}$. In that case we have [3]

$$\max_{p\in\mathscr{P}_n}\frac{|\sin(\theta)p'(\cos\theta)|}{\max\limits_{0\leqslant j\leqslant n}|p(\cos j\pi/n)|}=\frac{1}{2}\sum_{j=0}^{n}\left|\frac{\mathrm{d}}{\mathrm{d}\theta}\mathrm{d}_n(j;\theta)\right|$$

with

$$d_n(j,\theta)=\begin{cases}\dfrac{(-1)^j}{2n}\dfrac{\sin(n\theta)\sin(\theta)}{\cos(j\pi/n)-\cos(\theta)} & \text{if} \quad j=0,n\\[2ex] \dfrac{(-1)^j}{n}\dfrac{\sin(n\theta)\sin(\theta)}{\cos(j\pi/n)-\cos(\theta)} & \text{if} \quad 0<j<n\end{cases}$$

and there is numerical evidence that

$$\max_{\theta}\max_{P\in\mathscr{P}_n}\frac{|\sin(\theta)p'(\cos\theta)|}{\max\limits_{0\leqslant j\leqslant n}|p(\cos j\pi/n)}\simeq n\ln(n), \quad n\to\infty.$$

Our aim in this note is to obtain, amongst other results, discrete versions of (5) and (6). We shall prove:

THEOREM 1. *For any $p\in\mathscr{P}_n$, θ real,*

$$|\sin(\theta)p'(\cos\theta)|\leqslant n\max_{1\leqslant j\leqslant 2n}\left|p\left(\cos\left(\theta+\frac{(2j-1)\pi}{2n}\right)\right)\right|.$$

Given a set of distinct nodes $\{\theta_j\}_{j=0}^n \subset [0,\pi]$, let us define for $p \in \mathscr{P}_n$

$$|p|_n = \max_{0\leqslant j\leqslant n} \left| \frac{p(\mathrm{e}^{i\theta_j}) + p(\mathrm{e}^{-i\theta_j})}{2} \right|.$$

It has recently been obtained [7] that

$$\max_{p\in\mathscr{P}_n} \frac{|p'|_{\mathbb{D}}}{|p|_n} = O\big(n\ln(n)\big), \quad n \to \infty,$$

in the case of nodes $\{\theta_j\}_{j=0}^n = \{j\pi/n\}_{j=0}^n$. This of course can be seen as a discrete version of (1). Our next result has a similar flavor. We define for $p \in \mathscr{P}_n$,

$$\|p\|_n = \max_{0\leqslant j\leqslant n} \big|p(\cos(\theta_j))\big|.$$

This is a norm over $\mathscr{P}_n$ (as well as $|\cdot|_n$) and we shall prove :

THEOREM 2. *There exist distinct nodes $\{\theta_j\}_{j=0}^n \subset [0,\pi]$ such that*

$$\max_{\substack{p\in\mathscr{P}_n \\ \theta \text{ real}}} \frac{|\sin(\theta)p'(\cos\theta)|}{\|p\|_n} = O\big(n\ln(n)\big), \quad n \to \infty.$$

It is a well-known result due to De Bruijn [1] and perhaps others that for $p \in \mathscr{P}_n$, $z \in \mathbb{D}$ and $|\zeta| = 1$,

$$\left| p(z) + \frac{\zeta - 1}{n} z p'(z) \right| \leqslant |p|_{\mathbb{D}}.$$

We shall also obtain a discrete version of this inequality :

THEOREM 3. *Let $\{r_j\}_{j=1}^n$ be the n distinct n-roots of unity. Then for p, ζ, as above and an arbitrary complex number z,*

$$\left| p(z) + \frac{\zeta - 1}{n} z p'(z) \right| \leqslant \max_{1\leqslant j\leqslant n} |p(r_j \zeta^{1/n} z)|.$$

2. Proof of Theorem 1

We use the notation

$$\sum_{j=0}^{n}{}'' \alpha_j := \frac{\alpha_0}{2} + \sum_{j=1}^{n-1} \alpha_j + \frac{\alpha_n}{2}.$$

It has been obtained in [8] that for any $q \in \mathscr{P}_N$, θ real

$$\frac{q(\mathrm{e}^{i\theta}) - q(\mathrm{e}^{-i\theta})}{\mathrm{e}^{i\theta} - \mathrm{e}^{-i\theta}} = \sum_{j=0}^{N}{}''(-1)^j \frac{(-1)^j - \cos(N\theta)}{N(\cos(j\pi/N) - \cos(\theta))} \frac{q(\mathrm{e}^{ij\pi/N}) + q(\mathrm{e}^{-ij\pi/N})}{2}. \quad (10)$$

A simple computation leads to

$$q(1)-\frac{2}{N}q'(1)=\sum_{\substack{j=1\\ j\text{ odd}}}^{N}{}''\frac{1}{N^2\sin(j\pi/2N)}\big(q(e^{ij\pi/N})+q(e^{-ij\pi/N})\big)$$

and more generally for any complex number z,

$$q(z)-\frac{2}{N}zq'(z)=\sum_{\substack{j=1\\ j\text{ odd}}}^{N}{}''\frac{1}{N^2\sin(j\pi/2N)}\big(q(e^{ij\pi/N}z)+q(e^{-ij\pi/N}z)\big). \tag{11}$$

Given $p\in\mathscr{P}_n$, we define $q\in\mathscr{P}_{2n}$ by $q(z)\equiv z^n p\big((z+1/z)/2\big)$ and apply (10) with $N=2n$. This yields

$$\sin(\theta)p'(\cos\theta)=\frac{1}{n}\sum_{k=1}^{2n}(-1)^k\frac{p(\cos(\theta+(2k-1)\pi/2n))}{(2\sin((2k-1)\pi/4n))^2} \tag{12}$$

and choosing p as the nth Chebyshev polynomial we see that

$$\sum_{k=1}^{2n}1/\big[2\sin((2k-1)\pi/(4n))\big]^2=n^2.$$

We therefore obtain, $0\leqslant\theta\leqslant 2\pi$,

$$|\sin(\theta)p'(\cos\theta)|\leqslant n\max_{1\leqslant k\leqslant 2n}\left|p\left(\cos(\theta+\frac{(2k-1)\pi}{2n}\right)\right| \tag{13}$$

and it is easily seen from (12) that equality shall hold for some real θ in (13) if and only if for all $1\leqslant j\leqslant 2n$,

$$p\left(\cos\left(\theta+\frac{(2j-1)\pi}{2n}\right)\right)=(-1)^j\max_{1\leqslant k\leqslant 2n}\left|p\left(\cos\left(\theta+\frac{(2k-1)\pi}{2n}\right)\right)\right|.$$

Since the above already holds for any multiple of the nth Chebyshev polynomial, it is clear that equality shall hold in (13) only for such polynomials.

Given a trigonometric polynomial $t\in\mathscr{T}_n$, we may apply (11) to the polynomial $q\in\mathscr{P}_{2n}$ defined by $q(e^{i\theta})=e^{in\theta}t(\theta)$ and obtain, as a corollary of our argument above

$$|t'(\theta)|\leqslant n\max_{1\leqslant k\leqslant 2n}\left|t\left(\theta+\frac{(2k-1)\pi}{2n}\right)\right|,\quad 0\leqslant\theta<2n, \tag{14}$$

where equality holds, for a given θ, for trigonometric polynomials of the type $t(\varphi)\equiv Ae^{in\varphi}+Be^{-in\varphi}$.

The knowledgeable reader of course noticed that the inequalities (13) and (14) are also direct consequences of the classical Marcel Riesz interpolation for trigonometric polynomials [11]. This is not surprising since our formula (10) contains the Riesz formula; the later is as a matter of fact equivalent, for each even integer n, to the identity

$$\sum_{k=0}^{n} \frac{n-2k}{n} z^k = \frac{1}{n^2} \sum_{j=1}^{n} \frac{\lambda_{j,n}}{1-\zeta, z} + o(z^n) \tag{15}$$

where $\lambda_{j,n} = \csc^2\big((2j-1)\pi/2n\big)$ and $\zeta_j = \mathrm{e}^{i(2j-1)\pi/n}$ while our formula (10) implies that (15) holds for any positive integer. It follows in particular that for any complex number z and any $p \in \mathscr{P}_n$,

$$\left| zp'(z) - \frac{n}{2} p(z) \right| \leqslant \frac{n}{2} \max_{1 \leqslant j \leqslant n} |p(\zeta_j z)|,$$

which is of course another discrete improvement of the Bernstein inequality (1).

We shall end this section with an "aesthetical" remark. There is something unsatisfactory when comparing (3) on (4) with the results we have so far obtained since these results do not display a uniform estimate for the derivative of a polynomial. We remark that (3) can be used to provide such an estimate. Let $q \in \mathscr{P}_n$ and $p \in \mathscr{P}_{2n}$ defined by $p(z) \equiv z^n q\big((z+1/z)/2\big)$; it follows (3) that

$$|p'|_{\mathbb{D}} \leqslant 2n \max_{1 \leqslant j \leqslant 4n} |p(\mathrm{e}^{ij\pi/n})|$$

or equivalently, for any real θ,

$$|nq(\cos\theta) \pm i \sin(\theta) q'(\cos\theta)| \leqslant 2n \max_{1 \leqslant k \leqslant 2n} \left| q\left(\cos\frac{k\pi}{2n}\right)\right|.$$

In particular

$$|\sin(\theta) q'(\cos\theta)|_{[0,2\pi]} \leqslant 2n \max_{1 \leqslant k \leqslant 2n} \left| q\left(\cos\frac{k\pi}{2n}\right)\right|$$

but equality shall hold if and only if $q \equiv 0$!

3. Proof of Theorem 2

According to the Lagrange interpolation formula, we have for any $p \in \mathscr{P}_n$ and any collection $\{\theta_j\}_{j=0}^n$ of distinct nodes in $[0,\pi]$

$$p(\cos\theta) = \sum_{j=0}^{n} L_j(\cos\theta) p(\cos\theta_j) \tag{16}$$

where, together with $w(z) = \prod_{j=0}^{n}(z - \cos\theta_j)$, we set

$$L_j(z) = \frac{w(z)}{(z-\cos\theta_j) w'(\cos\theta_j)}.$$

It is an old result due to Erdös ([13, p. 109]) that nodes $\{\theta_j\}_{j=0}^n$ can be chosen such that for all real θ,

$$\sum_{j=0}^{n} |L_j(\cos\theta)| \leqslant M \ln(n)$$

for some constant $M > \frac{2}{\pi}$. For an arbitrary choice of the $\pm$ sign structure, $t(\theta) := \sum_{j=0}^{n} \pm L_j(\cos\theta)$ is a trigonometric polynomial in $\mathscr{T}_n$ for which $|t|_{[0,2\pi]} \leqslant M\ln(n)$ and by the Bernstein inequality (5) we obtain

$$|t'(\theta)| = \left|\frac{\mathrm{d}}{\mathrm{d}\theta}\sum_{j=0}^{n} \pm L_j(\cos\theta)\right| \leqslant Mn\ln(n)$$

and $\sum_{j=0}^{n} |\mathrm{d}/\mathrm{d}\theta L_j(\cos\theta)| \leqslant Mn\ln(n)$. The conclusion follows from (16). There is computer evidence that the order of growth $O(n\ln n)$ is sharp and $\{\theta_j\}_{j=0}^{n} = \{j\pi/n\}_{j=0}^{n}$ is a set of extremal nodes.

4. Proof of Theorem 3

After the work of Ruscheweyh [12, chapter 4] we may interpret the inequality of De Bruinj as meaning that the polynomial

$$b_n(z) := \sum_{k=0}^{n}\left(1+\left(\frac{\zeta-1}{n}\right)k\right)z^k$$

is bound-preserving over $\mathscr{P}_n$; that is to say

$$|b_n \star p|_{\mathbb{D}} \leqslant |p|_{\mathbb{D}}, \quad p \in \mathscr{P}_n$$

where $\star$ stands for the Hadamard product of the polynomials b_n and p. Any such bound-preserving polynomial admits a representation

$$b_n(z) = F(z) + o(z^n)$$

where F is a function analytic in $\mathbb{D}$ with real part greater than half there. Further, because the leading coefficient of b_n equals ζ which has modulus 1, we obtain [9, chapter 7]

$$b_n(z) = \sum_{j=1}^{n}\frac{\ell_j}{1-\zeta^{1/n}r_j z} + o(z^n), \quad |z| < 1 \tag{17}$$

where $\zeta^{1/n}$ is an arbitrary nth root of ζ and $\ell_j \geqslant 0$. It follows from (17) that for any $p \in \mathscr{P}_n$

$$p(z) + \frac{\zeta-1}{n}zp'(z) = \sum_{j=1}^{n}\ell_j p(\zeta^{1/n}r_j z)$$

and this last identity must extend to all complex numbers z and we therefore obtain

$$\left|p(z) + \frac{\zeta-1}{n}zp'(z)\right| \leqslant \max_{1\leqslant j\leqslant n}|p(\zeta^{1/n}r_j z)|. \tag{18}$$

A discussion of equality cases in (18) shall be possible once an explicit representation of the coefficients ℓ_j becomes available. We have to solve the linear system

$$\sum_{j=1}^{n} r_j^k \ell_j = \left(1+\frac{\zeta-1}{n}k\right)\bar{\zeta}^{k/n}, \quad k = 0,1,\ldots,n-1.$$

This clearly involves the inverse of the Vandermonde matrix

$$(r_j^k)_{0\leqslant k\leqslant n-1,\ 1\leqslant j\leqslant n}.$$

We shall state the result (details may be found in [6]) without proof when $\zeta \neq 1$:

$$\ell_j = \frac{-|1-\zeta|^2}{n^2}\frac{r_j\zeta^{1/n}}{(1-r_j\zeta^{1/n})^2}, \quad 1\leqslant j\leqslant n.$$

It is in particular clear that each ℓ_j is strictly positive and in that case equality shall hold in (18) for a given z if and only if $p(\zeta^{1/n}r_jz)$ is constant for $1\leqslant j\leqslant n$. This implies of course that $p(Z)\equiv AZ^n+B$ for some complex constants A, B. The case $\zeta=-1$ is of special interest since it leads again to the Marcel Riesz interpolation formula while the case $\zeta=1$ is of course degenerated, every $p\in\mathscr{P}_n$ being in that case extremal!

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