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WEIGHTED NORM INEQUALITIES FOR THE g–LITTLEWOOD–PALEY OPERATORS ASSOCIATED WITH LAPLACE–BESSEL DIFFERENTIAL OPERATORS

A. AKBULUT, V. S. GULIYEV AND M. DZIRI

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Abstract. In this work we define and study Poisson integral associated with Laplace-Bessel differential operators. We establish weighted inequalities with a general weight for the g-Littlewood-Paley functions and the commutator $g_{b,k}$ defined by (1.2) associated with Laplace Bessel differential operator.

1. Introduction

The study of the g-Littlewood-Paley theory enjoys a natural motivation and arises a great interest. Many works and topic have been studied. To our knowledge, for Euclidean analysis it is investigated, at first, by Stein in [8]. In his study of these operators, Stein uses two approaches. The first is the theory of singular integrals in the context of Hilbert space-valued functions, and the second in the theory of harmonics functions. Later, these operators play an important role in questions related to multipliers, Sobelev spaces and Hardy spaces.

Over the past 20 years considerable effort has been made to extend the classical Littlwood-Paley theory on the Bessel-Kingman hypergroups [12] and the Chebli-Trimeche hypergroups [1].

In this paper we consider harmonic analysis associated with the following system of partial differential operators

$$\begin{cases} D_j &= \frac{\partial}{\partial x_j}, \ 1 \leqslant j \leqslant n \\ \Delta_{n,\alpha} &= \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r} + \sum_{j=1}^n \frac{\partial^2}{\partial x_j{}^2}, \ (r,x) \in]0,\infty[\times\mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\alpha > -\frac{1}{2}$.

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Some problems of harmonic analysis that are associated with Laplace-Bessel operator $\Delta_{n,\alpha}$ are investigated, for example, [3, 5, 6, 7]. We point out that in [7] the author proved the $L_{p,w}$-boundedness of B-maximal functions.

In this paper, we are interested in problems related to weighted inequalities for the g-Littlewood-Paley functions associated to the Laplace-Bessel differential operators. More precisely, building on the results of harmonic analysis associated with $\Delta_{n,\alpha}$ we establish the $L_{p,w}$ inequalities with a general weight for the g-Littlewood-Paley functions in connection with the Laplace-Bessel differential operator $\Delta_{n,\alpha}$. Also, we give an application of great importance which deals with the $L_{p,w}$ boundedness of the commutator

$$g_{b,k}(f)(x) = g((b(x)-b(\cdot))^k f)(x), \tag{1.2}$$

where k is a positive integer and $b \in BMO(\mathbb{R}^{n+1}_+)$.

The article is organized as follows: In section 2 we include definitions and auxiliary results of harmonic analysis associated with the Laplace-Bessel differential operator. In section 3 we define and establish some estimates and properties of the Poisson integral related with the operator $\Delta_{n,\alpha}$. Section 4 deals with maximal operator associated with $\Delta_{n,\alpha}$. Also we establish some results for this operators that are essential to investigating the g-Littlewood-Paley functions . The subject of section 5 is to establish weighted inequalities with a general weight for the g-Littlewood-Paley functions and the commutator $g_{b,k}$ defined by (1.2) associated with Laplace Bessel differential operator.

Throughout the paper C denotes a positive constant whose value may vary from line to line.

2. Harmonic analysis related with $D_j; 1 \leqslant j \leqslant n$ and $\Delta_{n,\alpha}$

In this section we recall basic definitions and some facts. We consider the system of partial differential operators

$$\begin{cases} D_j &= \frac{\partial}{\partial x_j}, \ 1 \leqslant j \leqslant n \\ \Delta_{n,\alpha} &= l_\alpha + \Delta, \end{cases} \tag{2.1}$$

where l_α is the Bessel operator with respect to the first variable r given by

$$l_\alpha = \frac{\partial^2}{\partial r^2} + \frac{2\alpha+1}{r}\frac{\partial}{\partial r}$$

and $\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ is the Laplacian operator on $\mathbb{R}^n$. On the other hand, if $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{C}^n$ and $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, we put $\langle \lambda, x\rangle = \sum_{i=1}^n \lambda_i x_i$, $\|\lambda\| = \sqrt{\langle \lambda, \lambda\rangle}$.

By [13, 14] we have

PROPOSITION 2.1. *For $(\mu, \lambda) \in \mathbb{R}^{n+1}$, the following system of equations*

$$D_j v(r,x) = -i\lambda_j v(r,x),$$

$$\Delta_{n,\alpha}v(r,x) = -(\mu^2+\lambda^2)v(r,x),$$

$$v(0,0)=1; \qquad \frac{\partial v}{\partial r}(0,x)=0 \tag{2.2}$$

has a unique infinitely differentiable solution on $\mathbb{R}^{n+1}$ *even with respect to the first variable given by*

$$\varphi_{\mu,\lambda}(r,x) = j_\alpha(r\mu)e^{-i\langle\lambda,x\rangle}, \tag{2.3}$$

where

$$j_\alpha(s) = \frac{2^\alpha\Gamma(\alpha+1)J_\alpha(s)}{s^\alpha}, \ \text{if } s\neq 0, \text{ and } j_\alpha(s)=1, \ \text{if } s=0,$$

the functions J_α *are the Bessel functions of the first kind and order* α *and* Γ *is the Euler function.*

We have for all $(\mu,\lambda)\in\mathbb{R}^{n+1}$,

$$\sup_{(r,x)\in\mathbb{R}^{n+1}} |\varphi_{\mu,\lambda}(r,x)| = 1.$$

The shift operator $\mathscr{T}_{(r,x)}$ associated with Laplace Bessel operator $\Delta_{n,\alpha}$ is defined on the space of continuous functions even with respect to the first variable by

$$\mathscr{T}_{(r,x)}f(s,y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi f(\sqrt{r^2+s^2+2rs\cos\theta}, x+y)\sin^{2\alpha}\theta d\theta. \tag{2.4}$$

Denote by

- $d\nu_\alpha(r,x)$ the measure defined on $\mathbb{R}^{n+1}_+$ by

$$d\nu_\alpha(r,x) = r^{2\alpha+1}drdx. \tag{2.5}$$

- $L_{p,\alpha}(\mathbb{R}^{n+1}_+)$, $1\leqslant p\leqslant\infty$, the space of measurable functions f on $[0,\infty[\times\mathbb{R}^n$ satisfying

$$\|f\|_{L_{p,\alpha}} = \Big(\int_{\mathbb{R}^n}\int_0^\infty |f(r,x)|^p d\nu_\alpha(r,x)\Big)^{\frac{1}{p}} < \infty, \ \text{for } 1\leqslant p<\infty$$

and

$$\|f\|_{L_{\infty,\alpha}} = \|f\|_{L_\infty} = esssup_{(r,x)\in\mathbb{R}^{n+1}_+}|f(r,x)| < \infty \text{ for } p=\infty.$$

It is natural to define the convolution product generated by the shift operator.

DEFINITION 2.2. The convolution product of f,g in $L_{1,\alpha}(\mathbb{R}^{n+1}_+)$ associated with $\Delta_{n,\alpha}$ is defined, $\forall(r,x)\in[0,+\infty[\times\mathbb{R}^n$, as follows

$$(f*_\alpha g)(r,x) = \int_{\mathbb{R}^n}\int_0^\infty \mathscr{T}_{(r,x)}f(s,y)g(s,y)d\nu_\alpha(s,y).$$

Note that, the following properties are valid:

i) For all $(r,x),(s,y) \in \mathbb{R}^{n+1}_+$, $(\mu,\lambda) \in \mathbb{R}^{n+1}$, we have

$$\varphi_{(\mu,\lambda)}(r,x)\varphi_{(\mu,\lambda)}(s,y) = \mathscr{T}_{(r,x)}\varphi_{\mu,\lambda}(s,y).$$

ii) Let f be in $L_{1,\alpha}(\mathbb{R}^{n+1}_+)$, then for all $(s,y) \in \mathbb{R}^{n+1}_+$, we have

$$\int_{\mathbb{R}^n}\int_0^\infty \mathscr{T}_{(s,y)}f(r,x)d\nu_\alpha(r,x) = \int_{\mathbb{R}^n}\int_0^\infty f(r,x)d\nu_\alpha(r,x).$$

iii) If $f \in L_{p,\alpha}(\mathbb{R}^{n+1}_+)$, $1 \leqslant p \leqslant \infty$, then for all $(s,y) \in \mathbb{R}^{n+1}_+$, the function $\mathscr{T}_{(s,y)}f$ belongs to $L_{p,\alpha}(\mathbb{R}^{n+1}_+)$ and we have

$$\|\mathscr{T}_{(s,y)}f\|_{L_{p,\alpha}} \leqslant \|f\|_{L_{p,\alpha}}.$$

iv) $\lim\limits_{(r,x)\to(0,0} \|\mathscr{T}_{(r,x)}f - f\|_{L_{p,\alpha}} = 0.$

v) If $f \in L_{1,\alpha}(\mathbb{R}^{n+1}_+)$ and $g \in L_{1,\alpha}(\mathbb{R}^{n+1}_+)$, then $f *_\alpha g$ belongs to $L_{1,\alpha}(\mathbb{R}^{n+1}_+)$ and the convolution product is commutative and associative.

vi) For $p,q,r \in [0,\infty]$ such that $\frac{1}{p}+\frac{1}{q}-1=\frac{1}{r}$, the map

$$(f,g) \to f *_\alpha g$$

extends to a continuous map from $L_{p,\alpha}(\mathbb{R}^{n+1}_+) \times L_{q,\alpha}(\mathbb{R}^{n+1}_+)$ to $L_{r,\alpha}(\mathbb{R}^{n+1}_+)$ and we have

$$\|f *_\alpha g\|_{L_{r,\alpha}} \leqslant \|f\|_{L_{p,\alpha}}\|g\|_{L_{q,\alpha}}. \tag{2.5}$$

DEFINITION 2.3. The Fourier transform associated with the partial differential operators D_j and $L_{n,\alpha}$ is defined on $L_{1,\alpha}(\mathbb{R}^{n+1}_+)$ as follows, for all $(\mu,\lambda) \in \mathbb{R}^{n+1}_+$,

$$\mathscr{F}_\alpha(f)(\mu,\lambda) = \int_{\mathbb{R}^n}\int_0^\infty f(r,x)\varphi_{(\mu,\lambda)}(r,x)d\nu_\alpha(r,x).$$

The following properties are valid.

i) Let $f \in L_{1,\alpha}(\mathbb{R}^{n+1}_+)$. Then for all $(r,x) \in [0,+\infty[\times\mathbb{R}^n$, we have, $\forall(\mu,\lambda) \in \mathbb{R}^{n+1}_+$;

$$\mathscr{F}_\alpha(\mathscr{T}_{(r,x)}(f))(\mu,\lambda) = \varphi_{(\mu,\lambda)}(r,x)\mathscr{F}_\alpha(f)(\mu,\lambda).$$

ii) For $f,g \in L_{1,\alpha}(\mathbb{R}^{n+1}_+)$

$$\mathscr{F}_\alpha(f *_\alpha g)(\mu,\lambda) = \mathscr{F}_\alpha(f)(\mu,\lambda)\cdot\mathscr{F}_\alpha(g)(\mu,\lambda).$$

PROPOSITION 2.4. (see [14])

1) For all $f \in L_{1,\alpha}(\mathbb{R}^{n+1}_+)$, we have

$$\|\mathscr{F}_\alpha(f)\|_{\infty,\alpha} \leqslant \|f\|_{1,\alpha}.$$

2) The Fourier transform $\mathscr{F}_\alpha$ is a topological isomorphism from $\mathrm{S}_*(\mathbb{R}^{n+1})$ (the space of infinitely differentiable functions on $\mathbb{R}^{n+1}$, even with respect to the first variable, rapidly decreasing together with all their derivatives) onto itself.

3) (Plancherel theorem): The Fourier transform $\mathscr{F}_\alpha$ is an isometric automorphism of $L_{2,\alpha}(\mathbb{R}^{n+1}_+)$. In particular

$$\|\mathscr{F}_\alpha(f)\|_{2,\alpha} = \|f\|_{2,\alpha}.$$

4) (Inversion formula): Let $f \in L_{1,\alpha}(\mathbb{R}^{n+1}_+)$ such that $\mathscr{F}_\alpha(f) \in L_{1,\alpha}(\mathbb{R}^{n+1}_+)$, then

$$\mathscr{F}_\alpha^{-1}(f)(\mu,\lambda) = C_{\alpha,n}\mathscr{F}_\alpha(f)(\mu,-\lambda),$$

where

$$C_{\alpha,n} = \frac{1}{(2\pi)^n 2^{2\alpha}(\Gamma(\alpha+1))^2}.$$

3. Poisson kernels and Poisson integrals related with $\Delta_{n,\alpha}$

The main goal of this section is to define the Poisson integral associated with the Laplace-Bessel differential operator $\Delta_{n,\alpha}$ and give some estimates that are useful in the sequel of the paper.

Consider p_t, $t > 0$ to be the function defined on $\mathbb{R}^{n+1}_+$ by

$$p_t(r,x) = \int_{\mathbb{R}^n}\int_0^\infty e^{-t(\mu^2+\lambda^2)^{\frac{1}{2}}} e^{i\langle x,\lambda\rangle} j_\alpha(r\mu)d\gamma_\alpha(\mu,\lambda), \tag{3.1}$$

where

$$d\gamma(\mu,\lambda) = \frac{\mu^{2\alpha+1}}{(2\pi)^n 2^{2\alpha}(\Gamma(\alpha+1))^2}d\mu d\lambda.$$

The function p_t may be called Poisson kernel.

PROPOSITION 3.1. *For all $t > 0$ and $(r,x) \in \mathbb{R}^{n+1}_+$, we have*

$$p_t(r,x) = \frac{2^{\alpha+1}\Gamma(\alpha+\frac{n+3}{2})}{\pi^{(n+1)/2}\Gamma(\alpha+1)} \frac{t}{(t^2+r^2+|x|^2)^{\alpha+\frac{n+3}{2}}}.$$

Proof. By the following well known relation

$$\forall a > 0,\ e^{-a} = \int_0^\infty \frac{e^{-t}e^{-\frac{a^2}{4t}}}{\sqrt{\pi t}}dt$$

identity (3.1) becomes

$$p_t(r,x) = \frac{1}{(\Gamma(\alpha+1))^2(2\pi)^n 2^{2\alpha}}\int_{\mathbb{R}^n}\int_0^\infty \Big(\int_0^\infty \frac{e^{-s}e^{-t^2\frac{\mu^2+\lambda^2}{4s}}}{\sqrt{\pi s}}ds\Big)e^{i\langle x,\lambda\rangle} j_\alpha(r\mu)\mu^{2\alpha+1}d\mu d\lambda.$$

Therefore, from the fact that

$$\int_0^\infty e^{-\frac{t^2}{4s}\mu^2} j_\alpha(r\mu)\mu^{2\alpha+1}d\mu = \frac{2^{3\alpha+1}\Gamma(\alpha+1)s^{\alpha+1}e^{-s\frac{r^2}{t^2}}}{t^{2\alpha+2}},$$

we obtain

$$p_t(r,x) = \frac{2^{\alpha+1}}{(2\pi)^n t^{2\alpha+2}\Gamma(\alpha+1)} \int_0^\infty \Big(\int_{\mathbb{R}^n} e^{-\frac{t^2\lambda^2}{4s}} e^{i\langle x,\lambda\rangle} d\lambda\Big) \frac{e^{-s} e^{-\frac{sr^2}{t^2}}}{\sqrt{\pi s}} s^{\alpha+1} ds.$$

Thus the following relation, true for all $b > 0$,

$$\frac{1}{2\pi}\int_{\mathbb{R}^n} e^{-b\lambda^2} e^{i\lambda y} = \frac{1}{\sqrt{4\pi b}} e^{-\frac{y^2}{4b}}$$

yields

$$p_t(r,x) = \frac{2^{\alpha+1}}{\pi^{(n+1)/2}\Gamma(\alpha+1)\, t^{n+2\alpha+2}} \int_0^\infty e^{-s\left(\frac{r^2+|x|^2}{t^2}+1\right)} s^{\alpha+\frac{n+1}{2}} ds.$$

So, the result is deduced by using the change of variable $\tau = s\Big(\frac{r^2+|x|^2}{t^2}+1)\Big)$. □

PROPOSITION 3.2. *1. For all $t > 0$, the kernel $p_t > 0$ and*

$$\int_{\mathbb{R}^n}\int_0^\infty p_t(r,x) d\nu_\alpha(r,x) = 1.$$

2. For all $(r,x) \in \mathbb{R}_+^{n+1}$ and $t > 0$, the function

$$p_t(r,x) = t^{-n-2\alpha-2} p_1\big(\frac{1}{t}(r,x)\big).$$

3. For all $(\mu,\lambda) \in \mathbb{R}^{n+1}$, we have

$$\mathscr{F}_\alpha(p_t)(\mu,\lambda) = e^{-t(\mu^2+|\lambda|^2)^{\frac{1}{2}}}.$$

4. The function $p_t(r,x)$ satisfies the following equation

$$\Big(\Delta_{n,\alpha} + \frac{\partial^2}{\partial t^2}\Big) p_t(r,x) = 0.$$

5.

$$p_t *_\alpha p_s = p_{t+s}.$$

Proof. 1. It is clear that $p_t > 0$. On the other hand

$$\|p_t\|_{L_{1,\alpha}} = d_{\alpha,n} \int_{\mathbb{R}^n}\int_0^\infty \frac{t\, r^{2\alpha+1}}{(t^2+r^2+|x|^2)^{\alpha+\frac{n+3}{2}}} drdx,$$

where

$$d_{\alpha,n} = \frac{2^{\alpha+1}\Gamma(\alpha+\frac{n+3}{2})}{\pi^{\frac{n+1}{2}}\Gamma(\alpha+1)}.$$

By change of variables we can see that

$$\|p_t\|_{L_{1,\alpha}} = d_{\alpha,n} I_1 \cdot I_2,$$

with

$$I_1 = \int_0^\infty (1+s^2)^{-(\alpha+\frac{3}{2})} s^{2\alpha+1} ds$$

and

$$I_2 = \int_{\mathbb{R}^n} \frac{1}{(1+|u|^2)^{\alpha+\frac{n+3}{2}}} du$$

or, by easy calculation,

$$I_1 = \frac{\pi^{\frac{1}{2}}\Gamma(\alpha+1)}{2\Gamma(\alpha+\frac{3}{2})} \text{ and } I_2 = \frac{\pi^{\frac{n}{2}}\Gamma(\alpha+\frac{3}{2})}{\Gamma(\alpha+\frac{n+3}{2})}.$$

Therefore

$$\|p_t\|_{L_{1,\alpha}} = d_{\alpha,n} I_1 . I_2 = 1.$$

3. From relation (3.1) we have $p_t(r,x) = C_{\alpha,n}\mathscr{F}_\alpha(h_t)(r,-x)$, with $h_t(\mu,\lambda) = e^{-t(\mu^2+\lambda^2)^{1/2}}$. Thus,

$$\mathscr{F}_\alpha(p_t)(\mu,\lambda) = C_{\alpha,n}\mathscr{F}_\alpha(\check{\mathscr{F}_\alpha}(h_t)(\mu,\lambda),$$

with

$$\check{\mathscr{F}_\alpha}(h_t)(r,x) = \mathscr{F}_\alpha(h_t)(r,-x).$$

Therefore, using the proposition (2.4), 4) we obtain

$$\mathscr{F}_\alpha(p_t)(\mu,\lambda) = e^{-t(\mu^2+\lambda^2)^{1/2}}.$$

The assertion 4) is obtained by applying the Fourier transform $\mathscr{F}_\alpha$. Finally the assertion 5) is obtained from assertion 3) by the following relation.

$$\begin{aligned}
p_{t+s}(r,x) &= \mathscr{F}_\alpha^{-1}(e^{-(t+s)\left((r^2+|x|^2)^{1/2}\right)}) \\
&= \mathscr{F}_\alpha^{-1}(e^{-t\left((r^2+|x|^2)^{1/2}\right)} e^{-s\left((r^2+|x|^2)^{1/2}\right)}) \\
&= \mathscr{F}_\alpha^{-1}(e^{-t\left((r^2+|x|^2)^{1/2}\right)}) * \mathscr{F}_\alpha^{-1}(e^{-s\left((r^2+|x|^2)^{1/2}\right)}) \\
&= p_t(r,x) * p_s(r,x). \quad \square
\end{aligned}$$

However, for $t > 0$ and for all $f \in L_{p,\alpha}(\mathbb{R}_+^{n+1})$ we put

$$u((r,x),t) = p_t *_\alpha f(r,x),$$

the function u is called the Poisson integral of f associated with Laplace-Bessel differential operators.

LEMMA 3.3. *For all measurable function f bounded on $\mathbb{R}_+^{n+1}$ and continuous in $(0,0)$ we have*

$$\lim_{t\to 0}\int_{\mathbb{R}^n}\int_0^\infty p_t(r,x)f(r,x)r^{2\alpha+1}drdx = f(0,0).$$

Proof. By proposition 3.2, 1) we deduce

$$\int_{\mathbb{R}^n}\int_0^\infty p_t(r,x)f(r,x)d\nu(r,x) - f(0,0) = \int_{\mathbb{R}^n}\int_0^\infty p_t(r,x)(f(r,x)-f(0,0))d\nu(r,x).$$

But f is bounded on $\mathbb{R}_+^{n+1}$ and continuous in $(0,0)$ then, for all $\beta > 0$ there exists $\alpha > 0$ such that

$$\Big|\int_{\mathbb{R}^n}\int_0^\infty p_t(r,x)f(r,x)d\nu(r,x) - f(0,0)\Big| \leqslant \frac{\beta}{2} + 2||f||_{\infty,\alpha}\int\int_{r^2+|||x||^2>\alpha} p_t(r,x)d\nu(r,x).$$

So, the lemma is obtained from the fact that

$$\lim_{t\longrightarrow 0}\int\int_{r^2+|||x||^2>\alpha} p_t(r,x)d\nu(r,x) = 0. \quad \square$$

By using this lemma, properties of the shift operators and Fourier transforms associated with the Laplace-Bessel differential operator $\Delta_{n,\alpha}$, we deduce the following propositions.

PROPOSITION 3.4. *For all $f \in L_{p,\alpha}(\mathbb{R}_+^{n+1})$, $1 \leqslant p < \infty$, the function u defined on $\mathbb{R}_+^{n+1}$ by*

$$u((r,x),t) = (f *_\alpha p_t)(r,x)$$

satisfies the equation

$$\Big(\Delta_{n,\alpha} + \frac{\partial^2}{\partial t^2}\Big)p_t(r,x) = 0$$

and

$$\lim_{t\to 0}\|u(.,t) - f\|_{L_{p,\alpha}} = 0.$$

PROPOSITION 3.5. *Let f be continuous and bounded on $\mathbb{R}_+^{n+1}$, then the function u defined on $\mathbb{R}_+^{n+1}$ by*

$$u((r,x),t) = \big(f *_\alpha p_t\big)(r,x)$$

satisfies the equation

$$\left(\Delta_{n,\alpha} + \frac{\partial^2}{\partial t^2}\right)p_t(r,x) = 0$$

and

$$\lim_{t\to 0} u((r,x),t) = f(r,x) \ \textit{uniformly}.$$

The Poisson integral also has the following property:
For all $f \in S_*(\mathbb{R}^{n+1})$,

$$u((r,x),t) = \int_{\mathbb{R}^n}\int_0^\infty e^{-t(\mu^2+\lambda^2)^{\frac{1}{2}}} \mathscr{F}_\alpha(f)(\mu,\lambda)\overline{\psi_{(\mu,\lambda)}}(r,x)d\gamma_\alpha(\mu,\lambda).$$

Now, we will give some estimates of the Poisson integral and its partial derivatives. We denote by $\mathscr{D}_*(\mathbb{R}_+^{n+1})$ the space of infinitely differentiable functions on $\mathbb{R}_+^{n+1}$, even with respect to the first variable, and with compact support.

PROPOSITION 3.6. *Let $f \in \mathscr{D}_*(\mathbb{R}^{n+1})$ be a positive function and $p > 1$.*
i) For $|(r,x)| = \sqrt{r^2+x^2}$ large we have

$$u((r,x),t) \leqslant C(t^2+r^2+|x|^2)^{-(\alpha+\frac{n+2}{2})}. \tag{3.2}$$

ii)

$$\frac{\partial u}{\partial t}((r,x),t) \leqslant C\, t^{-(2\alpha+n+3)}. \tag{3.3}$$

iii)

$$\frac{\partial u}{\partial r}((r,x),t) \leqslant C(t^2+r^2+|x|^2)^{-(\alpha+\frac{n+3}{2})}. \tag{3.4}$$

iv)

$$\frac{\partial u}{\partial x_i}((r,x),t) \leqslant C(t^2+r^2+|x|^2)^{-(\alpha+\frac{n+3}{2})},\ 1 \leqslant i \leqslant n. \tag{3.5}$$

Proof. Since $f \in \mathscr{D}_*(\mathbb{R}^{n+1})$, then there exists $A > 0$ such that $\operatorname{supp}(f) \subset [0,A] \times B(0,A)$.

$$u((r,x),t) = p_t *_\alpha f(r,x) = \int_0^\infty \int_{B(0,A)} \mathscr{T}_{(r,x)} p_t(s,y) f(s,y) d\nu_\alpha(s,y).$$

On the other hand

$$\begin{aligned}\mathscr{T}_{(r,x)}p_t(s,y) &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi p_t(\sqrt{r^2+s^2+2rs\cos\theta}, x+y)\sin^{2\alpha}\theta d\theta \\ &= \frac{\Gamma(\alpha+1)C_\alpha}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2}}\int_0^\pi \frac{t\sin^{2\alpha}\theta}{(r^2+s^2+2rs\cos\theta+t^2+|x+y|^2)^{\alpha+\frac{n+3}{2}}}d\theta,\end{aligned}$$

where

$$C_\alpha = \frac{2^{2\alpha+\frac{n}{2}+1}\Gamma(\alpha+\frac{n+3}{2})}{\sqrt{\pi}}.$$

Thus $i)$ is obtained by using the fact that for $\|(r,x)\|$ large and $(s,y) \in [0,A] \times B(0,A)$ and $\theta \in [0,\pi]$

$$\left|\frac{t\sin^{2\alpha}\theta}{(r^2+s^2+2rs\cos\theta+t^2+|x+y|^2)^{\alpha+\frac{n+3}{2}}}\right| \leqslant \frac{t}{(r^2+t^2+|x|^2)^{\alpha+\frac{n+3}{2}}},$$

and

$$t \leqslant (r^2+t^2+|x|^2)^{\frac{1}{2}}.$$

ii) Relation (2.7) allows us to obtain

$$\left|\frac{\partial u}{\partial t}((r,x),t)\right| \leqslant \|f\|_{L_{1,\alpha}} \left\|\frac{\partial p_t}{\partial t}\right\|_{L_\infty} \leqslant Ct^{-(2\alpha+n+3)}.$$

The assertions *iii*) and *iv*) are obtained the same way as *ii*). □

PROPOSITION 3.7. *Let $f \in \mathscr{D}_*(\mathbb{R}^{n+1})$ be a positive function, $p>1$ and a weight function w satisfying the following condition*

$$\int_{B(0,A)}\int_0^A w(r,x)d\nu_\alpha(r,x) = O(A^{n+\alpha+2}), A \to \infty. \tag{3.6}$$

Then we have

i)

$$\lim_{A\to\infty}\int_{B(0,A)}\int_0^A \frac{\partial^2 u^p}{\partial^2 t}((r,x),t)tdtw(r,x)d\nu_\alpha(r,x) = \int_{\mathbb{R}^n}\int_0^\infty f^p(r,x)w(r,x)d\nu_\alpha(r,x).$$

ii)

$$\lim_{A\to\infty}\int_{B(0,A)}\int_0^A \Delta_{n,\alpha}u^p((r,x),t)tdtw(r,x)d\nu_\alpha(r,x) = 0.$$

Proof. Let $f \in \mathscr{D}_*(\mathbb{R}^{n+1})$,

$$\int_{B(0,A)}\int_0^A \frac{\partial^2 u^p}{\partial t^2}((r,x),t)tw(r,x)dtd\nu_\alpha(r,x) = \int_{B(0,A)}\int_0^A H(r,x,A)w(r,x)d\nu_\alpha(r,x)$$

and a positive function

$$H(r,x,A) = \int_0^A \frac{\partial^2 u^p}{\partial t^2}((r,x),t)tdt$$

such that

$$\begin{aligned} H(r,x,A) &= \left[t\frac{\partial u^p}{\partial t}((r,x),t)\right]_0^A - \int_0^A \frac{\partial u^p}{\partial t}((r,x),t)dt \\ &= A\frac{\partial u^p}{\partial t}((r,x),A) - u^p((r,x),A) + u^p((r,x),0) \\ &= A\frac{\partial u^p}{\partial t}((r,x),A) - u^p((r,x),A) + f^p(r,x). \end{aligned}$$

On the other hand by relations (3.2) and (3.3) we have

$$|u^p(r,x,A)| \leqslant CA^{-(n+2\alpha+2)p}$$

and

$$A\left|\frac{\partial u^p}{\partial t}((r,x),A)\right| \leqslant p\left|\frac{\partial u}{\partial t}((r,x),A)\right| |u^{p-1}((r,x),A)| \leqslant CA^{-(n+2\alpha+2)p}.$$

From the above estimates and the condition of the weight functions (3.6) we obtain

$$\left|\int_{B(0,A)}\int_0^A \left[A\frac{\partial u^p}{\partial t}((r,x),A) - u^p(r,x,A)\right] w(r,x)d\nu_\alpha(r,x)\right| \leqslant CA^{(1-p)(n+2\alpha+2)}.$$

Thus

$$\lim_{A\to\infty}\int_{B(0,A)}\int_0^A \left[A\frac{\partial u^p}{\partial t}((r,x),A) - u^p(r,x,A)\right] w(r,x)d\nu_\alpha(r,x) = 0.$$

Therefore we get

$$\lim_{A\to\infty}\int_{B(0,A)}\int_0^A \frac{\partial^2 u^p}{\partial t^2}((r,x),t)tdtw(r,x)d\nu_\alpha(r,x) = \int_{\mathbb{R}^n}\int_0^\infty f^p(r,x)w(r,x)d\nu_\alpha(r,x).$$

This achieved i).

To prove ii) we see that

$$\Delta_{n,\alpha} = \frac{1}{r^{2\alpha+1}}\frac{\partial}{\partial r}\left[r^{2\alpha+1}\frac{\partial}{\partial r}\right] + \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}.$$

Then

$$\int_{B(0,A)}\int_0^A \Delta_{n,\alpha}u^p((r,x),t)tdtw(r,x)d\nu_\alpha(r,x)$$

$$= \int_{B(0,A)}\int_0^A \frac{1}{r^{2\alpha+1}}\frac{\partial}{\partial r}\left[r^{2\alpha+1}\frac{\partial u^p}{\partial r}\right]((r,x),t)tdtw(r,x)d\nu_\alpha(r,x)$$

$$+\sum_{i=1}^n \int_{B(0,A)}\int_0^A \frac{\partial^2 u^p}{\partial x_i^2}((r,x),t)tdtw(r,x)d\nu_\alpha(r,x).$$

Thus the assertion ii) is obtained, similarly to i), by using Proposition 3.6. □

4. Maximal operators associated with Laplace-Bessel differential operators

We will make use of the Laplace Bessel maximal function associated with the differential operators $\Delta_{n,\alpha}$. The maximal function is defined by (see [5, 6, 7])

$$\mathscr{M}_\alpha f(r,x) = \sup_{\varepsilon>0}|B(0,\varepsilon)|_\alpha^{-1}\int_{B(0,\varepsilon)} \mathscr{T}_{(s,y)}(|f(r,x)|)d\nu_\alpha(s,y),$$

where

$$B(0,\varepsilon) = \{(s,y)\in\mathbb{R}_+^{n+1} : s^2+|y|^2 \leqslant \varepsilon^2\}.$$

An almost everywhere positive and locally integrable function $\omega: \mathbb{R}_+^{n+1} \to \mathbb{R}$ will be called a weight. Denote by $L_{p,\omega,\alpha}(\mathbb{R}_+^{n+1})$ the set of measurable functions $f(r,x)$, $(r,x) \in \mathbb{R}_+^{n+1}$, with finite norm

$$\|f\|_{L_{p,\omega,\alpha}} = \left(\int_{\mathbb{R}_+^{n+1}} |f(r,x)|^p w(r,x) r^{2\alpha+1} dr dx\right)^{\frac{1}{p}} < \infty, \ 1 \leqslant p < \infty.$$

DEFINITION 4.1. [7] The weight function ω belongs to the class $A_{p,\alpha}(\mathbb{R}_+^{n+1})$ for $1 < p < \infty$, if

$$\sup_{x \in \mathbb{R}_+^{n+1}, r>0} |B((r,x),t)|_\alpha^{-1} \int_{B((r,x),t)} \omega(s,y) s^{2\alpha+1} ds dy$$
$$\left(|B((r,x),t)|_\alpha^{-1} \int_{B((r,x),t)} \omega^{-\frac{1}{p-1}}(s,y) s^{2\alpha+1} ds dy\right)^{p-1} < \infty$$

and ω belongs to $A_{1,\alpha}(\mathbb{R}_+^{n+1})$, if there exists a positive constant C such that for any $(r,x) \in \mathbb{R}_+^{n+1}$ and $t > 0$

$$|B((r,x),t)|_\alpha^{-1} \int_{B((r,x),t)} w(s,y) s^{2\alpha+1} ds dy \leqslant C \operatorname*{ess\,inf}_{(s,y) \in B((r,x),t)} \omega(s,y).$$

The properties of the class $A_{p,\alpha}(\mathbb{R}_+^{n+1})$ are analogous to those of the B. Muckenhoupt classes. In particular, if $w \in A_{p,\alpha}(\mathbb{R}_+^{n+1})$, then $w \in A_{p-\varepsilon,\alpha}(\mathbb{R}_+^{n+1})$ for a certain sufficiently small $\varepsilon > 0$ and $w \in A_{p_1,\alpha}(\mathbb{R}_+^{n+1})$ for any $p_1 > p$.

Note that, $|(r,x)|^\alpha \in A_{p,\alpha}(\mathbb{R}_+^{n+1})$, $1 < p < \infty$, if and only if $-(n+2\alpha+2) < \alpha < (n+2\alpha+2)(p-1)$ and $|(r,x)|^\alpha \in A_{1,\alpha}(\mathbb{R}_+^{n+1})$, if and only if $-(n+2\alpha+2) < \alpha \leqslant 0$.

The following theorem was proved in [7].

THEOREM 4.2. *1) Let $1 < p < \infty$. Then the following two conditions are equivalent:*

(i) There is a constant $C > 0$ such that for any $f \in L_{p,\omega,\alpha}(\mathbb{R}_+^{n+1})$ the inequality

$$\int_{\mathbb{R}_+^{n+1}} (\mathscr{M}_\alpha(f)(r,x))^p \omega(r,x) r^{2\alpha+1} dr dx \leqslant C \int_{\mathbb{R}_+^{n+1}} |f(r,x)|^p \omega(r,x) r^{2\alpha+1} dr dx$$

holds.

(ii) $\omega \in A_{p,\alpha}(\mathbb{R}_+^{n+1})$.

2) Let $p = 1$. Then the following two conditions are equivalent:

(i) There is a constant $C > 0$ such that for any $f \in L_{1,\omega,\alpha}(\mathbb{R}_+^{n+1})$ the inequality

$$\int_{\{(r,x) \in B((0,0),t)\,:\, \mathscr{M}_\alpha f(r,x) > \lambda\}} \omega(r,x) r^{2\alpha+1} dr dx \leqslant \frac{C}{\lambda} \|f\|_{L_{1,\omega,\alpha}}$$

holds.

(ii) $\omega \in A_{1,\alpha}(\mathbb{R}_+^{n+1})$.

5. The Littlewood-Paley g-function

In this section we define the g-Littlewood functions and establish the $L_{p,w}$ inequalities.

DEFINITION 5.1. We define the g-Littlewood-Paley functions associated with the Laplace-Bessel differential operators for $f \in S_*(\mathbb{R}^{n+1})$ by the following

$$g(f)(r,x) = \left[\int_0^\infty |\nabla u((r,x),t)|^2 t dt\right]^{1/2}, \ (r,x) \in \mathbb{R}_+^{n+1},$$

where $u((r,x),t)$ is the Poisson integral defined by $u((r,x),t) = p_t * f(r,x)$ and

$$|\nabla u((r,x),t)|^2 = \left|\frac{\partial u}{\partial t}((r,x),t)\right|^2 + \left|\frac{\partial u}{\partial r}((r,x),t)|^2 + \sum_{i=1}^n\right|\frac{\partial u}{\partial x_i}((r,x),t)|^2.$$

In the Bessel case ($n=0$) the L_p norm of $g(f)$ is comparable with L_p norm of f for $p \in]1,\infty[$ (see [1, 12]). In the following we prove the same results for our g-function when $p \in]1,2]$.

PROPOSITION 5.2. *Let Φ a positive, non-increasing in $L_1(d\nu_\alpha)$ locally integrable function on $\mathbb{R}_+^{n+1}$, we have*

$$\sup_{t>0} |\Phi_t *_\alpha f(r,x)| \leqslant \|\Phi\|_{L_{1,\alpha}} \mathscr{M}_\alpha f(r,x),$$

where Φ_t is the dilation of Φ given by

$$\Phi_t(r,x) = t^{-(n+2\alpha+2)}\Phi(r/t,x/t).$$

Proof. As in the Euclidian case ([11], p. 57) we prove the proposition. □

THEOREM 5.3. *Let $p \in]1,2]$ and $\omega \in A_{p,\alpha}(\mathbb{R}_+^{n+1})$. Then there exist two constants $B_{p,\alpha}$ and $C_{p,\alpha}$ such that for all $f \in L_{p,\omega,\alpha}$ we have*

$$B_{p,\alpha}\|f\|_{L_{p,\omega,\alpha}} \leqslant \|g(f)\|_{L_{p,\omega,\alpha}} \leqslant C_{p,\alpha}\|f\|_{L_{p,\omega,\alpha}}.$$

Proof. First step. The inequality $B_{p,\alpha}\|f\|_{L_{p,\omega,\alpha}} \leqslant \|g(f)\|_{L_{p,\omega,\alpha}}$.

Let $f \in L_{2,\omega,\alpha}(\mathbb{R}_+^{n+1}) \cap L_{p,\omega,\alpha}(\mathbb{R}_+^{n+1})$, $h \in L_{2,\omega,\alpha}(\mathbb{R}_+^{n+1}) \cap L_{q,\omega,\alpha}(\mathbb{R}_+^{n+1})$, $\frac{1}{p}+\frac{1}{q}=1$.

Let $f \in \mathscr{D}_*(\mathbb{R}^{n+1})$. Since the operator g defined by $g(f)(r,x) = \left[\int_0^\infty |\nabla u((r,x),t)|^2 t dt\right]^{1/2}$, $(r,x) \in \mathbb{R}_+^{n+1}$, where u is the Poisson integral satisfies

$$g(f+h) \leqslant g(f) + g(h),$$

then it suffices to proof the theorem for $f \geqslant 0$.

Let us start proving the inequality on the right side.

Case $p<2$. By using the fact that

$$\Delta_\alpha u^p((r,x),t)=\left(\Delta_{n,\alpha}+\frac{\partial^2}{\partial t^2}\right)u^p((r,x),t)=0,$$

we obtain

$$\Delta_\alpha u^p((r,x),t)=p(p-1)u^{p-2}((r,x),t)|\nabla u((r,x),t)|^2. \tag{5.1}$$

Then

$$\begin{aligned}|g(f)((r,x))|^2 &\leqslant \frac{1}{p(p-1)}\int_0^\infty u^{2-p}((r,x),t)|\Delta_\alpha u^p((r,x),t)|tdt\\ &\leqslant \frac{1}{p(p-1)}\left(\sup_{t>0}|u((r,x),t)|\right)^{2-p}\int_0^\infty |\Delta_\alpha u^p((r,x),t)|tdt\\ &\leqslant \frac{1}{p(p-1)}(\widetilde{\mathscr{M}}(f)(r,x))^{2-p}\, I_\alpha(f)(r,x),\end{aligned}$$

where

$$\widetilde{\mathscr{M}}(f)(r,x)=\sup_{t>0}|u((r,x),t)|$$

and

$$I_\alpha(f)(r,x)=\int_0^\infty |\Delta_\alpha u^p((r,x),t)|tdt.$$

It follows that

$$|g(f)((r,x))|^p\leqslant\left(\frac{1}{p(p-1)}\right)^{p/2}(\widetilde{\mathscr{M}}(f)(r,x))^{(2-p)p/2}\,(I_\alpha(f)(r,x))^{p/2}.$$

Applying Hölder inequality we get

$$\|g(f)\|^p_{L_{p,\omega,\alpha}}\leqslant\left(\frac{1}{p(p-1)}\right)^{p/2}\|\widetilde{\mathscr{M}}(f)\|^{(2-p)p/2}_{L_{p,\omega,\alpha}}\,\|I_\alpha(f)\|^{p/2}_{L_{1,\omega,\alpha}}. \tag{5.2}$$

From Propositions 5.2 and 3.2 we have

$$|\widetilde{\mathscr{M}}(f)(r,x)|^p\leqslant\|p_t\|^p_{L_{1,\alpha}}\,\mathscr{M}_\alpha|f(r,x)|^p=\mathscr{M}_\alpha|f(r,x)|^p. \tag{5.3}$$

Therefore from Theorem 4.2 we deduce

$$\|\widetilde{\mathscr{M}}(f)\|_{L_{p,\omega,\alpha}}\leqslant C_{p,\alpha}\|f\|_{L_{p,\omega,\alpha}}. \tag{5.4}$$

Furthermore. Proposition 3.7 allows us to deduce

$$\begin{aligned}\|I_\alpha(f)\|_{L_{1,\omega,\alpha}} &\leqslant \int_{\mathbb{R}^n}\int_0^\infty\int_0^\infty \Delta_{n,\alpha}u^p((r,x),t)tdt\,\omega(r,x)d\nu_\alpha(r,x)\\ &\leqslant \int_{\mathbb{R}^n}\int_0^\infty |f(r,x)|^p\,\omega(r,x)d\nu_\alpha(r,x).\end{aligned} \tag{5.5}$$

So, from relations (5.1), (5.4) and (5.5) we have

$$\|g(f)\|_{L_{p,\omega,\alpha}} \leqslant C_{p,\alpha}\|f\|_{L_{p,\omega,\alpha}}.$$

Case $p=2$. By relation (5.1) and Proposition 3.7 we have

$$\|g(f)\|^2_{2,\omega,\alpha} \leqslant \frac{1}{2}\int_{\mathbb{R}^n}\int_0^\infty |f(r,x)|^2 d\nu_\alpha(r,x) = \frac{1}{2}\|f\|^2_{L_{2,\omega,\alpha}}.$$

We have shown that

$$\|g(f)\|_{L_{p,\omega,\alpha}} \leqslant C_{p,\alpha}\|f\|_{L_{p,\omega,\alpha}}, \ \text{for}\, 1<p\leqslant 2.$$

Using a standard duality argument, we can prove the converse inequality. In fact for $f \in L_{p,\omega,\alpha}(\mathbb{R}^{n+1}_+)$ we define the g_1-function for f by

$$g_1(f)(r,x) = \Big(\int_0^\infty t\big|\frac{\partial u}{\partial t}(r,x)\big|^2 dt\Big)^{1/2}.$$

Obviously

$$g_1(f)(r,x) \leqslant g(f)(r,x). \tag{5.6}$$

As in [1, 9] we prove that for $f \in L_{2,\omega,\alpha}(\mathbb{R}^{n+1}_+)$

$$\|g_1(f)\|_{L_{2,\omega,\alpha}} = 2\|f\|_{L_{2,\omega,\alpha}}.$$

From this relation and by polarization identity we get for all $f \in L_{2,\omega,\alpha}(\mathbb{R}^{n+1}_+) \cap L_{p,\omega,\alpha}(\mathbb{R}^{n+1}_+)$ and $h \in L_{2,\omega,\alpha}(\mathbb{R}^{n+1}_+) \cap L_{q,\omega,\alpha}(\mathbb{R}^{n+1}_+)$ such that $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_0^\infty f(r,x)\overline{h(r,x)}\omega(r,x)d\nu_\alpha(r,x)\\ &= 4\int_{\mathbb{R}^n}\int_0^\infty\int_0^\infty t\frac{\partial u}{\partial t}((r,x),t)\frac{\partial \overline{v}}{\partial t}((r,x),t)dt\,\omega(r,x)d\nu_\alpha(r,x),\end{aligned}$$

where u and v are the Poisson integral of f respectively of h.

Applying Hölder inequality and the relation (5.6), we obtain

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n}\int_0^\infty f(r,x)\overline{h(r,x)}\omega(r,x)d\nu_\alpha(r,x)\Big|\\ &\leqslant 4\int_{\mathbb{R}^n}\int_0^\infty g_1(f)(r,x)g_1(h)(r,x)\omega(r,x)d\nu_\alpha(r,x)\\ &\leqslant 4\,C_{p,\alpha}\|h\|_{q,\omega,\alpha}\|g_1(f)\|_{L_{p,\omega,\alpha}}.\end{aligned}$$

Since

$$\|f\|_{L_{p,\omega,\alpha}} = \sup\Big\{\Big|\int_{\mathbb{R}^n}\int_0^\infty f(r,x)\overline{h(r,x)}d\nu_\alpha(r,x)\Big|,\ \|h\|_{q,\omega,\alpha}\leqslant 1\Big\},$$

then we get

$$A_{p,\alpha}\|f\|_{L_{p,\omega,\alpha}} \leqslant \|g(f)\|_{L_{p,\omega,\alpha}}$$

where

$$A_{p,\alpha} = (4\,C_{p,\alpha})^{-1}.$$

This completes the proof of theorem 5.3. □

Application

Coifman and Meyer in [4] proved that the L_p boundedness of the commutator $[b,T]$ defined by

$$[b,T]f(x) = b(x)Tf(x) - T(bf)(x)$$

could be obtained from the weighted L_p estimate for T with A_p weight when $b \in$ BMO and T is a standard Calderón-Zygmund singular integral operator, where A_p is the weight function class of Muckenhoupt. In 1993, Alvarez, Babgy, Kurtz and Pérez [2] developed the idea of Coifman and Meyer, and established a general boundedness criterion for the commutators of linear operators. Their results can be stated as follows.

THEOREM 5.4. *Let E a Banach space, $1 < p,q < \infty$. Suppose that the linear operator $T : C_0^\infty \to M(E)$ satisfies the weight estimate*

$$||T(f)||_{L_{p,w}} \leqslant \overline{C}||f||_{L_{p,w}}$$

for all $w \in A_q$ and $\overline{C}$ depends only on n,p and $\widetilde{C}_q(w)$ (the A_p constant of w), but not on the weight w. Then for any positive integer k and $b \in BMO(\mathbb{R}^n)$, the commutator

$$T_{b,k}(f)(x) = T((b(x) - b(\cdot))^k f)(x)$$

is bounded on $L_{p,u}(E)$ for all $u \in A_q$ with norm

$$C(p,n,k,\widetilde{C}_q(u))||b||_{BMO}^k.$$

THEOREM 5.5. *Let $p \in]1,2]$ and $\omega \in A_{p,\alpha}(\mathbb{R}_+^{n+1})$. Then for any positive k and $b(x) \in BMO(\mathbb{R}_+^{n+1})$, the commutator*

$$g_{b,k}(f)(x) = g((b(x) - b(\cdot))^k f)(x)$$

is bounded on $L_{p,\omega,\alpha}(\mathbb{R}_+^{n+1})$ with norm

$$C(p,n,k,\widetilde{C}_{p,\alpha}(w))||b||_{BMO}^k,$$

where $\widetilde{C}_{p,\alpha}(w)$ is (the $A_{p,\alpha}$ constant of w).

Proof. From Theorem 5.3 we have for all $p \in]1,2]$ and $w \in A_{p,\alpha}(\mathbb{R}_+^{n+1})$

$$||g(f)||_{L_{p,w,\alpha}} \leqslant C_{p,\alpha}||f||_{L_{p,w,\alpha}},$$

then the result is obtained by Theorem 5.4. □

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REFERENCES

[1] A. ACHOUR, K. TRIMECHE, *La g-function de Littlewood-Paley associée à un opérateur différentiel singulier sur* $(0,\infty)$, Ann. Inst. Fourier, Grenoble **33** (1983), 203–226.
[2] J. ALVAREZ, R. BADGY, D. KURTZ AND C. PÉREZ, *Weighted estimates for commutators of linear operators*, Studia Math. **104** (1993), 195–209.
[3] J. J. BETANCOR, A. J. CASTRO, AND J. CURBELO, *Harmonic analysis operators associated with multidimentional Bessel operators*, Arxiv (2010).
[4] R. COIFMAN AND Y. MEYER, *Au déla des opérateurs pseudo-differentieles*, Asterrique **57** (1978), 1–185.
[5] V. S. GULIEV, *On the maximal function and fractional integral associated with the Bessel differential operator*, Math. Inequal. Appl. **6**, 2 (2003), 317–330.
[6] V. S. GULIEV, N. N. GARAKHANOVA, AND YU. ZEREN, *Pointwise and integral estimates for the B-Riesz potentiel in terms of B-maximal and B-factional maximal functions*, Siberian Mathematical Journal **49**, 6 (2008), 1008–1022.
[7] E. V. GULIEV, *Weighted inequality for fractional maximal functions and fractional integrals, associated with the Laplace-Bessel differential operator*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. **26**, 1 (2006), Math. Mech., 71–80.
[8] E. STEIN, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Ann. of math. Studies No. **63**, Princeton Univ. Press, N. J., 1970.
[9] E. STEIN AND G. WEISS, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Univ. Press, Princeton, N. J., 1971.
[10] E. STEIN AND S. WAIGNER, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), 1239–1295.
[11] E. STEIN, *Harmonic Analysis Real-variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, Princeton, NJ, 1993.
[12] K. STEMPAK, *La théorie de Littlewood-Paley pour la transformation de Fourier-Bessel*, C. R. Acad. Sci. Paris, Serie I, Math. **303** (1986), 15–18.
[13] K. TRIMÈCHE, *Transformation intégrale de Weyl et théorème de Paley-Wiener associés à un opérateur différentel singulier sur* $(0,\infty)$, J. Math Pures. Appl (9) **60** (1981), 51–98.
[14] K. TRIMÈCHE, *Inversion of the Lions transmutations operators using generalized wavelets*, Applied and computational Harmonic Analysis **4**, 1 (1997), 97–112.
[15] G. N. WATSON, *A treatise on the theory of Bessel functions*, Cambridge University Press, Cambridge, 1996.

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A. Akbulut
Ahi Evran University
Department of Mathematics
Kirsehir, Turkey
e-mail: aakbulut@ahievran.edu.tr

V. S. Guliyev
Ahi Evran University
Department of Mathematics
Kirsehir, Turkey
and
Institute of Mathematics and Mechanics
Baku, Azerbaijan
e-mail: vagif@guliyev.com

M. Dziri
Department of Mathematics
Faculty of Sciences of Bizerte
Bizerte, Tunisia
e-mail: Moncef.Dziri@iscae.rnu.tn