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OPERATOR INEQUALITIES ON HILBERT C^*–MODULES VIA THE CAUCHY–SCHWARZ INEQUALITY

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Abstract. By means of a new Cauchy-Schwarz inequality in the framework of a semi-inner product C^*-module over a unital C^*-algebra, we discuss some operator inequalities on a Hilbert C^*-module, for example, Kantorovich inequaity, Pólya-Szegö inequality, the covariance-variance inequality, Ozeki-Izumino-Mori-Seo inequality, Wielandt inequality, Hienz-Kato-Furuta inequality and Malamud inequality.

1. Introduction

Hilbert C^*-modules are mathematical objects which generalize the notions of a Hilbert space and a C^*-algebra. The theory of Hilbert C^*-modules is different from that of Hilbert spaces, for example, not all bounded linear operator between Hilbert C^*-modules is adjointable and not all closed submodule of a Hilbert C^*-module is complemented, see [24]. The theory of Hilbert C^*-modules over commutative C^*-algebras was first appeared in a work of Kaplansky [23] in 1953. Since then it has grown rapidly and has played significant roles in the theory of operator algebras and noncommutative geometry, also see [24].

The Cauchy-Schwarz inequality is one of the most important inequalities in mathematics. Spreading out the idea of Kantorovich inequality, Dragomir [4] proposed several additive and multiplicative type reverses of the Cauchy–Schwarz inequality in a pre-inner product space. Niculescu [29], Ilišević-Varošanec [20] and J.I. Fujii [6], Moslehian–Persson [27] and Arambasić–Bakić–Moslehian [2] have investigated some Cauchy–Schwarz type inequalities and its various reverses in the framework of semi-inner product C^*-modules. In [7], we gave some reverse Cauchy–Schwarz inequalities and presented some Klamkin–Mclenaghan, Shisha–Mond, Cassels and Grüss type inequalities on semi-inner product C^*-modules. In [8], we presented a new Cauchy-Schwarz inequality in the framework of a semi-inner product C^*-module over a unital C^*-algebra, and as an application we obtained a Kantorovich type inequality on a Hilbert C^*-module.

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In this paper, as a continuation of [8], by means of a new Cauchy-Schwarz inequality in a semi-inner product C^*-module, we discuss some operator inequalities on a Hilbert C^*-module, for example, Kantorovich inequality, Pólya-Szegö inequality, the covariance-variance inequality, Ozeki-Izumino-Mori-Seo inequality, Wielandt inequality, Hienz-Kato-Furuta inequality and Malamud inequality.

2. Preliminaries

Let us fix our notation and terminology. Let $\mathscr{B}(H)$ be the C^*-algebra of all bounded linear operators on a Hilbert space H, and $\mathscr{A}$ be a unital C^*-algebra of $\mathscr{B}(H)$ with the center $\mathscr{Z}(\mathscr{A})$. Since a unit of $\mathscr{A}$ differs from the identity operator of $\mathscr{B}(H)$, we denote the unit element of $\mathscr{A}$ by e. For $a \in \mathscr{A}$, we denote the real part of a by $\mathrm{Re}\, a = \frac{1}{2}(a+a^*)$, and the absolute value of a by $|a| = (a^*a)^{\frac{1}{2}}$. For positive elements $a, b \in \mathscr{A}$, the operator geometric mean of a and b is defined by

$$a \sharp b = a^{\frac{1}{2}} \left(a^{-\frac{1}{2}} b a^{-\frac{1}{2}}\right)^{\frac{1}{2}} a^{\frac{1}{2}}$$

for invertible a. If a and b are non invertible, then $a \sharp b \in \mathscr{A}'' \subset \mathscr{B}(H)$, where $\mathscr{A}''$ is the double commutant or equivalently the closure of $\mathscr{A}$ in the strong operator topology on $\mathscr{B}(H)$. In fact, since $a \sharp b$ satisfies the upper semicontinuity, it follows that $a \sharp b = \lim_{\varepsilon \to +0}(a+\varepsilon e) \sharp (b+\varepsilon e)$ in the strong operator topology. If $\mathscr{A}$ is monotone complete, in the sense that every bounded increasing net in the self-adjoint part has a supremum with respect to the usual partial order, then we have $a \sharp b \in \mathscr{A}$, see [18]. The operator geometric mean has the symmetric property: $a \sharp b = b \sharp a$. In the case that a and b commute, we have $a \sharp b = \sqrt{ab}$. The operator geometric mean has the following characterization (see [1]):

$$a \sharp b = \max\left\{ X \in \mathscr{A}'' \subset \mathscr{B}(H) : X = X^*,\ \begin{pmatrix} a & X \\ X & b \end{pmatrix} \geqslant 0 \right\}. \tag{2.1}$$

A complex linear space $\mathscr{X}$ is said to be an inner product $\mathscr{A}$-module (or a pre-Hilbert $\mathscr{A}$-module) if $\mathscr{X}$ is a right $\mathscr{A}$-module together with a C^*-valued map $(x,y) \mapsto \langle x,y \rangle : \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ such that

(i) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x,y \rangle + \beta \langle x,z \rangle \quad (x,y,x \in \mathscr{X}, \alpha, \beta \in \mathbb{C})$,

(ii) $\langle x, ya \rangle = \langle x,y \rangle a \quad (x,y \in \mathscr{X}, a \in \mathscr{A})$,

(iii) $\langle y,x \rangle = \langle x,y \rangle^* \quad (x,y \in \mathscr{X})$,

(iv) $\langle x,x \rangle \geqslant 0$ $(x \in \mathscr{X})$ and if $\langle x,x \rangle = 0$, then $x = 0$.

The linear structures of $\mathscr{A}$ and $\mathscr{X}$ are assumed to be compatible. If $\mathscr{X}$ satisfies all conditions for an inner-product $\mathscr{A}$-module except for the second part of (iv), then we call $\mathscr{X}$ a semi-inner product $\mathscr{A}$-module. For such modules, the following Cauchy-Schwarz inequality [24, Proposition 1.1] holds:

$$\langle x,y \rangle^* \langle x,y \rangle \leqslant \| \langle x,x \rangle \| \langle y,y \rangle \qquad \text{for all } x,y \in \mathscr{X}. \tag{2.2}$$

In this case, we write $\| x \| := \sqrt{\| \langle x,x \rangle \|}$, where the latter norm denotes the C*-norm of $\mathscr{A}$. It follows from (2.2) that if $\mathscr{X}$ is an inner-product $\mathscr{A}$-module, then $\| \cdot \|$ is a norm on $\mathscr{X}$. If $\mathscr{X}$ is complete with respect to this norm, then $\mathscr{X}$ is called a *Hilbert $\mathscr{A}$-module*.

In [8], from a viewpoint of operator theory, we presented the following new Cauchy-Schwarz inequality in the framework of a semi-inner product C*-module over a unital C*-algebra:

THEOREM 2.1. (Cauchy-Schwarz inequality) *Let $\mathscr{X}$ be a semi-inner product $\mathscr{A}$-module over a unital C^*–algebra $\mathscr{A}$. If $x,y \in \mathscr{X}$ such that the inner product $\langle x,y \rangle$ has a polar decomposition $\langle x,y \rangle = u|\langle x,y \rangle|$ with a partial isometry $u \in \mathscr{A}$, then*

$$|\langle x,y \rangle| \leqslant u^* \langle x,x \rangle u \sharp \langle y,y \rangle. \tag{2.3}$$

Under the assumption that $\mathscr{X}$ is an inner product $\mathscr{A}$-module and $\langle y,y \rangle$ is invertible, the equality in (2.3) holds if and only if $xu = yb$ for some $b \in \mathscr{A}$.

REMARK 2.2. Let a be an element in a unital C*-algebra $\mathscr{A}$. We have the polar decomposition $a = u|a|$, where $|a| \in \mathscr{A}$ and $u \in \mathscr{A}''$. In general, $u \notin \mathscr{A}$. We present an example of an infinite dimensional C*-algebra $\mathscr{A}$ such that $u \in \mathscr{A}$ (of course, it is known that if $\mathscr{A}$ is the full matrix algebra $\mathscr{M}_n(\mathbb{C})$, then $u \in \mathscr{A}$ is a unitary), see [28].

For $n = 1,2,\cdots$, let H_n be a finite dimensional Hilbert space. Put

$$\mathscr{A} = \{(t_n)_{n=1}^\infty \mid t_n \in \mathscr{B}(H_n),\ \sup \| t_n \| < +\infty\}.$$

Then $\mathscr{A}$ is a unital C*-algebra. For each $a = (t_n)_{n=1}^\infty \in \mathscr{A}$, since $\mathscr{B}(H_n)$ is finite dimensional, we have the polar decomposition $t_n = u_n|t_n|$ with a unitary $u_n \in \mathscr{B}(H_n)$. Put $u = (u_n)_{n=1}^\infty$. Then u is unitary in $\mathscr{A}$. Since $|a| = (|t_n|)_{n=1}^\infty$, it follows that $a = u|a|$ with a unitary $u \in \mathscr{A}$.

We now review the basic concepts of adjointable operators on a Hilbert C*-module. Let $\mathscr{X}$ be a Hilbert C*-module over a unital C*-algebra $\mathscr{A}$. We define $\mathscr{L}(\mathscr{X})$ to be the set of all maps $T : \mathscr{X} \mapsto \mathscr{X}$ for which there is a map $T^* : \mathscr{X} \mapsto \mathscr{X}$ such that $\langle Tx,y \rangle = \langle x,T^*y \rangle$ for $x,y \in \mathscr{X}$. For $T \in \mathscr{L}(\mathscr{X})$, using the closed graph theorem, it is easy to see that T is $\mathscr{A}$-linear and bounded. We call $\mathscr{L}(\mathscr{X})$ the set of adjointable operators on $\mathscr{X}$. Moreover, we define its norm by $\| T \| = \sup\{\| \langle Tx,Tx \rangle \|^{\frac{1}{2}} : \| x \| \leqslant 1\}$. Then $\mathscr{L}(\mathscr{X})$ is a C*-algebra. As usual, the symbol I stands for the identity operator in $\mathscr{L}(\mathscr{X})$. In addition, T is positive in $\mathscr{L}(\mathscr{X})$ if and only if $\langle x,Tx \rangle \geqslant 0$ for all $x \in \mathscr{X}$. For each element $a \in \mathscr{Z}(\mathscr{A})$, we define $R_a : \mathscr{X} \mapsto \mathscr{X}$ by $R_a x = xa$ for all $x \in \mathscr{X}$. Then it follows that $R_a \in \mathscr{L}(\mathscr{X})$ and $TR_a = R_a T$.

By virtue of Theorem 2.1, we obtain the following generalized Cauchy–Schwarz inequality on a Hilbert C*-module:

THEOREM 2.3. (generalized Cauchy-Schwarz inequality) *Let T be a positive operator in $\mathscr{L}(\mathscr{X})$. If $x,y \in \mathscr{X}$ such that $\langle x,Ty \rangle$ has a polar decomposition $\langle x,Ty \rangle = u|\langle x,Ty \rangle|$ with a partial isometry $u \in \mathscr{A}$, then*

$$|\langle x,Ty \rangle| \leqslant u^* \langle x,Tx \rangle u \sharp \langle y,Ty \rangle. \tag{2.4}$$

Under the assumption that $\langle y,Ty\rangle$ *is invertible, the equality in* (2.4) *holds if and only if* $T^{\frac{1}{2}}(xu)=T^{\frac{1}{2}}(yb)$ *for some* $b\in\mathscr{A}$.

Proof. We have this theorem by replacing x and y by $T^{\frac{1}{2}}x$ and $T^{\frac{1}{2}}y$, respectively in Theorem 2.1. □

Throughout the paper we follow the terminology and nonation of the book [24].

3. Kantorovich inequality

In this section, we discuss Kantorovich type inequalities on a Hilbert C^*-module $\mathscr{X}$ over a unital C^*-algebra $\mathscr{A}$ and present its applications. We refer to [22, 17, 16] on the Kantorovich inequality on a Hilbert space. Let T be a positive invertible operator in $\mathscr{L}(\mathscr{X})$. Then it follows from Theorem 2.1 that

$$\langle x,x\rangle \leqslant \langle x,Tx\rangle \,\sharp\, \langle x,T^{-1}x\rangle \tag{3.1}$$

for all $x\in\mathscr{X}$. As reverses of (3.1), we show the following Kantorovich type inequalities on a Hilbert C^*-module by means of the operator geometric mean, also see [8, Theorem 3.3]:

THEOREM 3.1. (Kantorovich inequality) *Let* T *be a positive invertible operator in* $\mathscr{L}(\mathscr{X})$ *such that* $R_a\leqslant T\leqslant R_b$ *for some positive invertible elements* $a,b\in\mathscr{Z}(\mathscr{A})$. *Then*

$$\langle x,Tx\rangle \,\sharp\, \langle x,T^{-1}x\rangle \leqslant \frac{1}{2}(a+b)(ab)^{-1/2}\langle x,x\rangle \tag{3.2}$$

and

$$\langle x,Tx\rangle \,\sharp\, \langle x,T^{-1}x\rangle - \langle x,x\rangle \leqslant \frac{1}{4}(a-b)^2(a(a+b))^{-1} \tag{3.3}$$

for all $x\in\mathscr{X}$.

Proof. Since $(R_b-T)(T-R_a)T^{-1}\geqslant 0$, we have $T+R_{ab}T^{-1}\leqslant R_{(a+b)}$ and this implies

$$\langle x,Tx\rangle + ab\,\langle x,T^{-1}x\rangle \leqslant (a+b)\,\langle x,x\rangle \tag{3.4}$$

for all $x\in\mathscr{X}$. For a given $\varepsilon>0$, it follows from the arithmetic-geometric mean inequality [16, Theorem 1.27] that

$$\begin{aligned}\sqrt{ab}\,(\langle x,Tx\rangle+\varepsilon e)\,\sharp\,\langle x,T^{-1}x\rangle &= (\langle x,Tx\rangle+\varepsilon e)\,\sharp\, ab\,\langle x,T^{-1}x\rangle\\ &\leqslant \frac{1}{2}\left(\langle x,Tx\rangle + ab\,\langle x,T^{-1}x\rangle+\varepsilon e\right) \leqslant \frac{1}{2}\left((a+b)\,\langle x,x\rangle+\varepsilon e\right).\end{aligned}$$

Since the operator geometric mean satisfies the upper semicontinuity, we have (3.2) as $\varepsilon\downarrow 0$.

To prove (3.3), we may assume that $\langle x, T^{-1}x\rangle$ is invertible by observation above, since $\langle x, T^{-1}x\rangle + \varepsilon e$ is invertible for all $\varepsilon > 0$.

If we put $X = \langle x, T^{-1}x\rangle^{-\frac{1}{2}} \langle x, Tx\rangle \langle x, T^{-1}x\rangle^{-\frac{1}{2}}$, then it follows from (3.4) and the symmetric property of the operator geometric mean that

$$\begin{aligned}
&\langle x, Tx\rangle \sharp \langle x, T^{-1}x\rangle - \langle x, x\rangle = \langle x, T^{-1}x\rangle \sharp \langle x, Tx\rangle - \langle x, x\rangle \\
&\leqslant \langle x, T^{-1}x\rangle^{\frac{1}{2}} X^{\frac{1}{2}} \langle x, T^{-1}x\rangle^{\frac{1}{2}} - (a+b)^{-1}\langle x, Tx\rangle - ab(a+b)^{-1}\langle x, T^{-1}x\rangle \\
&= \langle x, T^{-1}x\rangle^{\frac{1}{2}} \left(-(a+b)^{-1} \left(X^{\frac{1}{2}} - \frac{a+b}{2} \right)^2 + \frac{1}{4}(a-b)^2(a+b)^{-1} \right) \langle x, T^{-1}x\rangle^{\frac{1}{2}} \\
&\leqslant \frac{1}{4}(a-b)^2(a+b)^{-1}\langle x, T^{-1}x\rangle \\
&\leqslant \frac{1}{4}(a-b)^2(a(a+b))^{-1}
\end{aligned}$$

and whence we have (3.3). □

REMARK 3.2. We point out that there are some cases where the equality holds for (3.3) in Theorem 3.1. As a matter of fact, suppose that there exist $y, z \in \mathscr{X}$ such that $Ty = ya$ and $Tz = zb$, and $\langle y, y\rangle = \langle z, z\rangle = e$ and $\langle y, z\rangle = 0$. For example, let $\mathscr{A}_0$ be a unital C*-algebra of $B(H_0)$. Let $\mathscr{X} = \mathscr{A}_0 \oplus \mathscr{A}_0$ and $\mathscr{A} = \mathscr{A}_0 \oplus M_2$. For two positive invertible elements $a, b \in \mathscr{Z}(\mathscr{A}_0)$ such that $a < b$, put $T = R_a \oplus R_b$, $y = e \oplus 0$ and $z = 0 \oplus e$. Then T is positive invertible and $R_{a\oplus a} \leqslant T \leqslant R_{b\oplus b}$. In this case, $y, z \in \mathscr{X}$ satisfies the desired conditions. Now, if we put

$$x = \frac{\sqrt[4]{b}(\sqrt{a}+\sqrt{b})}{2\sqrt[4]{a}\sqrt{a+b}}(y+z),$$

then we have

$$\langle x, Tx\rangle = \frac{\sqrt{b}(\sqrt{a}+\sqrt{b})^2}{4\sqrt{a}}, \quad \langle x, T^{-1}x\rangle = \frac{(\sqrt{a}+\sqrt{b})^2}{4a\sqrt{ab}} \text{ and } \langle x, x\rangle = \frac{\sqrt{b}(\sqrt{a}+\sqrt{b})^2}{2\sqrt{a}(a+b)}.$$

Hence it follows that

$$\langle x, Tx\rangle \sharp \langle x, T^{-1}x\rangle - \langle x, x\rangle = \frac{1}{4}(a-b)^2(a(a+b))^{-1}$$

as desired. Similarly we have some cases where the equality holds for (3.2), also see [8, Remark 4.5].

As an application, we discuss a Cauchy type inequality and its reverses on a Hilbert C*-module by means of the operator geometric mean. Let A and B be positive in $\mathscr{L}(\mathscr{X})$. Then we have the following Cauchy type inequality on a Hilbert C*-module:

$$\langle x, A^2 \sharp B^2 x\rangle \leqslant \langle x, A^2 x\rangle \sharp \langle x, B^2 x\rangle \tag{3.5}$$

for all $x \in \mathscr{X}$. In fact, we have (3.5) by replacing x and T by $(A^{-1}B^2A^{-1})^{\frac{1}{2}}Ax$ and $(A^{-1}B^2A^{-1})^{\frac{1}{2}}$ in (3.1), respectively.

As applications of the Kantorovich inequality in Theorem 3.1, we show a Pólya-Szegö type inequality and an Ozeki-Izumino-Mori-Seo type inequality in the setting of Hilbert C^*-modules by means of the operator geometric mean. We refer to [31, 30, 21] on the Pólya-Szegö inequality and the Ozeki-Izumino-Mori-Seo inequality on a Hilbert space.

THEOREM 3.3. (Pólya-Szegö inequality) *Let A and B be positive invertible operators in $\mathscr{L}(\mathscr{X})$ such that $R_{a_1} \leqslant A \leqslant R_{b_1}$ and $R_{a_2} \leqslant B \leqslant R_{b_2}$ for some positive invertible elements $a_1, b_1, a_2, b_2 \in \mathscr{Z}(\mathscr{A})$. Then*

$$\langle x, A^2x \rangle \sharp \langle x, B^2x \rangle \leqslant \frac{1}{2}(a_1a_2 + b_1b_2)(a_1a_2b_1b_2)^{-\frac{1}{2}} \langle x, A^2 \sharp B^2x \rangle \tag{3.6}$$

for all $x \in \mathscr{X}$.

Proof. Let T be a positive invertible element in $\mathscr{L}(\mathscr{X})$ such that $R_a \leqslant T \leqslant R_b$ for some positive invertible elements $a, b \in \mathscr{Z}(\mathscr{A})$. If we replace x by $T^{\frac{1}{2}}x$ in (3.2) of Theorem 3.1, then we get

$$\langle x, T^2x \rangle \sharp \langle x, x \rangle \leqslant \frac{1}{2}(a+b)(ab)^{-\frac{1}{2}} \langle x, Tx \rangle \tag{3.7}$$

for all $x \in \mathscr{X}$. Moreover, replacing x and T by Ax and $\left(A^{-1}B^2A^{-1}\right)^{\frac{1}{2}}$ in (3.7) respectively, we get the desired inequality (3.6) since $R_{a_2b_1^{-1}} \leqslant \left(A^{-1}B^2A^{-1}\right)^{\frac{1}{2}} \leqslant R_{b_2a_1^{-1}}$. $\square$

THEOREM 3.4. (Ozeki-Izumino-Mori-Seo inequality) *Let A and B be positive invertible operators in $\mathscr{L}(\mathscr{X})$ such that $R_{a_1} \leqslant A \leqslant R_{b_1}$ and $R_{a_2} \leqslant B \leqslant R_{b_2}$ for some positive invertible elements $a_1, b_1, a_2, b_2 \in \mathscr{Z}(\mathscr{A})$. Then*

$$\langle x, A^2x \rangle \sharp \langle x, B^2x \rangle - \langle x, A^2 \sharp B^2x \rangle \leqslant \frac{1}{4}(b_1b_2 - a_1a_2)^2(a_1a_2(a_1a_2 + b_1b_2))^{-1} \tag{3.8}$$

for all $x \in \mathscr{X}$.

Proof. If we replace x by $T^{\frac{1}{2}}x$ in (3.3) of Theorem 3.1, then we get

$$\langle x, T^2x \rangle \sharp \langle x, x \rangle - \langle x, Tx \rangle \leqslant \frac{1}{4}(a-b)^2(a(a+b))^{-1} \tag{3.9}$$

for all $x \in \mathscr{X}$. Moreover, replacing x and T by Ax and $\left(A^{-1}B^2A^{-1}\right)^{\frac{1}{2}}$ in (3.9), respectively, we get the desired inequality (3.8) since $R_{a_2b_1^{-1}} \leqslant \left(A^{-1}B^2A^{-1}\right)^{\frac{1}{2}} \leqslant R_{b_2a_1^{-1}}$. $\square$

4. Covariance-variance inequality

Let S and T be two operators in $\mathscr{L}(\mathscr{X})$. The covariance of S and T at $x \in \mathscr{X}$ with $\langle x,x\rangle = e$ is introduced by

$$\mathrm{Cov}_x(S,T) = \langle x, S^*Tx\rangle - \langle x, Sx\rangle^* \langle x, Tx\rangle \tag{4.1}$$

and the variance of S at $x \in \mathscr{X}$ with $\langle x,x\rangle = e$ by

$$\mathrm{Var}_x(S) = \langle x, S^*Sx\rangle - |\langle x, Sx\rangle|^2; \tag{4.2}$$

also see [2, 10, 11, 32]. Notice that $\mathrm{Var}_x(S)$ is a positive element in $\mathscr{A}$. In fact, it follows from (2.2) that

$$|\langle x, Sx\rangle|^2 = \langle x, Sx\rangle^* \langle x, Sx\rangle \leqslant \| \langle x,x\rangle \| \langle Sx, Sx\rangle = \langle x, S^*Sx\rangle .$$

By the new Cauchy-Schwarz inequality (Theorem 2.1) on a Hilbert C*-module, we have the following covariance-variance inequality:

THEOREM 4.1. (Covariance-variance inequality) *Let S and T be two operators in $\mathscr{L}(\mathscr{X})$. Suppose that $x \in \mathscr{X}$ with $\langle x,x\rangle = e$ such that the covariance* $\mathrm{Cov}_x(S,T)$ *has a polar decomposition* $\mathrm{Cov}_x(S,T) = u|\mathrm{Cov}_x(S,T)|$ *with a partial isometry $u \in \mathscr{A}$. Then*

$$|\mathrm{Cov}_x(S,T)| \leqslant u^*\mathrm{Var}_x(S)u \sharp \mathrm{Var}_x(T). \tag{4.3}$$

Under the assumption that $\mathrm{Var}_x(T)$ *is invertible, the equality in* (4.3) *holds if and only if $(Sx - x\langle x, Sx\rangle)u = (Tx - x\langle x, Tx\rangle)b$ for some $b \in \mathscr{A}$.*

Proof. By the definition (4.2) and $\langle x,x\rangle = e$, we have

$$\mathrm{Var}_x(S) = \langle Sx - x\langle x, Sx\rangle, Sx - x\langle x, Sx\rangle\rangle$$

and hence it follows from (4.1) and Theorem 2.1 that

$$\begin{aligned}
|\mathrm{Cov}_x(S,T)| &= |\langle x, S^*Tx\rangle - \langle x, Sx\rangle^* \langle x, Tx\rangle| \\
&= |\langle Sx - x\langle x, Sx\rangle, Tx - x\langle x, Tx\rangle\rangle| \\
&\leqslant u^* \langle Sx - x\langle x, Sx\rangle, Sx - x\langle x, Sx\rangle\rangle u \sharp \langle Tx - x\langle x, Tx\rangle, Tx - x\langle x, Tx\rangle\rangle \\
&= u^*\mathrm{Var}_x(S)u \sharp \mathrm{Var}_x(T). \quad \square
\end{aligned}$$

REMARK 4.2. The variance of S at $x \in \mathscr{X}$ with $\langle x,x\rangle = e$ has the following characterization:

$$\mathrm{Var}_x(S) = \inf_{c \in \mathscr{A}} \langle Sx - xc, Sx - xc\rangle .$$

In fact, for each $c \in \mathscr{A}$

$$\langle Sx - xc, Sx - xc\rangle - \mathrm{Var}_x(S) = (\langle x, Sx\rangle - c)^* (\langle x, Sx\rangle - c) \geqslant 0$$

and hence

$$\mathrm{Var}_x(S) \leqslant \langle Sx - xc, Sx - xc\rangle \tag{4.4}$$

for all $x \in \mathscr{X}$ with $\langle x,x\rangle = e$. If we put $c = \langle x, Sx\rangle$, then the equality in (4.4) holds.

An operator $S \in \mathscr{L}(\mathscr{X})$ is said to be accretive if $\operatorname{Re}\langle x, Sx\rangle \geqslant 0$ for all $x \in \mathscr{X}$. The symbol $C_{a,b}(S)$ stands for $C_{a,b}(S) = (S - R_a)^*(R_b - S)$ for some $a, b \in \mathscr{Z}(\mathscr{A})$, also see [4, 5].

LEMMA 4.3. (Variance inequality) *Let S be an operator in $\mathscr{L}(\mathscr{X})$ and $a, b \in \mathscr{Z}(\mathscr{A})$. The operator $C_{a,b}(S)$ is accretive if and only if*

$$\operatorname{Var}_x(S) \leqslant \frac{1}{4}|a-b|^2 - \left|\langle x, Sx\rangle - \frac{a+b}{2}\right|^2 \quad \left(\leqslant \frac{1}{4}|a-b|^2\right)$$

for all $x \in \mathscr{X}$ with $\langle x, x\rangle = e$.

Proof. If $C_{a,b}(S)$ is accretive, then

$$\operatorname{Re}\left\langle x, C_{a,b}(S)x\right\rangle = \operatorname{Re}\left[(a+b)^*\langle x, Sx\rangle\right] - \langle Sx, Sx\rangle - \operatorname{Re}(a^*b) \geqslant 0. \tag{4.5}$$

We therefore have

$$\begin{aligned}\operatorname{Var}_x(S) &= \langle Sx, Sx\rangle - \langle x, Sx\rangle^*\langle x, Sx\rangle \\ &\leqslant \operatorname{Re}\left[(a+b)^*\langle x, Sx\rangle\right] - \operatorname{Re}(a^*b) - \langle x, Sx\rangle^*\langle x, Sx\rangle \\ &= -\left(\langle x, Sx\rangle - \frac{a+b}{2}\right)^*\left(\langle x, Sx\rangle - \frac{a+b}{2}\right) + \frac{1}{4}|a+b|^2 - \operatorname{Re}(a^*b) \\ &= \frac{1}{4}|a-b|^2 - \left|\langle x, Sx\rangle - \frac{a+b}{2}\right|^2.\end{aligned}$$

The converse implication can be easily proved. □

By the accretivity, we have the following corollary which is regarded as a ratio type reverse of (4.2)

COROLLARY 4.4. *Let S be an operator in $\mathscr{L}(\mathscr{X})$ and $a, b \in \mathscr{Z}(\mathscr{A})$ such that* $\operatorname{Re}(a^*b) > 0$. *If $C_{a,b}(S)$ is accretive, then*

$$\left\langle x, |S|^2 x\right\rangle \leqslant \frac{1}{4}|a+b|^2(\operatorname{Re} a^*b)^{-1}|\langle x, Sx\rangle|^2$$

for all $x \in \mathscr{X}$ with $\langle x, x\rangle = e$.

Proof. Since $C_{a,b}(S)$ is accretive and

$$\begin{aligned}&\frac{1}{4}|a+b|^2(\operatorname{Re} a^*b)^{-1}\langle x, Sx\rangle^*\langle x, Sx\rangle - \operatorname{Re}\left[(a+b)^*\langle x, Sx\rangle\right] + \operatorname{Re}(a^*b) \\ &= \left(\frac{1}{2}(a+b)^*(\sqrt{\operatorname{Re} a^*b})^{-1}\langle x, Sx\rangle - \sqrt{\operatorname{Re} a^*b}\right)^* \\ &\quad \times \left(\frac{1}{2}(a+b)^*(\sqrt{\operatorname{Re} a^*b})^{-1}\langle x, Sx\rangle - \sqrt{\operatorname{Re} a^*b}\right) \\ &\geqslant 0,\end{aligned}$$

it follows from (4.5) that

$$\langle x, S^*Sx\rangle \leqslant \mathrm{Re}\,[(a+b)^*\langle x,Sx\rangle] - \mathrm{Re}(a^*b) \leqslant \frac{1}{4}|a+b|^2(\mathrm{Re}\, a^*b)^{-1}\langle x,Sx\rangle^*\langle x,Sx\rangle$$

as desired. □

THEOREM 4.5. *Let S and T be two operators in $\mathscr{L}(\mathscr{X})$ and $a,b,c,d \in \mathscr{Z}(\mathscr{A})$ such that $C_{a,b}(S)$ and $C_{c,d}(T)$ are accretive. Suppose that $x \in \mathscr{X}$ with $\langle x,x\rangle = e$ such that the covariance $\mathrm{Cov}_x(S,T)$ has a polar decomposition $\mathrm{Cov}_x(S,T) = u|\mathrm{Cov}_x(S,T)|$ with a partial isometry $u \in \mathscr{A}$. Then*

$$\begin{aligned}
&|\mathrm{Cov}_x(S,T)| \\
&\leqslant \frac{1}{4}\left(u^*|a-b|^2u \,\sharp\, |c-d|^2\right) - \left(u^*|\left\langle x,(S-R_{\frac{a+b}{2}})x\right\rangle|^2u \,\sharp\, |\left\langle x,(T-R_{\frac{c+d}{2}})x\right\rangle|^2\right) \\
&\left(\leqslant \frac{1}{4}\left(u^*|a-b|^2u \,\sharp\, |c-d|^2\right)\right).
\end{aligned}$$

Proof. It follows from Theorem 4.1 and Lemma 4.3 that

$$\begin{aligned}
|\mathrm{Cov}_x(S,T)| &\leqslant u^*\mathrm{Var}_x(S)u \,\sharp\, \mathrm{Var}_x(T) \\
&\leqslant \left(\frac{1}{4}u^*|a-b|^2u - u^*\left|\langle x,Sx\rangle - \frac{a+b}{2}\right|^2 u\right) \sharp \left(\frac{1}{4}|c-d|^2 - \left|\langle x,Tx\rangle - \frac{c+d}{2}\right|^2\right).
\end{aligned}$$

Since the operator geometric mean is subadditive, we have

$$\begin{aligned}
&\frac{1}{4}\left(u^*|a-b|^2u \,\sharp\, |c-d|^2\right) \\
&= \left(\frac{1}{4}u^*|a-b|^2u - u^*\left|\langle x,Sx\rangle - \frac{a+b}{2}\right|^2 u + u^*\left|\langle x,Sx\rangle - \frac{a+b}{2}\right|^2 u\right) \\
&\qquad \sharp \left(\frac{1}{4}|c-d|^2 - \left|\langle x,Tx\rangle - \frac{c+d}{2}\right|^2 + \left|\langle x,Tx\rangle - \frac{c+d}{2}\right|^2\right) \\
&\geqslant \left(\frac{1}{4}u^*|a-b|^2u - u^*\left|\langle x,Sx\rangle - \frac{a+b}{2}\right|^2 u\right) \sharp \left(\frac{1}{4}|c-d|^2 - \left|\langle x,Tx\rangle - \frac{c+d}{2}\right|^2\right) \\
&\qquad + \left(u^*|\left\langle x,(S-R_{\frac{a+b}{2}})x\right\rangle|^2u \,\sharp\, |\left\langle x,(T-R_{\frac{c+d}{2}})x\right\rangle|^2\right).
\end{aligned}$$

Therefore, combining two results above, we have Theorem 4.5. □

As a corollary, we have the following Kantorovich type inequality on a Hilbert C^*-module which somewhat differs from (3.2) in Theorem 3.1:

COROLLARY 4.6. *Let A be a positive invertible operator in $\mathscr{L}(\mathscr{X})$ such that $R_a \leqslant A \leqslant R_b$ for some positive invertible elements a and b in $\mathscr{Z}(\mathscr{A})$. Suppose that $x \in$*

$\mathscr{X}$ with $\langle x,x\rangle = e$ such that the covariance $\mathrm{Cov}_x(A,A^{-1})$ has a polar decomposition $\mathrm{Cov}_x(A,A^{-1}) = u|\mathrm{Cov}_x(A,A^{-1})|$ with a partial isometry $u \in \mathscr{A}$. Then

$$|e - \langle x,Ax\rangle \langle x,A^{-1}x\rangle | \leqslant \frac{1}{4}(b-a)^2(ab)^{-1}.$$

In particular, if $\mathscr{A}$ is an abelian C^*-algebra, then

$$\langle x,Ax\rangle \langle x,A^{-1}x\rangle \leqslant \frac{1}{4}(a+b)^2(ab)^{-1}. \tag{4.6}$$

Proof. The assumption $R_a \leqslant A \leqslant R_b$ and $a,b \in \mathscr{Z}(\mathscr{A})$ implies $C_{a,b}(A)$ and $C_{b^{-1},a^{-1}}(A^{-1})$ are accretive. Hence it follows from Theorem 4.5 that

$$|e - \langle x,Ax\rangle \langle x,A^{-1}x\rangle | = |\mathrm{Cov}_x(A,A^{-1})| \leqslant \frac{1}{4}\left(u^*(b-a)^2 u \,\sharp\, (b^{-1}-a^{-1})^2\right)$$
$$= \frac{1}{4}(b-a)^2(ab)^{-1}.$$

If $\mathscr{A}$ is abelian, then the fact $e = \langle x,x\rangle \leqslant \langle x,Ax\rangle \langle x,A^{-1}x\rangle$ implies the desired inequality (4.6). □

By virtue of Lemma 4.3 (Variance inequality), we obtain the following Ozeki-Izumino-Mori-Seo type inequality which somewhat differs from Theorem 3.4 (see also [26]):

THEOREM 4.7. *Let A and B be positive invertible operators in $\mathscr{L}(\mathscr{X})$ such that $R_{a_1} \leqslant A \leqslant R_{b_1}$ and $R_{a_2} \leqslant B \leqslant R_{b_2}$ for some positive invertible $a_1,b_1,a_2,b_2 \in \mathscr{Z}(\mathscr{A})$. Then*

$$|\langle x,B^2x\rangle^{\frac{1}{2}} \langle x,A^2x\rangle^{\frac{1}{2}}|^2 - |\langle x,A^2x\rangle^{-\frac{1}{2}} \langle x,A^2\sharp B^2x\rangle \langle x,A^2x\rangle^{\frac{1}{2}}|^2 \leqslant \frac{1}{4}(a_1a_2-b_1b_2)^2a_1^{-2}b_1^2$$

for every $x \in \mathscr{X}$ with $\langle x,x\rangle = e$.

In particular, if $\mathscr{A}$ is an abelian C^*-algebra, then

$$\langle x,A^2x\rangle \langle x,B^2x\rangle - \langle x,A^2\sharp B^2x\rangle^2 \leqslant \frac{1}{4}(a_1a_2-b_1b_2)^2a_1^{-2}b_1^2.$$

Proof. If C is a positive operator in $\mathscr{L}(\mathscr{X})$ such that $R_a \leqslant C \leqslant R_b$ for some positive invertible $a,b \in \mathscr{Z}(\mathscr{A})$, then $C_{a,b}(C)$ is accretive and it follows from Lemma 4.3 (the variance inequality) that

$$\langle x,C^2x\rangle - \langle x,Cx\rangle^2 \leqslant \frac{1}{4}(a-b)^2 \tag{4.7}$$

for every $x \in \mathscr{X}$ with $\langle x,x\rangle = e$. If we replace x by $x\langle x,x\rangle^{-\frac{1}{2}}$ in (4.7) where $\langle x,x\rangle$ is invertible, then we have

$$\langle x,C^2x\rangle - \langle x,Cx\rangle \langle x,x\rangle^{-1} \langle x,Cx\rangle \leqslant \frac{1}{4}(a-b)^2 \langle x,x\rangle. \tag{4.8}$$

Since A is positive invertible in $\mathscr{L}(\mathscr{X})$, it follows that $A \geqslant \delta I > 0$ for some positive scalar $\delta > 0$ and this implies $\langle x, A^2x \rangle \geqslant \delta^2 \langle x,x \rangle = \delta^2 > 0$ for all $x \in \mathscr{X}$ with $\langle x,x \rangle = e$. Hence $\langle x, A^2x \rangle$ is invertible. If we replace C by $(A^{-1}B^2A^{-1})^{\frac{1}{2}}$ and x by Ax with $\langle x,x \rangle = e$ in (4.8), then $R_{a_2b_1^{-1}} \leqslant (A^{-1}B^2A^{-1})^{\frac{1}{2}} \leqslant R_{b_2a_1^{-1}}$ and we have

$$\langle x, B^2x \rangle - \langle x, A^2 \sharp B^2 x \rangle \langle x, A^2 x \rangle^{-1} \langle x, A^2 \sharp B^2 x \rangle \leqslant \frac{1}{4} \frac{(a_1a_2 - b_1b_2)^2}{a_1^2 b_1^2} \langle x, A^2x \rangle . \quad (4.9)$$

Multiplying $\langle x, A^2x \rangle^{\frac{1}{2}} \cdot \langle x, A^2x \rangle^{\frac{1}{2}}$ on both sides of (4.9), we have

$$\begin{aligned}
&|\langle x, B^2x \rangle^{\frac{1}{2}} \langle x, A^2x \rangle^{\frac{1}{2}}|^2 - |\langle x, A^2x \rangle^{-\frac{1}{2}} \langle x, A^2 \sharp B^2 x \rangle \langle x, A^2x \rangle^{\frac{1}{2}}|^2 \\
&= \langle x, A^2x \rangle^{\frac{1}{2}} \langle x, B^2x \rangle \langle x, A^2x \rangle^{\frac{1}{2}} - \langle x, A^2x \rangle^{\frac{1}{2}} \langle x, A^2 \sharp B^2 x \rangle \langle x, A^2x \rangle^{-1} \langle x, A^2 \sharp B^2 x \rangle \langle x, A^2x \rangle^{\frac{1}{2}} \\
&\leqslant \frac{1}{4} \frac{(a_1a_2 - b_1b_2)^2}{a_1^2 b_1^2} \langle x, A^2x \rangle^2 \leqslant \frac{1}{4}(a_1a_2 - b_1b_2)^2 \cdot a_1^{-2} b_1^2. \quad \square
\end{aligned}$$

5. Wielandt inequality

In this section, we consider a Wielandt type inequality on a Hilbert C*-module [19, 12], which is regarded as an improvement of the generalized Cauchy-Schwarz inequality (Theorem 2.3):

THEOREM 5.1. (Wielandt inequality) *Let T be a positive invertible operator in $\mathscr{L}(\mathscr{X})$ such that $R_a \leqslant T \leqslant R_b$ for positive invertible $a, b \in \mathscr{Z}(\mathscr{A})$. Suppose that $x, y \in \mathscr{X}$ such that $\langle x, Ty \rangle$ has a polar decomposition $\langle x, Ty \rangle = u|\langle x, Ty \rangle|$ with a partial isometry $u \in \mathscr{A}$. If x and y are an orthogonal pair, then*

$$|\langle x, Ty \rangle| \leqslant (b-a)(b+a)^{-1}(u^* \langle x, Tx \rangle u \,\sharp\, \langle y, Ty \rangle). \quad (5.1)$$

Proof. By the assumption $R_a \leqslant T \leqslant R_b$, we have

$$a \langle xc+yd, xc+yd \rangle \leqslant \langle xc+yd, T(xc+yd) \rangle \leqslant b \langle xc+yd, xc+yd \rangle \quad (5.2)$$

for all $c, d \in \mathscr{A}$. Then it follows from the orthogonality of x and y that

$$0 \leqslant c^*(\langle x, Tx \rangle - a\langle x,x \rangle)c + d^* \langle y, Tx \rangle c + c^* \langle x, Ty \rangle d + d^*(\langle y, Ty \rangle - a\langle y,y \rangle)d \quad (5.3)$$

and

$$0 \leqslant c^*(b\langle x,x \rangle - \langle x, Tx \rangle)c + d^* \langle y, Tx \rangle c + c^* \langle x, Ty \rangle d + d^*(b\langle y,y \rangle - \langle y, Ty \rangle)d \quad (5.4)$$

for all $c, d \in \mathscr{A}$. Calculating (5.3) $\times b$ + (5.4) $\times a$, we have

$$c^*(b-a)\langle x, Tx \rangle c + (b+a)(d^* \langle y, Ty \rangle c + c^* \langle x, Ty \rangle d) + (b-a)d^* \langle y, Ty \rangle d \geqslant 0$$

and hence

$$\begin{pmatrix} c^* & d^* \end{pmatrix} \begin{pmatrix} (b-a)\langle x, Tx\rangle & (b+a)\langle x, Ty\rangle \\ (b+a)\langle x, Ty\rangle^* & (b-a)\langle y, Ty\rangle \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \geqslant 0$$

for all $c, d \in \mathscr{A}$. Therefore, we have

$$\begin{pmatrix} (b-a)\langle x, Tx\rangle & (b+a)\langle x, Ty\rangle \\ (b+a)\langle x, Ty\rangle^* & (b-a)\langle y, Ty\rangle \end{pmatrix} \geqslant 0.$$

Since $\langle x, Ty\rangle = u|\langle x, Ty\rangle|$ and $\langle x, Ty\rangle^* = |\langle x, Ty\rangle| u^*$, we have

$$\begin{aligned} &\begin{pmatrix} (b-a)u^*\langle x, Tx\rangle u & (b+a)|\langle x, Ty\rangle| \\ (b+a)|\langle x, Ty\rangle| & (b-a)\langle y, Ty\rangle \end{pmatrix} \\ &= \begin{pmatrix} u^* & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} (b-a)\langle x, Tx\rangle & (b+a)\langle x, Ty\rangle \\ (b+a)\langle x, Ty\rangle^* & (b-a)\langle y, Ty\rangle \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & e \end{pmatrix} \geqslant 0. \end{aligned}$$

By the property (2.1) of the operator geometric mean, we have the desired inequality (5.1). □

The second proof of the Wielandt inequality is along with the one stated in [3].

Proof 2. If we put $c = 2ab(b+a)^{-1} \in \mathscr{Z}(\mathscr{A})$, then

$$\begin{aligned} |\langle x, Ty\rangle| &= |\langle x, Ty\rangle - \langle x, y\rangle c| = |\langle x, (T-R_c)y\rangle| \\ &= \left|\left\langle T^{\frac{1}{2}}x, T^{-\frac{1}{2}}(T-R_c)T^{-\frac{1}{2}}T^{\frac{1}{2}}y\right\rangle\right| = \left|\left\langle T^{\frac{1}{2}}x, (I-R_cT^{-1})T^{\frac{1}{2}}y\right\rangle\right| \\ &\leqslant u^*\left\langle T^{\frac{1}{2}}x, T^{\frac{1}{2}}x\right\rangle u \,\sharp\, \left\langle (I-R_cT^{-1})T^{\frac{1}{2}}y, (I-R_cT^{-1})T^{\frac{1}{2}}y\right\rangle. \end{aligned}$$

Since $0 < R_a \leqslant T \leqslant R_b$ implies $R_{b^{-1}} \leqslant T^{-1} \leqslant R_{a^{-1}}$, it follows that

$$R_{(b-a)(b+a)^{-1}} - (I - R_{2ab(b+a)^{-1}}T^{-1}) \geqslant R_{(b-a)(b+a)^{-1}-e+2ab(b+a)^{-1}b^{-1}} = 0.$$

By the commutativity, we have $(I-R_cT^{-1})^2 \leqslant R_{(b-a)^2(b+a)^{-2}}$ and this implies

$$\begin{aligned} \left\langle (I-R_cT^{-1})^2T^{\frac{1}{2}}y, T^{\frac{1}{2}}y\right\rangle &\leqslant \left\langle R_{(b-a)^2(b+a)^{-2}}T^{\frac{1}{2}}y, T^{\frac{1}{2}}y\right\rangle \\ &= (b-a)^2(b+a)^{-2}\langle y, Ty\rangle. \end{aligned}$$

Hence we have

$$\begin{aligned} |\langle x, Ty\rangle| &\leqslant u^*\left\langle T^{\frac{1}{2}}x, T^{\frac{1}{2}}x\right\rangle u \,\sharp\, \left\langle (I-R_cT^{-1})T^{\frac{1}{2}}y, (I-R_cT^{-1})T^{\frac{1}{2}}y\right\rangle \\ &\leqslant u^*\left\langle T^{\frac{1}{2}}x, T^{\frac{1}{2}}x\right\rangle u \,\sharp\, \left\langle R_{(b-a)^2(b+a)^{-2}}T^{\frac{1}{2}}y, T^{\frac{1}{2}}y\right\rangle \\ &= (b-a)(b+a)^{-1}(u^*\langle x, Tx\rangle u \,\sharp\, \langle y, Ty\rangle). \quad \square \end{aligned}$$

Based on a proof in Theorem 5.1, we can generalize the Wielandt inequality without assumption on the orthogonality of vectors.

THEOREM 5.2. (generalized Wielandt inequality) *Let T be a positive invertible operator in $\mathscr{L}(\mathscr{X})$ such that $0 < R_a \leqslant T \leqslant R_b$ for positive invertible $a,b \in \mathscr{Z}(\mathscr{A})$. Suppose that $x,y \in \mathscr{X}$ such that $(b+a)\langle x,Ty\rangle - 2ab\langle x,y\rangle$ has a polar decomposition $(b+a)\langle x,Ty\rangle - 2ab\langle x,y\rangle = u|(b+a)\langle x,Ty\rangle - 2ab\langle x,y\rangle|$ with a partial isometry $u \in \mathscr{A}$. Then*

$$|\langle x,Ty\rangle - 2ab(b+a)^{-1}\langle x,y\rangle| \leqslant (b-a)(b+a)^{-1}(u^*\langle x,Tx\rangle u \sharp \langle y,Ty\rangle). \tag{5.5}$$

Proof. Without assumption on the orthogonality of x and y, the inequality (5.2) implies the following two inequalities

$$\begin{aligned} 0 \leqslant\ & c^*(\langle x,Tx\rangle - a\langle x,x\rangle)c + d^*(\langle y,Tx\rangle - a\langle y,x\rangle)c + c^*(\langle x,Ty\rangle \\ & -a\langle x,y\rangle)d + d^*(\langle y,Ty\rangle - a\langle y,y\rangle)d \end{aligned}$$

and

$$\begin{aligned} 0 \leqslant\ & c^*(b\langle x,x\rangle - \langle x,Tx\rangle)c + d^*(\langle y,Tx\rangle - b\langle y,x\rangle)c + c^*(\langle x,Ty\rangle \\ & -b\langle x,y\rangle)d + d^*(b\langle y,y\rangle - \langle y,Ty\rangle)d \end{aligned}$$

for all $c,d \in \mathscr{A}$. Hence we have

$$\begin{pmatrix} c^* & d^* \end{pmatrix} \begin{pmatrix} (b-a)\langle x,Tx\rangle & (b+a)\langle x,Ty\rangle - 2ab\langle x,y\rangle \\ ((b+a)\langle x,Ty\rangle - 2ab\langle x,y\rangle)^* & (b-a)\langle y,Ty\rangle \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \geqslant 0$$

for all $c,d \in \mathscr{A}$. A similar argument as in the proof of Theorem 5.1 induce the desired inequality (5.5). □

REMARK 5.3. In the case that $\mathscr{A}$ is an abelian unital C^*-algebra, we consider relations among the Wielandt inequality, the Cauchy-Schwarz inequality and the Kantorovich inequality on a Hilbert C^*-module. Let T be a positive invertible operator in $\mathscr{L}(\mathscr{X})$ such that $R_a \leqslant T \leqslant R_b$ for positive invertible $a,b \in \mathscr{A}$. For given x and y in $\mathscr{X}$ with $\langle x,x\rangle = e$, we put $z = y - x\langle x,y\rangle$. Since $\langle x,z\rangle = 0$, it follows from Theorem 5.1 that

$$|\langle x,Tz\rangle|^2 \leqslant (b-a)^2(b+a)^{-2}\langle x,Tx\rangle\langle z,Tz\rangle$$

and hence

$$|\langle x,Ty\rangle|^2 \leqslant \langle x,Tx\rangle\langle y,Ty\rangle - 4ab(b-a)^{-2}|\langle x,Tx\rangle\langle x,y\rangle - \langle x,Ty\rangle|^2.$$

Moreover, if we replace y by $T^{-1}x$, then we have the Kantorovich inequality

$$\langle x,Tx\rangle\langle x,T^{-1}x\rangle \leqslant (b+a)^2(4ab)^{-1}$$

for all $x \in \mathscr{X}$ with $\langle x,x\rangle = e$.

6. Heinz-Kato-Furuta inequality

In this section, we discuss a Heinz-Kato-Furuta type inequality on a Hilbert C^*-module.

For $T \in \mathscr{L}(\mathscr{X})$, we denote the range of T and the kernel of T by $R(T)$ and $N(T)$, respectively. A closed submodule $\mathscr{M}$ of $\mathscr{X}$ is said to be complemented if $\mathscr{X} = \mathscr{M} \oplus \mathscr{M}^{\perp}$. Suppose that the closures of the ranges of T and T^* are both complemented. Then it follows from [24, Proposition 3.8] that T has a polar decomposition $T = U|T|$ with a partial isometry $U \in \mathscr{L}(\mathscr{X})$. We have the following lemma, also see [14, 15]:

LEMMA 6.1. *Let T be an operator in $\mathscr{L}(\mathscr{X})$ such that the closures of the ranges of T and T^* are both complemented. Then the following hold:*

(i) $N(|T|) = N(T)$.

(ii) $|T^*|^q = U|T|^q U^*$ *for any positive number q.*

Proof. For (i), since $\langle |T|x, |T|x\rangle = \left\langle x, |T|^2 x\right\rangle = \langle x, T^*Tx\rangle = \langle Tx, Tx\rangle$, we have $|T|x = 0 \Longleftrightarrow Tx = 0$ and hence $N(|T|) = N(T)$.

For (ii), since $N(|T|^q) = N(|T|)$ for all $q > 0$ and U^*U is the initial projection on $\overline{R(|T|)}$, it follows that $\overline{R(|T|^q)} = N(|T|^q)^{\perp} = N(|T|)^{\perp} = \overline{R(|T|)}$ and $U^*U|T|^q = |T|^q$. Since $|T^*|^2 = (U|T|U^*)^2$, it follows from $U^*U|T| = |T|$ that

$$p_n(|T^*|^2) = U p_n(|T|^2) U^*$$

for any polynomial $p_n(t)$. If $p_n(t) \mapsto \sqrt{t}$, then we have $|T^*| = U|T|U^*$. By induction, $|T^*|^{\frac{n}{m}} = U|T|^{\frac{n}{m}}$ holds for any positive integer m and n. Hence we have $|T^*|^q = U|T|^q U^*$ as $\frac{n}{m} \to q$. $\square$

THEOREM 6.2. (Heinz-Kato-Furuta inequality) *Let T be an operator in $\mathscr{L}(\mathscr{X})$ such that the closures of the ranges of T and T^* are both complemented. Let A and B be positive operators in $\mathscr{L}(\mathscr{X})$ such that $\langle T^*x, T^*x\rangle \leqslant \langle Ax, Ax\rangle$ and $\langle Ty, Ty\rangle \leqslant \langle By, By\rangle$ for all $x, y \in \mathscr{X}$. If $x, y \in \mathscr{X}$ such that $\left\langle x, T|T|^{\alpha+\beta-1}y\right\rangle$ has a polar decomposition $\left\langle x, T|T|^{\alpha+\beta-1}y\right\rangle = u\left|\left\langle x, T|T|^{\alpha+\beta-1}y\right\rangle\right|$ with a partial isometry $u \in \mathscr{A}$, then the following inequality holds*

$$\left|\left\langle x, T|T|^{\alpha+\beta-1}y\right\rangle\right| \leqslant u^*\left\langle x, A^{2\alpha}x\right\rangle u \,\sharp\, \left\langle y, B^{2\beta}y\right\rangle \tag{6.1}$$

for any $\alpha, \beta \in [0,1]$ and $\alpha + \beta \geqslant 1$.

Proof. By the assumption, we have $|T^*|^2 \leqslant A^2$ and $|T|^2 \leqslant B^2$. It follows from the Löwner-Heinz inequality that $\left\langle x, |T^*|^{2\alpha}x\right\rangle \leqslant \left\langle x, A^{2\alpha}x\right\rangle$ and $\left\langle y, |T|^{2\beta}y\right\rangle \leqslant \left\langle y, B^{2\beta}y\right\rangle$

for each $\alpha,\beta\in[0,1]$. In the case $\alpha,\beta\in[0,1]$ such that $\beta>0$ and $\alpha+\beta\geqslant 1$, by Lemma 6.1 we have $|T^*|^{2\beta}=U|T|^{2\beta}U^*$ for any $\beta>0$. Then for all $x,y\in\mathscr{X}$

$$\begin{aligned}\left|\left\langle x,T|T|^{\alpha+\beta-1}y\right\rangle\right| &= \left|\left\langle x,U|T|^{\alpha+\beta}y\right\rangle\right| = \left|\left\langle |T|^{\alpha}U^*x,|T|^{\beta}y\right\rangle\right| \\ &\leqslant u^*\left\langle |T|^{\alpha}U^*x,|T|^{\alpha}U^*x\right\rangle u \,\sharp\, \left\langle |T|^{\beta}y,|T|^{\beta}y\right\rangle \\ &= u^*\left\langle U|T|^{2\alpha}U^*x,x\right\rangle u \,\sharp\, \left\langle y,|T|^{2\beta}y\right\rangle \\ &= u^*\left\langle |T^*|^{2\alpha}x,x\right\rangle u \,\sharp\, \left\langle y,|T|^{2\beta}y\right\rangle \leqslant u^*\left\langle x,A^{2\alpha}x\right\rangle u \,\sharp\, \left\langle y,B^{2\beta}y\right\rangle\end{aligned}$$

and the result is trivial in the case $\beta=0$. $\square$

Next we discuss Wielandt type variations of the Heinz-Kato-Furuta inequality on a Hilbert C^*-module, also see [13].

THEOREM 6.3. *Let T be an operator in $\mathscr{L}(\mathscr{X})$ such that the closures of the ranges of T and T^* are both complemented, and satisfying $R_a\leqslant|T|\leqslant R_b$ for some positive invertible $a,b\in\mathscr{Z}(\mathscr{A})$. Suppose that $x,y\in\mathscr{X}$ and $\gamma>0$ such that*

$$\begin{aligned}&\left\langle x,T|T|^{\alpha+\beta-1}y\right\rangle - 2b^{\gamma}a^{\gamma}(b^{\gamma}+a^{\gamma})^{-1}\left\langle x,T|T|^{\alpha+\beta-\gamma-1}y\right\rangle \\ &\qquad = u\left|\left\langle x,T|T|^{\alpha+\beta-1}y\right\rangle - 2b^{\gamma}a^{\gamma}(b^{\gamma}+a^{\gamma})^{-1}\left\langle x,T|T|^{\alpha+\beta-\gamma-1}y\right\rangle\right|\end{aligned}$$

has a polar decomposition with a partial isometry $u\in\mathscr{A}$. Then

$$\begin{aligned}&\left|\left\langle x,T|T|^{\alpha+\beta-1}y\right\rangle - 2b^{\gamma}a^{\gamma}(b^{\gamma}+a^{\gamma})^{-1}\left\langle x,T|T|^{\alpha+\beta-\gamma-1}y\right\rangle\right| \\ &\qquad \leqslant (b^{\gamma}-a^{\gamma})(b^{\gamma}+a^{\gamma})^{-1}\left(u^*\left\langle x,|T^*|^{2\alpha}x\right\rangle u \,\sharp\, \left\langle y,|T|^{2\beta}y\right\rangle\right)\end{aligned}$$

for all $\alpha,\beta\in\mathbb{R}$.

Proof. Let $T=U|T|$ be the polar decomposition of T with a partial isometry $U\in\mathscr{L}(\mathscr{X})$. For given $x,y\in\mathscr{X}$, we put $x_1=|T|^{\alpha-\gamma/2}U^*x$ and $y_1=|T|^{\beta-\gamma/2}y$. Thus we use Theorem 5.2 for x_1,y_1 and $T=|T|^{\gamma}$. Since $0<R_{a^{\gamma}}\leqslant|T|^{\gamma}\leqslant R_{b^{\gamma}}$ for $\gamma>0$ and $U|T|^{\beta}U^*=|T^*|^{\beta}$ by Lemma 6.1, we have the desired inequality. $\square$

In particular, we take $\gamma=\alpha+\beta$ in above.

COROLLARY 6.4. *Let T, a,b, x,y be as in Theorem 6.3. Then*

$$\begin{aligned}&\left|\left\langle x,T|T|^{\alpha+\beta-1}y\right\rangle - 2b^{\alpha+\beta}a^{\alpha+\beta}(b^{\alpha+\beta}+a^{\alpha+\beta})^{-1}\left\langle x,T|T|^{-1}y\right\rangle\right| \\ &\qquad \leqslant (b^{\alpha+\beta}-a^{\alpha+\beta})(b^{\alpha+\beta}+a^{\alpha+\beta})^{-1}\left(u^*\left\langle x,|T^*|^{2\alpha}x\right\rangle u \,\sharp\, \left\langle y,|T|^{2\beta}y\right\rangle\right)\end{aligned}$$

for all $\alpha,\beta\in\mathbb{R}$ with $\alpha+\beta>0$.

Next we obtain the following Wielandt type Heinz-Kato-Furuta inequalities by the Löwner-Heinz inequality.

THEOREM 6.5. *Let T, a,b, x,y be as in Theorem 6.3. If A and B are positive invertible operators in $\mathscr{L}(\mathscr{X})$ such that $\langle T^*x,T^*x\rangle \leqslant \langle Ax,Ax\rangle$ and $\langle Ty,Ty\rangle \leqslant \langle By,By\rangle$ for all $x,y\in\mathscr{X}$, then each $\gamma>0$*

$$\begin{aligned}&\left|\left\langle x,T|T|^{\alpha+\beta-1}y\right\rangle - 2b^\gamma a^\gamma (b^\gamma+a^\gamma)^{-1}\left\langle x,T|T|^{\alpha+\beta-\gamma-1}y\right\rangle\right|\\&\qquad\leqslant (b^\gamma-a^\gamma)(b^\gamma+a^\gamma)^{-1}\left(u^*\left\langle x,A^{2\alpha}x\right\rangle u \,\sharp\, \left\langle y,B^{2\beta}y\right\rangle\right)\end{aligned}$$

for all $\alpha,\beta\in[0,1]$.

In particular,

$$\begin{aligned}&\left|\left\langle x,T|T|^{\alpha+\beta-1}y\right\rangle - 2b^{\alpha+\beta}a^{\alpha+\beta}(b^{\alpha+\beta}+a^{\alpha+\beta})^{-1}\left\langle x,T|T|^{-1}y\right\rangle\right|\\&\qquad\leqslant (b^{\alpha+\beta}-a^{\alpha+\beta})(b^{\alpha+\beta}+a^{\alpha+\beta})^{-1}\left(u^*\left\langle x,A^{2\alpha}x\right\rangle u \,\sharp\, \left\langle y,B^{2\beta}y\right\rangle\right)\end{aligned}$$

for all $\alpha,\beta\in[0,1]$.

COROLLARY 6.6. *Let T,A,B, a,b, x,y be as in Theorem 6.5, and $\alpha,\beta\in[0,1]$ and $\gamma>0$ be given. If $x,y\in\mathscr{X}$ moreover satisfy*

$$T|T|^{\alpha+\beta-\gamma-1}x\neq 0 \quad \textit{and} \quad \left\langle x,T|T|^{\alpha+\beta-\gamma-1}y\right\rangle = 0,$$

then

$$\left|\left\langle x,T|T|^{\alpha+\beta-1}y\right\rangle\right| \leqslant (b^{\alpha+\beta}-a^{\alpha+\beta})(b^{\alpha+\beta}+a^{\alpha+\beta})^{-1}\left(u^*\left\langle x,A^{2\alpha}x\right\rangle u \,\sharp\, \left\langle y,B^{2\beta}y\right\rangle\right).$$

Finally we mention some corollaries of Theorem 6.2 which are generalizations of the Wielandt inequality.

COROLLARY 6.7. *Let $T=U|T|$ be an operator in $\mathscr{L}(\mathscr{X})$ with a partial isometry $U\in\mathscr{L}(\mathscr{X})$ such that $R_a\leqslant|T|\leqslant R_b$ for some positive invertible $a,b\in\mathscr{Z}(\mathscr{A})$. If $x,y\in\mathscr{X}$ such that $\langle x,Ty\rangle - 2ab(a+b)^{-1}\langle x,Uy\rangle$ has a polar decomposition $\langle x,Ty\rangle - 2ab(a+b)^{-1}\langle x,Uy\rangle = u|\langle x,Ty\rangle - 2ab(a+b)^{-1}\langle x,Uy\rangle|$ with a partial isometry $u\in\mathscr{A}$, then*

$$|\langle x,Ty\rangle - 2ab(a+b)^{-1}\langle x,Uy\rangle| \leqslant (b-a)(a+b)^{-1}(u^*\langle x,|T^*|x\rangle u \,\sharp\, \langle y,|T|y\rangle).$$

COROLLARY 6.8. *Let $T=U|T|$ be an operator in $\mathscr{L}(\mathscr{X})$ with a partial isometry $U\in\mathscr{L}(\mathscr{X})$ such that $R_a\leqslant|T|\leqslant R_b$ for some positive invertible $a,b\in\mathscr{Z}(\mathscr{A})$. If $x,y\in\mathscr{X}$ such that $\langle x,Ty\rangle$ has a polar decomposition $\langle x,Ty\rangle = u|\langle x,Ty\rangle|$ and $\langle x,Uy\rangle=0$, then*

$$|\langle x,Ty\rangle| \leqslant (b-a)(a+b)^{-1}(u^*\langle x,|T^*|x\rangle u \,\sharp\, \langle y,|T|y\rangle).$$

7. Malamud inequality

Let A be a positive invertible operator in $\mathscr{L}(\mathscr{X})$ such that $R_a \leqslant A \leqslant R_b$ for some positive invertible $a, b \in \mathscr{Z}(\mathscr{A})$. Here we call the operator constant $(b-a)^2(b+a)^{-2}$ Wielandt's operator constant for A which is denoted by $W_m(A)$, that is,

$$W_m(A) = (b-a)^2(b+a)^{-2} \in \mathscr{Z}(\mathscr{A}),$$

see also [25]. As in the proof of Theorem 5.1, the Wielandt inequality is represented as follows:

Wielandt Theorem. If A is a positive invertible operator in $\mathscr{L}(\mathscr{X})$ such that $R_a \leqslant A \leqslant R_b$ for some positive invertible $a, b \in \mathscr{Z}(\mathscr{A})$, then

$$\begin{pmatrix} W_m(A)\langle x, Ax\rangle & \langle x, Ay\rangle \\ \langle x, Ay\rangle^* & \langle y, Ay\rangle \end{pmatrix} \geqslant 0 \tag{7.1}$$

for every orthogonal pair x and y.

In this section, we show the equivalence theorem including the Malamud and the Wielandt ones on a Hilbert C^*-module, also see [25, 9]. Let P be a projection in $\mathscr{L}(\mathscr{X})$. It follows that the range of P is orthogonally complemented, that is, $\mathrm{R}(P) \oplus \mathrm{R}(I-P) = \mathscr{X}$. For $x, y \in \mathscr{X}$, define $\theta_{x,y} : \mathscr{X} \mapsto \mathscr{X}$ by $\theta_{x,y}(z) = x\langle y, z\rangle$ $(z \in \mathscr{X})$ and $\theta_{x,y} \in \mathscr{L}(\mathscr{X})$. If $x \in \mathscr{X}$ with $\langle x, x\rangle = e$, then $\theta_{x,x}$ is a projection in $\mathscr{L}(\mathscr{X})$. Firstly, we have the following lemma:

LEMMA 7.1. *Let A be a positive invertible operator in $\mathscr{L}(\mathscr{X})$ and $c \in \mathscr{Z}(\mathscr{A})$ a positive element. Then the following conditions are mutually equivalent:*

(i) $A + (c-e)PAP \geqslant 0$ *for all projections P;*

(ii) $\begin{pmatrix} cPAP & PAP^{\perp} \\ P^{\perp}AP & P^{\perp}AP^{\perp} \end{pmatrix} \geqslant 0$ *for all projections P;*

(iii) $\begin{pmatrix} cPAP & PAQ \\ QAP & QAQ \end{pmatrix} \geqslant 0$ *for all mutually orthogonal projections P and Q;*

(iv) $\begin{pmatrix} c\langle x, Ax\rangle & \langle x, Ay\rangle \\ \langle y, Ax\rangle & \langle y, Ay\rangle \end{pmatrix} \geqslant 0$ *for all orthogonal pairs x and y in $\mathscr{X}$,*

where $P^{\perp} = I - P$.

Proof. Identifying an operator A on $\mathscr{X}$ as

$$\begin{pmatrix} PAP & PAP^{\perp} \\ P^{\perp}AP & P^{\perp}AP^{\perp} \end{pmatrix}$$

on the orthogonal direct sum $\mathrm{R}(P)\oplus\mathrm{R}(P^{\perp})$, we have $A+(c-e)PAP\geqslant 0$ corresponding to the operator matrix in (ii). Thus (i) and (ii) are equivalent. If P and Q are mutually orthogonal, then $QP^{\perp}=Q$ and hence the equation

$$\begin{pmatrix} I & 0\\ 0 & Q\end{pmatrix}\begin{pmatrix} cPAP & PAP^{\perp}\\ P^{\perp}AP & P^{\perp}AP^{\perp}\end{pmatrix}\begin{pmatrix} I & 0\\ 0 & Q\end{pmatrix}=\begin{pmatrix} cPAP & PAQ\\ QAP & QAQ\end{pmatrix}$$

assures that (ii) implies (iii), which shows the equivalence.

Suppose that (iii) holds. For all orthogonal pair x and y, we may assume that $\langle x,x\rangle=e$ and $\langle y,y\rangle=e$. We put two orthogonal projections P and Q in $\mathscr{L}(\mathscr{X})$ such that $P=\theta_{x,x}$ and $Q=\theta_{y,y}$. Then for all $v,w\in\mathscr{A}$

$$\begin{aligned}&\left\langle\begin{pmatrix} xv\\ yw\end{pmatrix},\begin{pmatrix} cPAP & PAQ\\ QAP & QAQ\end{pmatrix}\begin{pmatrix} xv\\ yw\end{pmatrix}\right\rangle\\ &=v^{*}c\langle x,Ax\rangle v+v^{*}\langle x,Ay\rangle w+w^{*}\langle y,Ax\rangle v+w^{*}\langle y,Ay\rangle w\\ &=\left\langle\begin{pmatrix} v\\ w\end{pmatrix},\begin{pmatrix} c\langle x,Ax\rangle & \langle x,Ay\rangle\\ \langle y,Ax\rangle & \langle y,Ay\rangle\end{pmatrix}\begin{pmatrix} v\\ w\end{pmatrix}\right\rangle\geqslant 0\end{aligned}$$

and hence we have (iv). Conversely, suppose that (iv) holds. For all orthogonal projections P and Q, put $x_1=Px$ and $y_1=Qy$ for each $x,y\in\mathscr{X}$ and we have $x_1\perp y_1$. Then

$$\begin{aligned}&\left\langle\begin{pmatrix} x\\ y\end{pmatrix},\begin{pmatrix} cPAP & PAP^{\perp}\\ P^{\perp}AP & P^{\perp}AP^{\perp}\end{pmatrix}\begin{pmatrix} x\\ y\end{pmatrix}\right\rangle=c\langle x_1,Ax_1\rangle+\langle x_1,Ay_1\rangle+\langle y_1,Ax_1\rangle+\langle y_1,Ay_1\rangle\\ &=\left\langle\begin{pmatrix} e\\ e\end{pmatrix},\begin{pmatrix} c\langle x_1,Ax_1\rangle & \langle x_1,Ay_1\rangle\\ \langle y_1,Ax_1\rangle & \langle y_1,Ay_1\rangle\end{pmatrix}\begin{pmatrix} e\\ e\end{pmatrix}\right\rangle\geqslant 0\end{aligned}$$

and hence we have (iii). Therefore we have the equivalence (iii) and (iv). □

Let A be a positive invertible operator in $\mathscr{L}(\mathscr{X})$ such that $R_a\leqslant A\leqslant R_b$ for some positive invertible $a,b\in\mathscr{Z}(\mathscr{A})$. The set $\{\langle x,Ax\rangle:\langle x,x\rangle=e\}$ is called the operator range of A. Suppose that a and b are *the least bounds* for the operator range of A in the sense that there exist two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathscr{X}$ such that $\langle x_n,x_n\rangle=\langle y_n,y_n\rangle=e$ and $\langle x_n,y_n\rangle=0$ for all n, and $\langle x_n,Ax_n\rangle\to a$ and $\langle y_n,Ay_n\rangle\to b$ as $n\to\infty$. For example, let $\mathscr{A}$ be a unital C*-algebra. Put $\mathscr{X}_i=\mathscr{A}$ for all $i\in\mathbb{Z}$. We define $\mathscr{X}=\oplus_{i\in\mathbb{Z}}\mathscr{X}_i$ to be the set of all sequences $x=(x_i)$, with x_i in $\mathscr{X}_i$, such that $\sum_{i\in\mathbb{Z}}\langle x_i,x_i\rangle$ converges in $\mathscr{A}$. It follows that $\mathscr{X}$ is a Hilbert C*-module. Take $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that $a_n\downarrow a$ and $b_n\downarrow b$, and $a_n,b_n\in\mathscr{Z}(\mathscr{A})$. Put $c_i=a_{i+1}$ for $i=0,1,2,\cdots$ and $c_{-i}=b_i$ for $i=1,2,\cdots$. Put $A=R_{\tilde{c}}$ for $\tilde{c}=(c_i)$. Since $c_i\in\mathscr{Z}(\mathscr{A})$, it follows that A is a positive invertible operator in $\mathscr{L}(\mathscr{X})$ such that $R_{\tilde{a}}\leqslant A\leqslant R_{\tilde{b}}$, where $\tilde{a}=(a_i)$ and $\tilde{b}=(b_i)$ for $a_i=a$ and $b_i=b$ for all $i\in\mathbb{Z}$. For $n\in\mathbb{N}$, $x_n=(x_n^i)_{i\in\mathbb{Z}}$ and $y_n=(y_n^i)_{i\in\mathbb{Z}}$ are defined by

$$x_n^i=\begin{cases} e & \text{if}\quad i=n\\ 0 & \text{otherwise}\end{cases}\qquad\text{and}\qquad y_n^i=\begin{cases} e & \text{if}\quad i=-n\\ 0 & \text{otherwise.}\end{cases}$$

In this case, two sequences $\{x_n\}$ and $\{y_n\}$ in $\mathscr{X}$ satisfies the desired conditions.

Now, we have the following Malamud type theorem:

THEOREM 7.2. *Let A be a positive invertible operator in $\mathscr{L}(\mathscr{X})$ such that $R_a \leqslant A \leqslant R_b$ for some positive invertible $a,b \in \mathscr{Z}(\mathscr{A})$. If a and b are the least bounds for the operator range of A, then for a positive element $c \in \mathscr{Z}(\mathscr{A})$, the following assertions are equivalent:*

(i) $A+(c-e)PAP \geqslant 0$ *for all projections* P*;*

(ii) $c \geqslant W_m(A)$.

Proof. Suppose that (ii) holds. By (7.1), we have

$$\begin{pmatrix} c\langle x,Ax\rangle & \langle x,Ay\rangle \\ \langle x,Ay\rangle^* & \langle y,Ay\rangle \end{pmatrix} \geqslant 0$$

for every orthogonal pair x and y. By Lemma 7.1, we have (i).

Conversely, suppose that (i) holds. Since $0 \leqslant I-R_{b^{-1}}A \leqslant I$, we have $(I-R_{b^{-1}}A)^2 \leqslant I-R_{b^{-1}}A$ and it follows that

$$\begin{aligned} 0 &\leqslant \langle (R_b-A)y_n,(R_b-A)y_n\rangle = b^2\langle (I-R_{b^{-1}}A)y_n,(I-R_{b^{-1}}A)y_n\rangle \\ &\leqslant b^2\langle y_n,(I-R_{b^{-1}}A)y_n\rangle = b\langle y_n,(R_b-A)y_n\rangle \end{aligned}$$

and hence we have

$$\begin{aligned} &\| \langle x_n,Ay_n\rangle \| = \| \langle x_n,(R_b-A)y_n\rangle \| \leqslant \| \langle x_n,x_n\rangle \|^{\frac{1}{2}} \| \langle (R_b-A)y_n,(R_b-A)y_n\rangle \|^{\frac{1}{2}} \\ &\leqslant \| b \|^{\frac{1}{2}} \| \langle y_n,(R_b-A)y_n\rangle \|^{\frac{1}{2}} \to 0. \end{aligned}$$

If we put $u_n = x_n+y_n$ and $v_n = x_n-y_n$, then we have $\| u_n \| = \| v_n \| = \sqrt{2}$ and $\langle u_n,v_n\rangle = 0$. Now, it follows that

$$\begin{aligned} &\| \langle u_n,Av_n\rangle - (a+b) \| \\ &\leqslant \| \langle x_n,Ax_n\rangle - a \| + \| \langle y_n,Ay_n\rangle - b \| + \| \langle x_n,Ay_n\rangle \| + \| \langle y_n,Ax_n\rangle \| \to 0 \end{aligned}$$

as $n\to\infty$. Similarly, we have $\langle v_n,Av_n\rangle \to a+b$ and $\langle u_n,Av_n\rangle \to a-b$ as $n\to\infty$. Since x_n and y_n are orthogonal for all n, it follows from Lemma 7.1 that

$$\begin{pmatrix} c\langle x_n,Ax_n\rangle & \langle x_n,Ay_n\rangle \\ \langle y_n,Ax_n\rangle & \langle y_n,Ay_n\rangle \end{pmatrix} \geqslant 0$$

and hence this implies

$$\begin{pmatrix} c(a+b) & a-b \\ a-b & a+b \end{pmatrix} \geqslant 0.$$

Therefore, we have the desired inequality (ii). □

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