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SOME COMPLETELY MONOTONIC FUNCTIONS INVOLVING THE q–GAMMA FUNCTION

PENG GAO

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Abstract. We present some completely monotonic functions involving the q-gamma function that are inspired by their analogues involving the gamma function.

1. Introduction

The q-gamma function is defined for a complex number z and $q \neq 1$ by

$$\Gamma_q(z) = \begin{cases} \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, & 0<q<1; \\ \frac{(q^{-1};q^{-1})_\infty}{(q^{-z};q^{-1})_\infty}(q-1)^{1-z}q^{\frac{1}{2}z(z-1)}, & q>1. \end{cases} \tag{1.1}$$

where the product $(a;q)_\infty$ is defined by

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n).$$

In what follows we restrict our attention to positive real numbers x. We note here [17] the limit of $\Gamma_q(x)$ as $q \to 1^-$ gives back the well-known Euler's gamma function:

$$\lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x) = \int_0^\infty t^x e^{-t}\frac{dt}{t}.$$

It follows from this and (1.1) that $\lim_{q\to 1}\Gamma_q(x) = \Gamma(x)$. For historical remarks on gamma and q-gamma functions, we refer the reader to [17], [2] and [3].

There exists an extensive and rich literature on inequalities for the gamma and q-gamma functions of positive real numbers. For the recent developments in this area, we refer the reader to the articles [14], [2]–[4], [20] and the references therein. Many of these inequalities follow from the monotonicity properties of functions which are closely related to Γ (resp. Γ_q) and its logarithmic derivative ψ (resp. ψ_q) as ψ' and ψ'_q are completely monotonic functions on $(0,+\infty)$ (see [15], [4]). Here we recall that a function $f(x)$ is said to be completely monotonic on (a,b) if it has derivatives of all

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orders and $(-1)^k f^{(k)}(x) \geqslant 0$, $x \in (a,b)$, $k \geqslant 0$. We further note that Lemma 2.1 of [7] asserts that $f(x) = e^{-h(x)}$ is completely monotonic on an interval if h' is. Following [13], we call such functions $f(x)$ logarithmically completely monotonic.

We note here that $\lim_{q \to 1} \psi_q(x) = \psi(x)$ (see [18]), hence in what follows we also write $\Gamma_1(x)$ for $\Gamma(x)$ and $\psi_1(x)$ for $\psi_q(x)$. Thus we may also regard the gamma function as a q-gamma function with $q=1$ and in this manner, many completely monotonic functions involving $\Gamma_q(x)$ and $\psi_q(x)$ are inspired by their analogues involving $\Gamma(x)$ and $\psi(x)$. It is our goal in this paper to present some completely monotonic functions involving Γ_q, ψ_q that are motivated by this point of view. In the remaining part of this introduction, we briefly mention the motivations for our results in the paper.

In [16], Kershaw proved the $q=1$ case of the following result for $0 < s < 1$, $x > 0$:

$$e^{(1-s)\psi_q(x+s^{1/2})} < \frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} < e^{(1-s)\psi_q(x+(s+1)/2)}. \tag{1.2}$$

A result of Ismail and Muldoon [14] establishes the second inequality in (1.2) for $0 < q < 1$. In [7], Bustoz and Ismail showed that when $q=1$, the function $(0 < s < 1)$

$$x \mapsto \frac{\Gamma_q(x+s)}{\Gamma_q(x+1)} e^{(1-s)\psi_q(x+(s+1)/2)}$$

is completely monotonic on $(0, +\infty)$. In [12], it is shown that the result of Bustoz and Ismail also holds for any $q > 0$.

In [3], Alzer asked to determine the best possible values of $a(q,s)$ and $b(q,s)$ such that the following inequalities hold for all $x > 0$, $0 < q \neq 1$, $0 < s < 1$:

$$e^{(1-s)\psi_q(x+a(q,s))} < \frac{\Gamma_q(x+1)}{\Gamma_q(x+s)} < e^{(1-s)\psi_q(x+b(q,s))}. \tag{1.3}$$

We shall determine the best possible values of $a(q,s)$ and $b(q,s)$ in Section 3. Another result given in Section 3 is motivated by the following result of Alzer and Batir [5], who showed that the function $(x > 0,\ c \geqslant 0)$

$$G_c(x) = \ln\Gamma(x) - x\ln x + x - \frac{1}{2}\ln(2\pi) + \frac{1}{2}\psi(x+c)$$

is completely monotonic if and only if $c \geqslant 1/3$ and $-G_c(x)$ is completely monotonic if and only if $c = 0$. We shall present a q-analogue in Section 3 for $G'_c(x)$.

Muldoon [19] studied the monotonicity property of the function

$$h_\alpha(x) = x^\alpha \Gamma(x)(e/x)^x.$$

He showed that $h_\alpha(x)$ is logarithmically completely monotonic on $(0, +\infty)$ for $\alpha \leqslant 1/2$. We point out here that as was shown in [6, Theorem 3.3], $1/2$ is the largest possible number to make the assertion hold for $h_\alpha(x)$. In [12, Proposition 3.7], it is shown that if one defines for $\alpha \geqslant 0$,

$$f_\alpha(x) = -\ln\Gamma(x) + (x - \frac{1}{2})\ln x - x + \frac{1}{12}\psi'(x+\alpha), \tag{1.4}$$

then $f'_\alpha(x)$ is completely monotonic on $(0,+\infty)$ if $\alpha \geqslant 1/2$ and $-f'_\alpha(x)$ is completely monotonic on $(0,+\infty)$ if $\alpha = 0$. As was pointed out in [12], this implies a result of Alzer [1, Theorem 1]. In Section 3, we shall establish a q-analogue of the above result.

It is shown in the proof of Theorem 2.2 in [8] that for $x > 0$ and $0 < q < 1$,

$$\psi'_q(x+1) < \frac{\ln(1/q)q^x}{1-q^x}. \tag{1.5}$$

The $q=1$ analogue of inequality (1.5) is $\psi'(x+1) \leqslant 1/x$, which reminds us the following asymptotic expansion [4, (1.5)] for the derivatives of $\psi(x)$:

$$(-1)^{n+1}\psi^{(n)}(x) = \frac{(n-1)!}{x^n} + \frac{n!}{2x^{n+1}} + O\left(\frac{1}{x^{n+2}}\right), \quad n \geqslant 1, \quad x \to +\infty. \tag{1.6}$$

We note that Lemma 2.2 of [11] asserts that for fixed $n \geqslant 1$, $a \geqslant 0$, the function $f_{a,n}(x) = x^n(-1)^{n+1}\psi^{(n)}(x+a)$ is increasing on $[0,+\infty)$ if and only if $a \geqslant 1/2$. It follows from this and (1.6) that we have $\psi'(x+1/2) \leqslant 1/x$ and this suggests that inequality (1.5) would still hold if one replaces $\psi'_q(x+1)$ with $\psi'_q(x+1/2)$. We shall show this is indeed the case in Section 3.

2. Lemmas

The following lemma gathers a few results on Γ_q and ψ_q. Equality (2.1) below is given in [3, (2.7)] and the rest can be easily derived from (1.1) and (2.1).

LEMMA 2.1. *For $0 < q < 1$, $x > 0$,*

$$\psi_q(x) = -\ln(1-q) + \ln q \sum_{n=1}^{\infty} \frac{q^{nx}}{1-q^n}, \tag{2.1}$$

$$\ln\Gamma_q(x+1) = \ln\Gamma_q(x) + \ln\frac{1-q^x}{1-q}, \tag{2.2}$$

$$\psi_q(x+1) = \psi_q(x) - \frac{(\ln q)q^x}{1-q^x}, \tag{2.3}$$

$$\psi'_q(x+1) = \psi'_q(x) - \frac{(\ln q)^2 q^x}{(1-q^x)^2}. \tag{2.4}$$

Our next lemma is a result in [21]:

LEMMA 2.2. *For positive numbers $x \neq y$ and real number r, we define*

$$E(r,0;x,y) = \left(\frac{1}{r}\cdot\frac{x^r-y^r}{\ln x - \ln y}\right)^{1/r}, \quad r \neq 0; \quad E(0,0;x,y) = \sqrt{xy}.$$

Then the function $r \mapsto E(r,0;x,y)$ is strictly increasing on $\mathbb{R}$.

LEMMA 2.3. *Let $0<q<1$, then for any integer $n \geqslant 1$,*

$$\frac{\ln q}{1-q^n}+\frac{1}{n}-\frac{\ln q}{2}-\frac{n^2(\ln q)^3q^{n/2}}{12(1-q^n)}<0, \tag{2.5}$$

$$\frac{\ln q}{1-q^n}+\frac{1}{n}-\frac{\ln q}{2}-\frac{n^2(\ln q)^3}{12(1-q^n)}>0. \tag{2.6}$$

Proof. On setting $\ln q^n = x$, one checks that inequality (2.5) follows from $f(x)<0$ for $x<0$, where

$$f(x)=6x(1+e^x)+12(1-e^x)-x^3e^{x/2}.$$

Now, $f''(x)=6xe^{x/2}h(x)$ with $h(x)=e^{x/2}-1-x/2-x^2/24$. Note that $h(0)=0$ and by the Taylor expansion, we see that $h(x)>0$ for x sufficiently close to 0. As $\lim_{x\to-\infty}h(x)<0$, it follows that $h(x)=0$ has an odd number of solutions on $(-\infty,0)$. If this number of solutions is at least three, then by Rolle's theorem (note that $h(0)=0$), $h''(x)=e^{x/2}/4-1/12=0$ has at least two solutions on $(-\infty,0)$. As this is not the case, it follows that there is a unique solution $x_0\in(-\infty,0)$ of the equation $e^{x/2}-1-x/2-x^2/24=0$, so that $f''(x)>0$ for $x<x_0$ and $f''(x)<0$ for $x_0<x<0$. As $\lim_{x\to-\infty}f'(x)>0$ and $f'(0)=0$, we deduce that $f'(x)>0$ for $x<0$. It follows from this and $f(0)=0$ that $f(x)<0$ for $x<0$.

Similarly, inequality (2.6) follows from $g(x)>0$ for $x<0$, where

$$g(x)=6x(1+e^x)+12(1-e^x)-x^3.$$

As $g''(x)=6x(e^x-1)>0$ for $x<0$ and $g'(0)=0$, we see that $g'(x)<0$ for $x<0$ and it follows from this and $g(0)=0$ that $g(x)>0$ for $x<0$ and this completes the proof. □

3. Main Results

We first determine the best possible value for $a(q,s)$ in (1.3). For this, for any $q>0$, $t>s>0$, we denote $I_{\psi_q}(s,t)$ as the integral ψ_q mean of s and t:

$$I_{\psi_q}(s,t)=\psi_q^{-1}\left(\frac{1}{t-s}\int_s^t\psi_q(u)du\right). \tag{3.1}$$

Then we have the following result:

THEOREM 3.1. *For every $q>0$, $x>0$, $t>s>0$, we have*

$$\psi_q\left(x+I_{\psi_q}(s,t)\right)<\frac{1}{t-s}\int_s^t\psi_q(x+u)du,$$

where the constant $I_{\psi_q}(s,t)$ is best possible.

Proof. We note that the case $q = 1$ of the assertion of the theorem is already established in [9, Thereom 4]. The general case can be established similarly, on noting that the function

$$x \mapsto I_{\psi_q}(x+s, x+t) - x$$

is increasing by Theorem 4 of [10], in view that ψ_q' is completely monotonic on $(0, +\infty)$. On considering the case $x \to 0^+$, we see immediately that the constant $I_{\psi_q}(s,t)$ is best possible and this completes the proof. □

On setting $t = 1$ in Theorem 3.1, we readily deduce the following result concerning the best possible value $a(q,s)$ in (1.3):

COROLLARY 3.1. *Let $q > 0$ and $0 < s < 1$. The first inequality of (1.3) holds for all $x > 0$ with the best possible value $a(q,s) = I_{\psi_q}(s,1)$, where I_{ψ_q} is defined as in (3.1).*

Now to determine the best possible value for $b(q,s)$ in (1.3), we note that it is easy to see on considering the case $x \to +\infty$ that the best possible value for $b(q,s)$ is $(1+s)/2$ when $q > 1$. When $0 < q < 1$, we have the following result:

THEOREM 3.2. *Let $0 < q < 1$ and $0 < s < 1$. Let*

$$b(q,s) = \frac{\ln \frac{q^s - q}{(s-1)\ln q}}{\ln q}.$$

For $x > 0$, let

$$f_{q,s,c}(x) = \ln\Gamma_q(x+1) - \ln\Gamma_q(x+s) - (1-s)\psi_q(x+c),$$

where $c > 0$. Then $-f_{q,s,c}(x)$ is completely monotonic on $(0, +\infty)$ if and only if $c \geqslant b(q,s)$.

Proof. We have, using (2.1), that

$$\begin{aligned} f'_{q,s,b(q,s)}(x) &= \psi_q(x+1) - \psi_q(x+s) - (1-s)\psi_q'(x+b(q,s)) \\ &= \ln q \sum_{n=1}^{\infty} \frac{q^{nx}}{1-q^n}\left(q^n - q^{ns} - (1-s)(\ln q^n)q^{nb(q,s)}\right). \end{aligned}$$

We want to show $q^n - q^{ns} - (1-s)(\ln q^n)q^{nb(q,s)} \leqslant 0$, which is equivalent to $E^{s-1}(n(s-1), 0; q, 1) \geqslant q^{b(q,s)-1}$, where E is defined as in Lemma 2.2. It also follows from Lemma 2.2 that $E^{s-1}(n(s-1), 0; q, 1) \geqslant E^{s-1}(s-1, 0; q, 1) = q^{b(q,s)-1}$. We then deduce that $f'_{q,s,c}(x)$ is completely monotonic on $(0, +\infty)$ when $c \geqslant b(q,s)$. This together with the observation that $\lim_{x\to+\infty} f_{q,s,c}(x) = 0$ implies the "if" part of the assertion of the theorem.

To show the "only if" part of the assertion of the theorem, we use (2.2) and (2.3) to deduce that

$$f_{q,s,c}(x+1)-f_{q,s,c}(x)=\ln\frac{1-q^{x+1}}{1-q^{x+s}}+(1-s)\ln q\frac{q^{x+c}}{1-q^{x+c}}.$$

If we set $z=q^x$ and consider the Taylor expansion of the above expression at $z=0$, then the first order term is:

$$(q^s-q+(1-s)(\ln q)\,q^c)\,z.$$

Note that the expression in the parenthesis above is <0 if $c<b(q,s)$ as it is 0 when $c=b(q,s)$. This implies that $f_{q,s,c}(x+1)<f_{q,s,c}(x)$ when x is large enough and this shows that $-f_{q,s,c}(x)$ can't be completely monotonic on $(0,+\infty)$ when $c<b(q,s)$ and this completes the proof of the "only if" part of the assertion of the theorem. □

Theorem 3.2 now allows us to determine the best possible value of $b(q,s)$ in (1.3) when $0<q<1$ in the following:

COROLLARY 3.2. *Let $0<q<1$ and $0<s<1$. The inequality*

$$\frac{\Gamma_q(x+1)}{\Gamma_q(x+s)}<e^{(1-s)\psi_q(x+b(q,s))} \tag{3.2}$$

holds for all $x>0$ with the best possible value $b(q,s)$ given as in the statement of Theorem 3.2.

Proof. Using the same notions in the proof of Theorem 3.2, we see from the proof of Theorem 3.2 that $f'_{q,s,b(q,s)}(x)>0$ for $x>0$, which implies the strict inequality in (3.2). To show $b(q,s)$ is best possible, we note that in the proof of Theorem 3.2, we've shown that $f_{q,s,c}(x+1)-f_{q,s,c}(x)<0$ for x large enough if $c<b(q,s)$. It follows that $f_{q,s,c}(x+k)-f_{q,s,c}(x)<0$ for any positive integer k when x is large enough and $c<b(q,s)$. On letting $k\to+\infty$, we see immediately that this implies that $-f_{q,s,c}(x)<0$, so that inequality (3.2) fails to hold with $b(q,s)$ being replaced by any $c<b(q,s)$ and this completes the proof. □

We note here that Corollary 3.2 refines a result of Ismail and Muldoon in [14], mentioned in the introduction of this paper, where $b(q,s)$ is replaced by $(1+s)/2$ in (3.2). One can also check directly that $b(q,s)\leqslant(1+s)/2$, as it follows from $E(s-1,0;q,1)\leqslant E(0,0;q,1)$. Moreover, it is easy to see that when $q\to1^-$, $b(q,s)\to(1+s)/2$ and in this case (3.2) gives back the second inequality in (1.2) for $q=1$.

Our next result is a q-analogue of the result of Alzer and and Batir [5] mentioned in Section 1.

THEOREM 3.3. *Let $0<q<1$ be fixed. Let $c\geqslant0$. Let $a_q=(q-1-\ln q)/(\ln q)^2$. The function*

$$g_{q,c}(x)=\psi_q(x)-\ln\frac{1-q^x}{1-q}+a_q\psi'_q(x+c)$$

is completely monotonic on $(0,+\infty)$ if and only if $c=0$.

Proof. We have, using (2.1), that

$$g_{q,c}(x)=\ln q\sum_{n=1}^{\infty}\frac{q^{nx}}{1-q^n}\left(1+\frac{1-q^n}{n\ln q}+a_q(\ln q^n)q^{nc}\right).$$

On setting $t=-\ln q^n$, we have $t\geqslant -\ln q$ and the expression in the parenthesis above when $c=0$ can be rewritten as

$$1-\frac{1-e^{-t}}{t}-a_qt=\frac{1}{t}(-1+t+e^{-t}-a_qt^2):=h_q(t)/t.$$

It suffices to show $h_q(t)\leqslant 0$ for $t\geqslant -\ln q$. For this, note that $h_q(-\ln q)=0$ and that

$$h_q'(t)=1-2a_qt-e^{-t},\quad h_q''(t)=-2a_q+e^{-t}.$$

We have

$$\frac{(\ln q)^2h_q''(-\ln q)}{2}=\frac{q(\ln q)^2}{2}+\ln q+1-q:=d(q).$$

As $d(1)=d'(1)=0$ and $d''(q)=q^{-1}\ln q+q^{-2}(q-1)<0$ for $0<q<1$, it follows that $d(q)<0$ for $0<q<1$. As $h_q^{(3)}(t)<0$ for $t\geqslant -\ln q$, we conclude that $h_q''(t)<0$ for $t\geqslant -\ln q$. Similarly, one shows that $h_q'(-\ln q)<0$ to deduce that $h_q'(t)<0$ for $t\geqslant -\ln q$ and this implies $h_q(t)\leqslant 0$ for $t\geqslant -\ln q$, which completes the proof of the "if" part of the assertion of the theorem.

For the "only if" part of the assertion of the theorem, note that we have by (2.3) and (2.4),

$$g_{q,c}(x+1)-g_{q,c}(x)=-\frac{(\ln q)q^x}{1-q^x}-\ln\frac{1-q^{x+1}}{1-q^x}-a_q\frac{(\ln q)^2q^{x+c}}{(1-q^{x+c})^2}.$$

If we set $z=q^x$ and consider the Taylor expansion of the above expression at $z=0$, then the first order term is:

$$(-\ln q+q-1-a_q(\ln q)^2q^c)z=(h_q(-\ln q)+a_q(\ln q)^2-a_q(\ln q)^2q^c)t>0,$$

if $c>0$. This implies that $g_{q,c}(x+1)>g_{q,c}(x)$ when x is large enough and this shows that $g_{q,c}(x)$ can not be completely monotonic on $(0,+\infty)$ when $c>0$ and this completes the proof of the "only if" part of the assertion of the theorem. □

Similar to Theorem 3.3, one can prove the following result, whose proof we leave to the reader.

THEOREM 3.4. *Let $0<q<1$ be fixed. Let $c\geqslant 0$. The function*

$$x\mapsto \psi_q(x)-\ln\frac{1-q^x}{1-q}+\frac{1}{2}\psi_q'(x+c)$$

is completely monotonic on $(0,+\infty)$ if $c=0$ and its negative is completely monotonic on $(0,+\infty)$ if $c\geqslant 1/3$.

Related to the function given in (1.4), we have the following q-analogue:

THEOREM 3.5. *Let* $0<q<1$ *be fixed, the functions*

$$-\psi_q(x)+\ln\Big(\frac{1-q^x}{1-q}\Big)+\frac{(\ln q)q^x}{2(1-q^x)}+\frac{1}{12}\psi_q''(x+1/2), \tag{3.3}$$

$$\psi_q(x)-\ln\Big(\frac{1-q^x}{1-q}\Big)-\frac{(\ln q)q^x}{2(1-q^x)}-\frac{1}{12}\psi_q''(x) \tag{3.4}$$

are completely monotonic on $(0,+\infty)$.

Proof. The function given in (3.3) being completely monotonic on $(0,+\infty)$ follows from (2.5) and (2.1). As by (2.1), we have

$$\psi_q(x)-\ln\Big(\frac{1-q^x}{1-q}\Big)-\frac{(\ln q)q^x}{2(1-q^x)}-\frac{1}{12}\psi_q''(x+1/2)$$
$$=\sum_{n=1}^{\infty}\Big(\frac{\ln q}{1-q^n}+\frac{1}{n}-\frac{\ln q}{2}-\frac{n^2(\ln q)^3q^{n/2}}{12(1-q^n)}\Big)q^{nx}.$$

Similarly, the function given in (3.4) being completely monotonic on $(0,+\infty)$ follows from (2.6) and (2.1). □

Our next result is motivated by (1.5) and (1.6):

THEOREM 3.6. *Let* $0<q<1$ *be fixed, the functions*

$$\psi_q'(x)-\frac{(\ln q)^2q^x}{(1-q)(1-q^x)}-\frac{(\ln q)^2q^{2x}}{(1+q)(1-q^x)^2}, \tag{3.5}$$

$$-\psi_q'(x+1/2)+\frac{(\ln q)^2q^{x+1/2}}{(1-q)(1-q^x)} \tag{3.6}$$

are completely monotonic on $(0,+\infty)$.

Proof. To show the function given in (3.5) is completely monotonic on $(0,+\infty)$, we note that

$$\frac{q^x}{(1-q^x)^2}=\sum_{n=1}^{\infty}nq^{nx}.$$

Using this and (2.1), we can recast (3.5) as

$$\psi_q'(x)-\frac{(\ln q)^2q^x}{(1-q)(1-q^x)}-\frac{(\ln q)^2q^{2x}}{(1+q)(1-q^x)^2}$$
$$=(\ln q)^2\Big(\sum_{n=1}^{\infty}\frac{nq^{nx}}{1-q^n}-\frac{1}{1-q}\sum_{n=1}^{\infty}q^{nx}-\frac{q^x}{1+q}\sum_{n=1}^{\infty}nq^{nx}\Big)$$
$$=(\ln q)^2\sum_{n=2}^{\infty}\Big(\frac{n}{1-q^n}-\frac{1}{1-q}-\frac{n-1}{1+q}\Big)q^{nx}$$
$$=(\ln q)^2\sum_{n=2}^{\infty}\Big(\frac{u_n(q)}{(1-q^n)(1-q)(1+q)}\Big)q^{nx},$$

where

$$u_n(q) = n(1-q^2) - (1+q)(1-q^n) - (n-1)(1-q)(1-q^n)$$
$$= n(1-q)(q+q^n) - 2q(1-q^n).$$

It suffices to show that $u_n(q) \geqslant 0$ for $n \geqslant 2$, $0 < q < 1$, or equivalently,

$$n(1+q^{n-1}) \geqslant 2\frac{1-q^n}{1-q} = 2\sum_{i=0}^{n-1} q^i.$$

Let $v(q) = 2\sum_{i=0}^{n-1} q^i - nq^{n-1}$. Then

$$v'(q) = q^{n-2}(2\sum_{i=0}^{n-1} iq^{i-n+1} - n(n-1)) \geqslant q^{n-2}(2\sum_{i=0}^{n-1} i - n(n-1)) = 0$$

when $0 < q \leqslant 1$. Thus $v(q)$ is an increasing function of q for $0 < q \leqslant 1$. As $v(1) = n$, we see that this implies $u_n(q) \geqslant 0$ for $n \geqslant 2$, $0 < q < 1$ and this establishes our assertion on the function given in (3.5).

To show the function given in (3.6) is completely monotonic on $(0, +\infty)$, we use (2.1) to get

$$\psi_q'(x+1/2) - \frac{(\ln q)^2 q^{x+1/2}}{(1-q)(1-q^x)} = (\ln q)^2 \sum_{n=1}^{\infty} \left(\frac{nq^{n/2}}{1-q^n} - \frac{q^{1/2}}{1-q}\right) q^{nx}.$$

It suffices to show that $nq^{n/2-1/2} \leqslant (1-q^n)/(1-q) = \sum_{i=0}^{n-1} q^i$ for $0 < q < 1$. This follows by noting that $2\sum_{i=0}^{n-1} q^i = \sum_{i=0}^{n-1}(q^i + q^{n-i-1})$ and that $q^i + q^{n-i-1} \geqslant 2q^{n/2-1/2}$ by the arithmetic-geometric inequality and this completes the proof. □

COROLLARY 3.3. *Let $0 < q < 1$ be fixed, then for $x > 0$, we have*

$$\psi_q'(x+1/2) \leqslant \frac{(\ln q)^2 q^{x+1/2}}{(1-q)(1-q^x)}.$$

The above inequality follows readily from Theorem 3.6 on considering the value of the function given in (3.6) as $x \to +\infty$. As it is easy to see that $-(\ln q)q^{1/2} < 1-q$ when $0 < q < 1$, the above inequality gives a refinement of inequality (1.5).

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Peng Gao
Division of Mathematical Sciences
School of Physical and Mathematical Sciences
Nanyang Technological University
637371 Singapore
current address:
Department of Mathematics
School of Mathematics and System Sciences
Beijing University of Aeronautics and Astronautics
Beijing 100083, PR China
e-mail: penggao@buaa.edu.cn