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SOME INEQUALITIES FOR L_p–DUAL AFFINE SURFACE AREA

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Abstract. Lutwak proposed the notion of L_p-affine surface area according to the L_p-mixed volume. Recently, Wang and He introduced the concept of L_p-dual affine surface area combing with the L_p-dual mixed volume. In this article, some inequalities for L_p-dual affine surface area are established.

1. Introduction and main results

Let $\mathscr{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors, the set of convex bodies whose centroid lie at the origin and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $\mathscr{K}_o^n$, $\mathscr{K}_c^n$ and $\mathscr{K}_s^n$, respectively. Let S_o^n denote the set of star bodies (about the origin) in $\mathbb{R}^n$. Let S^{n-1} denote the unit sphere in $\mathbb{R}^n$ and $V(K)$ the n-dimensional volume of body K. For the standard unit ball B in $\mathbb{R}^n$, denote $\omega_n = V(B)$.

The notion of affine surface area was given by Leichtweiß (see [6]). For $K \in \mathscr{K}^n$, the affine surface area, $\Omega(K)$, of K is defined by

$$n^{-\frac{1}{n}}\Omega(K)^{\frac{n+1}{n}} = \inf\{nV_1(K,Q^*)V(Q)^{\frac{1}{n}} : Q \in S_o^n\}.$$

Here Q^* denotes the polar of body Q.

According to the L_p-mixed volume, Lutwak introduced the notion of L_p-affine surface area in [9]. For $K \in \mathscr{K}_o^n$, $p \geqslant 1$, the L_p-affine surface area, $\Omega_p(K)$, of K is defined by

$$n^{-\frac{p}{n}}\Omega_p(K)^{\frac{n+p}{n}} = \inf\{nV_p(K,Q^*)V(Q)^{\frac{p}{n}} : Q \in S_o^n\}. \tag{1.1}$$

Obviously, if $p = 1$, $\Omega_1(K)$ is just the classical affine surface area $\Omega(K)$.

In addition, Lutwak gave the notion of L_p-mixed affine surface area in [9]. Further, Wang and Leng in [16] defined ith L_p-mixed affine surface area, $\Omega_{p,i}(K)$, of K (for $i = 0$, $\Omega_{p,i}(K)$ is just the L_p-affine surface area $\Omega_p(K)$) and extended Lutwak's some

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Paper MIA-17-32

results. Regarding the study of L_p-affine surface area, many results have been obtained in these articles (see [9, 15–17, 22, 23]).

According to the notion of L_p-affine surface area introduced by Lutwak. In 2008, Wang and He gave the notion of L_p-dual affine surface area associated with the L_p-dual mixed volume in [21]. For $K \in S_o^n$ and $1 \leqslant p < n$, L_p-dual affine surface area, $\widetilde{\Omega}_{-p}(K)$, of K is defined by

$$n^{\frac{p}{n}}\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}} = \inf\{n\widetilde{V}_{-p}(K,Q^*)V(Q)^{-\frac{p}{n}} : Q \in \mathscr{K}_c^n\}. \tag{1.2}$$

Associated with the definition of L_p-dual affine surface area, Wang and He (see [21]) proved the following results:

THEOREM 1.A. *If $K \in \mathscr{K}_c^n$, $1 \leqslant p < n$, then*

$$\widetilde{\Omega}_{-p}(K)^{n-p} \geqslant n^{n-p}\omega_n^{-2p}V(K)^{n+p},$$

with equality if and only if K is an ellipsoid.

THEOREM 1.B. *If $K,L \in \mathscr{K}_c^n$ and $1 \leqslant p < n$, then*

$$\widetilde{\Omega}_{-p}(K\tilde{+}_{n+p}L)^{\frac{n-p}{n}} \geqslant \widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}} + \widetilde{\Omega}_{-p}(L)^{\frac{n-p}{n}},$$

with equality if and only if K and L are dilates. Here $K\tilde{+}_{n+p}L$ denotes the L_{n+p}-radial linear combination of K and L (see [21]).

In this article, we shall continuously study the L_p-dual affine surface area $\widetilde{\Omega}_{-p}(K)$. Firstly, associated with Theorem 1.A, we give its dual form as follows:

THEOREM 1.1. *If $K \in \mathscr{K}_c^n$, $1 \leqslant p < n$, then*

$$\widetilde{\Omega}_{-p}(K)^{n-p} \leqslant n^{n-p}\omega_n^{2n}V(K^*)^{-(n+p)}, \tag{1.3}$$

with equality if and only if K is an ellipsoid.

Secondly, combining with L_p-curvature image, we obtain a kind of corresponding form of Theorem 1.A.

THEOREM 1.2. *If $K \in \mathscr{F}_s^n$, $1 \leqslant p < n$, then*

$$\widetilde{\Omega}_{-p}(\Lambda_p K)^{n-p} \geqslant n^{n-p}\omega_n^{-2p}V(\Lambda_p K)^{n+p}, \tag{1.4}$$

with equality if and only if K is an ellipsoid.

Further, corresponding to Theorem 1.B, we get a Brunn-Minkowski inequality of the L_p-dual affine surface area about the L_q-radial linear combination.

THEOREM 1.3. *For $K,L \in \mathscr{K}_c^n$, $\lambda,\mu \geqslant 0$ (not both zero) and $1 \leqslant p < n$, if $q > n+p$, then*

$$\widetilde{\Omega}_{-p}(\lambda \circ K\tilde{+}_q\mu \circ L)^{\frac{q(n-p)}{n(n+p)}} \geqslant \lambda\widetilde{\Omega}_{-p}(K)^{\frac{q(n-p)}{n(n+p)}} + \mu\widetilde{\Omega}_{-p}(L)^{\frac{q(n-p)}{n(n+p)}}, \tag{1.5}$$

with equality if and only if K and L are dilates.

Besides, associated with the L_q-harmonic radial combination of star bodies, we give another Brunn-Minkowski inequality for the L_p-dual affine surface area.

THEOREM 1.4. *If* $K,L \in \mathscr{K}_c^n$, $1 \leqslant p < n$, $q \geqslant 1$ *and* $\lambda,\mu \geqslant 0$ *(not both zero),*

$$\widetilde{\Omega}_{-p}(\lambda \star K +_{-q} \mu \star L)^{-\frac{q(n-p)}{n(n+p)}} \geqslant \lambda \widetilde{\Omega}_{-p}(K)^{-\frac{q(n-p)}{n(n+p)}} + \mu \widetilde{\Omega}_{-p}(L)^{-\frac{q(n-p)}{n(n+p)}}, \tag{1.6}$$

with equality if and only if K *and* L *are dilates.*

The proofs of Theorem 1.1–1.4 will be completed in section 3 of this paper.

2. Preliminaries

2.1. Support function, radial function and polar of convex bodies

If $K \in \mathscr{K}^n$, then its support function, $h_K = h(K,\cdot): \mathbb{R}^n \to (-\infty,\infty)$, is defined by (see[4, 14])

$$h(K,x) = \max\{x \cdot y : y \in K\}, \ \ x \in \mathbb{R}^n, \tag{2.1}$$

where $x \cdot y$ denotes the standard inner product of x and y.

From the definition of the support function, we easily obtain for $c > 0$ and any $u \in S^{n-1}$

$$h(cK,u) = ch(K,u), \tag{2.2}$$

where $cK = \{cx : x \in K\}$.

If K is a compact star-shaped (about the origin) in $\mathbb{R}^n$, then its radial function, $\rho_K = \rho(K,\cdot): \mathbb{R}^n \setminus \{0\} \to [0,\infty)$, is defined by (see[4, 14])

$$\rho(K,u) = \max\{\lambda \geqslant 0 : \lambda \cdot u \in K\}, \ \ u \in S^{n-1}. \tag{2.3}$$

Given $c > 0$, we can get for any $u \in S^{n-1}$

$$\rho(cK,u) = c\rho(K,u). \tag{2.4}$$

If ρ_K is continuous and positive, then K will be called a star body. Two star bodies K, L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If $K \in \mathscr{K}_o^n$, the polar body , K^*, of K is defined by (see [4, 14])

$$K^* = \{x \in \mathbb{R}^n : x \cdot y \leqslant 1, y \in K\}. \tag{2.5}$$

From (2.5), we easily have $(K^*)^* = K$ and

$$h_{K^*} = \frac{1}{\rho_K}, \ \ \rho_{K^*} = \frac{1}{h_K}. \tag{2.6}$$

For $K \in \mathscr{K}_c^n$ and its polar body, the well-known Blaschke-santaló inequality (see [10]) can be stated that:

THEOREM 2.A. *If* $K \in \mathscr{K}_c^n$, *then*

$$V(K)V(K^*) \leqslant \omega_n^2, \tag{2.7}$$

with equality if and only if K *is an ellipsoid.*

2.2. Some L_p-combinations

1. Firey L_p-combination. For $K,L \in \mathscr{K}_o^n$, $p \geqslant 1$ and $\lambda,\mu \geqslant 0$ (not both zero), the Firey L_p-combination (also called the L_p-Minkowski combination), $\lambda \cdot K +_p \mu \cdot L \in \mathscr{K}_o^n$, of K and L is defined by (see [1, 4, 11])

$$h(\lambda \cdot K +_p \mu \cdot L, \cdot)^p = \lambda h(K,\cdot)^p + \mu h(L,\cdot)^p, \tag{2.8}$$

where the operation "$+_p$" is called Firey addition and $\lambda \cdot K$ denotes the Firey scalar multiplication. From (2.2) and (2.8), we can get

$$\lambda \cdot K = \lambda^{\frac{1}{p}} K.$$

For $p = 1$, Firey L_p-combination (2.8) is the Minkowski combination (see [4]).

2. L_p-radial combination. For $K,L \in S_o^n$, $p \geqslant 1$ and $\lambda,\mu \geqslant 0$ (not both zero), the L_p-radial combination, $\lambda \circ K \tilde{+}_p \mu \circ L \in S_o^n$, of K and L is defined by (see [4, 14])

$$\rho(\lambda \circ K \tilde{+}_p \mu \circ L, \cdot)^p = \lambda \rho(K,\cdot)^p + \mu \rho(L,\cdot)^p, \tag{2.9}$$

where the operation "$\tilde{+}_p$" is called L_p-radial addition and $\lambda \circ K$ denotes the L_p-radial scalar multiplication. From (2.4) and (2.9), we easily get

$$\lambda \circ K = \lambda^{\frac{1}{p}} K.$$

For $p = 1$, L_p-radial combination (2.9) is the classical radial combination (see [4]).

3. L_p-harmonic radial combination. For $K,L \in S_o^n$, $p \geqslant 1$ and $\lambda,\mu \geqslant 0$ (not both zero), the L_p-harmonic radial combination, $\lambda \star K +_{-p} \mu \star L \in S_o^n$, of K and L is defined by (see [2, 3, 9])

$$\rho(\lambda \star K +_{-p} \mu \star L, \cdot)^{-p} = \lambda \rho(K,\cdot)^{-p} + \mu \rho(L,\cdot)^{-p}, \tag{2.10}$$

where the operation "$+_{-p}$" is called L_p-harmonic radial addition and $\lambda \star K$ denotes the L_p-harmonic radial scalar multiplication. From (2.4) and (2.10), we can obtain

$$\lambda \star K = \lambda^{-\frac{1}{p}} K.$$

For $p = 1$, L_p-harmonic radial combination (2.10) is the classical harmonic radial combination (see [9]).

2.3. L_p-mixed volume

If $K,L \in \mathscr{K}_o^n$, then for $p \geqslant 1$ and $\varepsilon > 0$, the L_p-mixed volume, $V_p(K,L)$, of K and L is defined by (see [11])

$$\frac{n}{p} V_p(K,L) = \lim_{\varepsilon \to 0^+} \frac{V(K +_p \varepsilon \cdot L) - V(K)}{\varepsilon}.$$

Corresponding to each $K,L\in\mathscr{K}_o^n$, there is a positive Borel measure, $S_p(K,\cdot)$, on S^{n-1} such that (see [11])

$$V_p(K,L)=\frac{1}{n}\int_{S^{n-1}}h_L^p(u)dS_p(K,\cdot). \tag{2.11}$$

From (2.11), we have

$$V_p(K,K)=V(K). \tag{2.12}$$

The Minkowski inequality for the L_p-mixed volume is called L_p-Minkowski inequality. The L_p-Minkowski inequality was given by Lutwak (see [9, 11]):

THEOREM 2.B. *If $K,L\in\mathscr{K}_o^n$ and $p\geqslant 1$, then*

$$V_p(K,L)\geqslant V(K)^{\frac{n-p}{n}}V(L)^{\frac{p}{n}}, \tag{2.13}$$

with equality for $p=1$ if and only if K and L are homothetic, for $p>1$ if and only if K and L are dilates.

2.4. L_p-dual mixed volume

Associated with the L_p-harmonic radial combination of star bodies, Lutwak in [9] introduced the notion of L_p-dual mixed volume as follows: For $K,L\in S_o^n$, $p\geqslant 1$ and $\varepsilon>0$, the L_p-dual mixed volume, $\widetilde{V}_{-p}(K,L)$, of K and L is defined by (see [9])

$$\frac{n}{-p}\widetilde{V}_{-p}(K,L)=\lim_{\varepsilon\to 0^+}\frac{V(K\widetilde{+}_{-p}\,\varepsilon\star L)-V(K)}{\varepsilon}.$$

The definition above and Hospital's role give the following integral representation of L_p-dual mixed volume (see [9]):

$$\widetilde{V}_{-p}(K,L)=\frac{1}{n}\int_{S^{n-1}}\rho_K^{n+p}(u)\rho_L^{-p}(u)dS(u), \tag{2.14}$$

where the integration is with respect to spherical Lebesgue measure S on S^{n-1}.

From the formula (2.14), we get

$$\widetilde{V}_{-p}(K,K)=V(K)=\frac{1}{n}\int_{S^{n-1}}\rho_K^n(u)dS(u). \tag{2.15}$$

The Minkowski's inequality for the L_p-dual mixed volume can be stated that (see [9]):

THEOREM 2.C. *If $K,L\in S_o^n$, $p\geqslant 1$, then*

$$\widetilde{V}_{-p}(K,L)\geqslant V(K)^{\frac{n+p}{n}}V(L)^{-\frac{p}{n}}, \tag{2.16}$$

with equality if and only if K and L are dilates.

2.5. L_p-curvature image

For $K\in\mathscr{K}_o^n$, and real $p\geqslant 1$, the L_p-surface area measure, $S_p(K)$, of K is defined by (see [11])

$$\frac{dS_p(K,\cdot)}{dS(K,\cdot)}=h(K,\cdot)^{1-p}. \tag{2.17}$$

Equation (2.17) is also called Radon-Nikodym derivative, it turns out that the measure $S_p(K,\cdot)$ is absolutely continuous with respect to surface area measure $S(K,\cdot)$.

A convex body $K\in\mathscr{K}_o^n$ is said to have L_p-curvature function (see [9]), $f_p(K,\cdot): S^{n-1}\to\mathbb{R}$, if its L_p-surface area measure $S_p(K,\cdot)$ is absolutely continuous with respect to spherical Lebesgue measures, and

$$f_p(K,\cdot)=\frac{dS_p(K,\cdot)}{dS}.$$

Let $\mathscr{F}_o^n$, $\mathscr{F}_s^n$ denote the set of all bodies in $\mathscr{K}_o^n$, $\mathscr{K}_s^n$, respectively, that have a positive continuous curvature function.

Lutwak showed the notion of L_p-curvature image in [9] as follows: For each $K\in\mathscr{F}_o^n$ and $p\geqslant 1$, defined, $\Lambda_pK\in S_o^n$, L_p-curvature image of K by

$$\rho(\Lambda_pK,\cdot)^{n+p}=\frac{V(\Lambda_pK)}{\omega_n}f_p(K,\cdot). \tag{2.18}$$

Note that for $p=1$, this definition differs from the definition of classical curvature image (see [9]). For the studies of classical curvature image and L_p-curvature image, one may see [7, 8, 12–14, 17–20].

3. The proofs of theorems

In this section, we complete the proofs of Theorem 1.1–1.4.

Proof of Theorem 1.1. From the definition (1.2) and the inequality (2.16), we have

$$\begin{aligned}&V(K^*)^{\frac{n+p}{n}}\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}}\\ \leqslant\,& n^{\frac{n-p}{n}}\widetilde{V}_{-p}(K,Q^*)V(K^*)^{\frac{n+p}{n}}V(Q)^{-\frac{p}{n}}\\ \leqslant\,& n^{\frac{n-p}{n}}\widetilde{V}_{-p}(K,Q^*)\widetilde{V}_{-p}(K^*,Q).\end{aligned} \tag{3.1}$$

Since $K\in\mathscr{K}_c^n$, taking $Q=K^*$ in (3.1), then from (2.7), (2.15) and (3.1), we get

$$V(K^*)^{\frac{n+p}{n}}\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}}\leqslant n^{\frac{n-p}{n}}V(K)V(K^*)\leqslant n^{\frac{n-p}{n}}\omega_n^2.$$

This gets (1.3).

According to the equality condition of (2.7), we see that equality holds in (1.3) if and only if K is an ellipsoid. □

COROLLARY 3.1. *If $K \in \mathscr{K}_c^n$ and $1 \leqslant p < n$, then*

$$\widetilde{\Omega}_{-p}(K)\widetilde{\Omega}_{-p}(K^*) \leqslant n^2 \omega_n^{\frac{4n}{n-p}} [V(K)V(K^*)]^{-\frac{n+p}{n-p}}, \tag{3.2}$$

with equality if and only if K is an ellipsoid.

Proof. From (1.3) and (2.5), we have

$$\widetilde{\Omega}_{-p}(K)^{n-p} \leqslant n^{n-p} \omega_n^{2n} V(K^*)^{-(n+p)},$$

$$\widetilde{\Omega}_{-p}(K^*)^{n-p} \leqslant n^{n-p} \omega_n^{2n} V(K)^{-(n+p)}. \tag{3.3}$$

Combining with (1.3) and (3.3), we get (3.2).

According to the equality condition of (1.3), we see that equality holds in (3.2) if and only if K is an ellipsoid. □

LEMMA 3.1. [11] *If $K \in \mathscr{F}_o^n$, $p \geqslant 1$, then for any $Q \in S_o^n$,*

$$\widetilde{V}_{-p}(\Lambda_p K, Q^*) = \frac{V(\Lambda_p K)}{\omega_n} V_p(K,Q). \tag{3.4}$$

LEMMA 3.2. [9] *If $K \in \mathscr{F}_s^n$ and $p \geqslant 1$, then*

$$V(\Lambda_p K) \leqslant \omega_n^{\frac{2p-n}{p}} V(K)^{\frac{n-p}{p}}, \tag{3.5}$$

with equality if and only if K is an ellipsoid.

Proof of Theorem 1.2. From the definition (1.2), (2.13) and (3.4), we know

$$\begin{aligned}
\widetilde{\Omega}_{-p}(\Lambda_p K)^{\frac{n-p}{n}} &= n^{\frac{n-p}{n}} \inf\{\widetilde{V}_{-p}(\Lambda_p K, Q^*) V(Q)^{-\frac{p}{n}} : Q \in \mathscr{K}_c^n\} \\
&= n^{\frac{n-p}{n}} \inf\left\{ \frac{V(\Lambda_p K)}{\omega_n} V_p(K,Q) V(Q)^{-\frac{p}{n}} : Q \in \mathscr{K}_c^n \right\} \\
&\geqslant n^{\frac{n-p}{n}} \inf\left\{ \frac{V(\Lambda_p K)}{\omega_n} V(K)^{\frac{n-p}{n}} V(Q)^{\frac{p}{n}} V(Q)^{-\frac{p}{n}} : Q \in \mathscr{K}_c^n \right\} \\
&= n^{\frac{n-p}{n}} \frac{V(\Lambda_p K)}{\omega_n} V(K)^{\frac{n-p}{n}}.
\end{aligned} \tag{3.6}$$

Combining with (3.5) and (3.6), hence we have (1.4).

According to the equality condition of (3.5), we see that equality holds in (1.4) if and only if K is an ellipsoid. □

LEMMA 3.3. *For $K, L \in S_o^n$, $\lambda, \mu \geqslant 0$ (not both zero) and $p \geqslant 1$, if $q > n+p$, then for any $Q \in S_o^n$,*

$$\widetilde{V}_{-p}(\lambda \circ K \tilde{+}_q \mu \circ L, Q)^{\frac{q}{n+p}} \geqslant \lambda \widetilde{V}_{-p}(K,Q)^{\frac{q}{n+p}} + \mu \widetilde{V}_{-p}(L,Q)^{\frac{q}{n+p}}, \tag{3.7}$$

with equality if and only if K *and* L *are dilates.*

Proof. Since $q > n+p$, thus $\frac{n+p}{q} < 1$. Hence from (2.9), (2.14) and Minkowski's integral inequality (see [5]), we get

$$\begin{aligned}
&\widetilde{V}_{-p}(\lambda \circ K \tilde{+}_q \mu \circ L, Q)^{\frac{q}{n+p}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} \rho(\lambda \circ K \tilde{+}_q \mu \circ L, u)^{n+p} \rho(Q,u)^{-p} dS(u)\right]^{\frac{q}{n+p}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} [\rho(\lambda \circ K \tilde{+}_q \mu \circ L, u)^{q} \rho(Q,u)^{-\frac{pq}{n+p}}]^{\frac{n+p}{q}} dS(u)\right]^{\frac{q}{n+p}} \\
&= \left[\frac{1}{n}\int_{S^{n-1}} [(\lambda\rho(K,u)^q + \mu\rho(L,u)^q)\rho(Q,u)^{-\frac{pq}{n+p}}]^{\frac{n+p}{q}} dS(u)\right]^{\frac{q}{n+p}} \\
&\geqslant \lambda\left[\frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n+p}\rho(Q,u)^{-p} dS(u)\right]^{\frac{q}{n+p}} + \mu\left[\frac{1}{n}\int_{S^{n-1}} \rho(L,u)^{n+p}\rho(Q,u)^{-p} dS(u)\right]^{\frac{q}{n+p}} \\
&= \lambda \widetilde{V}_{-p}(K,Q)^{\frac{q}{n+p}} + \mu \widetilde{V}_{-p}(L,Q)^{\frac{q}{n+p}}.
\end{aligned}$$

This yield (3.7).

According to the equality condition of Minkowski's integral inequality, we see that equality holds in (3.7) if and only if K and L are dilates. □

Proof of Theorem 1.3. From (1.2) and (3.7), we have

$$\begin{aligned}
&[n^{\frac{p}{n}} \widetilde{\Omega}_{-p}(\lambda \circ K \tilde{+}_q \mu \circ L)^{\frac{n-p}{n}}]^{\frac{q}{n+p}} \\
&= \inf\{[n\widetilde{V}_{-p}(\lambda \circ K \tilde{+}_q \mu \circ L, Q^*)V(Q)^{-\frac{p}{n}}]^{\frac{q}{n+p}} : Q \in \mathscr{K}_c^n\} \\
&= \inf\{[n\widetilde{V}_{-p}(\lambda \circ K \tilde{+}_q \mu \circ L, Q^*)]^{\frac{q}{n+p}} [V(Q)^{-\frac{p}{n}}]^{\frac{q}{n+p}} : Q \in \mathscr{K}_c^n\} \\
&\geqslant \inf\{[\lambda(n\widetilde{V}_{-p}(K,Q^*))^{\frac{q}{n+p}} + \mu(n\widetilde{V}_{-p}(L,Q^*))^{\frac{q}{n+p}}][V(Q)^{-\frac{p}{n}}]^{\frac{q}{n+p}} : Q \in \mathscr{K}_c^n\} \\
&\geqslant \inf\{\lambda[n\widetilde{V}_{-p}(K,Q^*)V(Q)^{-\frac{p}{n}}]^{\frac{q}{n+p}} : Q \in \mathscr{K}_c^n\} \\
&\quad + \inf\{\mu[n\widetilde{V}_{-p}(L,Q^*)V(Q)^{-\frac{p}{n}}]^{\frac{q}{n+p}} : Q \in \mathscr{K}_c^n\} \\
&= \lambda[n^{\frac{p}{n}}\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}}]^{\frac{q}{n+p}} + \mu[n^{\frac{p}{n}}\widetilde{\Omega}_{-p}(L)^{\frac{n-p}{n}}]^{\frac{q}{n+p}}.
\end{aligned}$$

So (1.5) is obtained.

According to the equality condition of (3.7), we see that equality holds in (1.5) if and only if K and L are dilates. □

Using the proof method of Lemma 3.3 and combining with L_q-harmonic radial combination (2.10), we easily obtain the following result for the L_p-dual mixed volume.

LEMMA 3.4. *If* $K, L \in S_o^n$, $p \geqslant 1$, $q \geqslant 1$ *and* $\lambda, \mu \geqslant 0$ *(not both zero), then for any* $Q \in S_o^n$,

$$\widetilde{V}_{-p}(\lambda \star K +_{-q} \mu \star L, Q)^{-\frac{q}{n+p}} \geqslant \lambda \widetilde{V}_{-p}(K,Q)^{-\frac{q}{n+p}} + \mu \widetilde{V}_{-p}(L,Q)^{-\frac{q}{n+p}}, \tag{3.8}$$

with equality if and only if K and L are dilates.

If $p=q$, then (3.8) can be found in [18].

Proof of Theorem 1.4. From (1.2) and (3.8), we have

$$\begin{aligned}
&[n^{\frac{p}{n}}\widetilde{\Omega}_{-p}(\lambda\star K+_{-q}\mu\star L)^{\frac{n-p}{n}}]^{-\frac{q}{n+p}}\\
=&\inf\{[n\widetilde{V}_{-p}(\lambda\star K+_{-q}\mu\star L,Q^*)V(Q)^{-\frac{p}{n}}]^{-\frac{q}{n+p}}:Q\in\mathscr{K}_c^n\}\\
=&\inf\{[n\widetilde{V}_{-p}(\lambda\star K+_{-q}\mu\star L,Q^*)]^{-\frac{q}{n+p}}[V(Q)^{-\frac{p}{n}}]^{-\frac{q}{n+p}}:Q\in\mathscr{K}_c^n\}\\
\geqslant&\inf\{[\lambda(n\widetilde{V}_{-p}(K,Q^*))^{-\frac{q}{n+p}}+\mu(n\widetilde{V}_{-p}(L,Q^*))^{-\frac{q}{n+p}}][V(Q)^{-\frac{p}{n}}]^{-\frac{q}{n+p}}:Q\in\mathscr{K}_c^n\}\\
\geqslant&\inf\{\lambda[n\widetilde{V}_{-p}(K,Q^*)V(Q)^{-\frac{p}{n}}]^{-\frac{q}{n+p}}:Q\in\mathscr{K}_c^n\}\\
&+\inf\{\mu[n\widetilde{V}_{-p}(L,Q^*)V(Q)^{-\frac{p}{n}}]^{-\frac{q}{n+p}}:Q\in\mathscr{K}_c^n\}\\
=&\lambda[n^{\frac{p}{n}}\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}}]^{-\frac{q}{n+p}}+\mu[n^{\frac{p}{n}}\widetilde{\Omega}_{-p}(L)^{\frac{n-p}{n}}]^{-\frac{q}{n+p}}.
\end{aligned}$$

So (1.6) is obtained.

According to the equality condition of (3.8), we see that the equality holds in (1.6) if and only if K and L are dilates. □

Finally, we give a concept called the L_p-harmonic Blaschke combination about star bodies and establish a Brunn-Minkowski type inequality for the L_p-dual affine surface area.

For $K,L\in S_o^n$, $p\geqslant 1$ and $\lambda,\mu\geqslant 0$ (not both zero), the L_p-harmonic Blaschke combination, $\lambda*K\hat{+}_p\mu*L$, of K and L is defined by

$$\frac{\rho(\lambda*K\hat{+}_p\mu*L,\cdot)^{n+p}}{V(\lambda*K\hat{+}_p\mu*L)}=\lambda\frac{\rho(K,\cdot)^{n+p}}{V(K)}+\mu\frac{\rho(L,\cdot)^{n+p}}{V(L)}. \tag{3.9}$$

Taking $\lambda=\mu=1$ in $\lambda*K\hat{+}_p\mu*L$, then $K\hat{+}_pL$ is just L_p-harmonic Blaschke addition introduced in [24]. Associated with (3.9), we obtain the following fact.

THEOREM 3.1. *If $K,L\in\mathscr{K}_c^n$, $\lambda,\mu\geqslant 0$ (not both zero) and $1\leqslant p<n$, then*

$$\frac{\widetilde{\Omega}_{-p}(\lambda*K\hat{+}_p\mu*L)^{\frac{n-p}{n}}}{V(\lambda*K\hat{+}_p\mu*L)}\geqslant\lambda\frac{\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}}}{V(K)}+\mu\frac{\widetilde{\Omega}_{-p}(L)^{\frac{n-p}{n}}}{V(L)}, \tag{3.10}$$

with equality if and only if K and L are dilates.

Proof. From the definition (1.2) and (3.9), we get

$$\begin{aligned}
&n^{\frac{p}{n}}\widetilde{\Omega}_{-p}(\lambda*K\hat{+}_p\mu*L)^{\frac{n-p}{n}}\\
=&\inf\{n\widetilde{V}_{-p}(\lambda*K\hat{+}_p\mu*L,Q^*)V(Q)^{-\frac{p}{n}}:Q\in\mathscr{K}_c^n\}\\
=&\inf\left\{\int_{S^{n-1}}\rho(\lambda*K\hat{+}_p\mu*L,u)^{n+p}\rho(Q^*,u)^{-p}dS(u)V(Q)^{-\frac{p}{n}}:Q\in\mathscr{K}_c^n\right\}.
\end{aligned}$$

Thus

$$
\begin{aligned}
&\frac{n^{\frac{p}{n}}\widetilde{\Omega}_{-p}(\lambda*K\hat{+}_p\mu*L)^{\frac{n-p}{n}}}{V(\lambda*K\hat{+}_p\mu*L)}\\
&=\inf\left\{\int_{S^{n-1}}\frac{\rho(\lambda*K\hat{+}_p\mu*L,u)^{n+p}}{V(\lambda*K\hat{+}_p\mu*L)}\rho(Q^*,u)^{-p}dS(u)V(Q)^{-\frac{p}{n}}:Q\in\mathscr{K}_c^n\right\}\\
&=\inf\left\{\int_{S^{n-1}}[\lambda\frac{\rho(K,u)^{n+p}}{V(K)}+\mu\frac{\rho(L,u)^{n+p}}{V(L)}]\rho(Q^*,u)^{-p}dS(u)V(Q)^{-\frac{p}{n}}:Q\in\mathscr{K}_c^n\right\}\\
&=\inf\left\{\frac{\lambda}{V(K)}\int_{S^{n-1}}\rho(K,u)^{n+p}\rho(Q^*,u)^{-p}dS(u)V(Q)^{-\frac{p}{n}}\right.\\
&\quad\left.+\frac{\mu}{V(L)}\int_{S^{n-1}}\rho(L,u)^{n+p}\rho(Q^*,u)^{-p}dS(u)V(Q)^{-\frac{p}{n}}:Q\in\mathscr{K}_c^n\right\}\\
&\geqslant\frac{\lambda}{V(K)}\inf\{n\widetilde{V}_{-p}(K,Q^*)V(Q)^{-\frac{p}{n}}:Q\in\mathscr{K}_c^n\}\\
&\quad+\frac{\mu}{V(L)}\inf\{n\widetilde{V}_{-p}(L,Q^*)V(Q)^{-\frac{p}{n}}:Q\in\mathscr{K}_c^n\}.
\end{aligned}
$$

This give (3.10).

The equality of (3.10) holds if and only if $\lambda*K\hat{+}_p\mu*L$ are dilates with K and L, respectively. This mean that the equality holds if and only if K and L are dilates. □

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