SHARP BOUNDS FOR SEIFFERT MEAN IN TERMS OF WEIGHTED POWER MEANS OF ARITHMETIC MEAN AND GEOMETRIC MEAN

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Abstract. For \( a, b > 0 \) with \( a \neq b \), let \( P = (a - b) / (4 \arctan \sqrt{a/b} - \pi) \), \( A = (a + b) / 2 \), \( G = \sqrt{ab} \) denote the Seiffert mean, arithmetic mean, geometric mean of \( a \) and \( b \), respectively. In this paper, we present new sharp bounds for Seiffert \( P \) in terms of weighted power means of arithmetic mean \( A \) and geometric mean \( G \):

\[
\left( \frac{2}{3} A^{p_1} + \frac{1}{3} G^{p_1} \right)^{1/p_1} < P < \left( \frac{2}{3} A^{p_2} + \frac{1}{3} G^{p_2} \right)^{1/p_2},
\]

where \( p_1 = 4/5 \) and \( p_2 = \log_{\pi/2} (3/2) \) are the best possible constants. Moreover, our sharp bounds for \( P \) are compared with other known ones, which yields a chain of inequalities involving Seiffert mean \( P \).

1. Introduction and main results

Throughout the paper, we assume that \( a, b > 0 \) with \( a \neq b \).

Let \( w \in (0, 1) \). The \( r \)-th weighted power mean of positive numbers \( a, b > 0 \) is defined as

\[
M_r(a, b; w) := (wa^r + (1 - w)b^r)^{1/r} \quad \text{if } r \neq 0 \quad \text{and} \quad M_0(a, b; w) = a^w b^{1-w}. \tag{1.1}
\]

It is well-known that \( M_r(a, b; w) \) is increasing with respect to \( r \) on \( \mathbb{R} \) (see [1]). In particular, \( M_r(a, b) := M_r(a, b; 1/2) \) is the standard power mean. As special cases, the arithmetic mean and geometric mean are \( A = A(a, b) = M_1(a, b) \) and \( G = G(a, b) = M_0(a, b) \), respectively. Let \( L = (a - b) / (\log a - \log b) \), \( I = e^{-1} (b^b / a^a) ^{1/(b-a)} \) denote the logarithmic mean and identric mean, respectively.

The Seiffert’s mean defined by

\[
P = P(a, b) = \frac{a - b}{4 \arctan \sqrt{a/b} - \pi} \tag{1.2}
\]
or

\[
P = P(a, b) = \frac{a - b}{2 \arcsin \frac{a-b}{a+b}} \tag{1.3}
\]


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was introduced in [17], it has attracted many scholars’ attention, and the inequalities involving \( P(a,b) \) have been the subject of intensive research. In [18], the author proved that
\[
L < P < I
\]  
(1.4)
and further showed that [19]:
\[
P > \frac{3AG}{2A + G},
\]  
(1.5)
\[
P > \frac{AG}{L},
\]  
(1.6)
\[
\frac{2}{\pi}A < P < A.
\]  
(1.7)

Jagers [9] and Hästo [5] gave bounds for \( P \) in terms of power means:
\[
M_{1/2} < P < M_{2/3},
\]  
(1.8)
\[
\frac{2\sqrt{2}}{\pi}M_{2/3} < P < M_{2/3},
\]  
(1.9)
respectively. Later, Hästo obtained a sharp lower bound for \( P \) [6]:
\[
P > M_{\log_{4/5}2}.
\]  
(1.10)

In 2001, Sándor [16] established the following
\[
\frac{A + G}{2} < P < \sqrt{\frac{A + G}{2}A},
\]  
(1.11)
\[
A^{2/3}G^{1/3} < P < \frac{2A + G}{3}.
\]  
(1.12)

The more results can be found in [4], [7], [12], [14], [20], [21].

The main purpose of this paper is to strengthen the inequalities (1.12), that is, to determine the best \( p \in (0,1) \) such that the inequality
\[
P > \left( \frac{2}{3}A^p + \frac{1}{3}G^p \right)^{1/p}
\]  
(1.13)
or its reverse inequality holds. Our main results are the following

**Theorem 1.** The inequality (1.13) holds for all \( a,b > 0 \) with \( a \neq b \) if and only if \( p \leq p_1 = 4/5 \). Moreover, we have
\[
\alpha_1 \left( \frac{2}{3}A^{4/5} + \frac{1}{3}G^{4/5} \right)^{5/4} < P < \alpha_2 \left( \frac{2}{3}A^{4/5} + \frac{1}{3}G^{4/5} \right)^{5/4},
\]  
(1.14)
where \( \alpha_1 = 1 \) and \( \alpha_2 = 3\sqrt{24}/(2\pi) = 1.0568... \) are the best possible constants.
Theorem 2. The inequality (1.13) is reversed for all $a, b > 0$ with $a \neq b$ if and only if $p \geq p_2 = \log_{\pi/2} (3/2) = 0.89788$. Moreover, we have
\[
\beta_1 \left( \frac{2}{3}A^{p_2} + \frac{1}{3}G^{p_2} \right)^{1/p_2} < P < \beta_2 \left( \frac{2}{3}A^{p_2} + \frac{1}{3}G^{p_2} \right)^{1/p_2},
\]
where $\beta_1 \approx 0.99237$ and $\beta_2 = 1$ are the best possible constants.

Due to (1.3) and with $x = \arcsin \frac{a-b}{a+b} \in (0, \pi/2)$, we have
\[
\frac{P}{A} = \frac{\sin x}{x}, \quad \frac{G}{A} = \cos x.
\]
Thus Theorems 1 and 2 can be changed as the following two equivalent theorems.

**Theorem A.** The inequality
\[
\frac{\sin x}{x} > \left( \frac{2}{3} + \frac{1}{3} \cos x \right)^{1/p}
\]
holds for $x \in (0, \pi/2)$ if and only if $p \leq p_1 = 4/5$. Moreover, we have
\[
\alpha_1 \left( \frac{2}{3} + \frac{1}{3} \cos x \right)^{4/5} < \frac{\sin x}{x} < \alpha_2 \left( \frac{2}{3} + \frac{1}{3} \cos x \right)^{4/5},
\]
where $\alpha_1 = 1$ and $\alpha_2 = 3\sqrt{24}/(2\pi) = 1.0568...$ are the best possible constants.

**Theorem B.** The inequality (1.16) is reversed for $x \in (0, \pi/2)$ if and only if $p \geq p_2 = \log_{\pi/2} (3/2) = 0.89788$. Moreover, we have
\[
\beta_1 \left( \frac{2}{3} + \frac{1}{3} \cos x \right)^{p_2} < \frac{\sin x}{x} < \beta_2 \left( \frac{2}{3} + \frac{1}{3} \cos x \right)^{p_2},
\]
where $\beta_1 \approx 0.99237$ and $\beta_2 = 1$ are the best possible constants.

**Remark 1.** Cusa-Huygens inequality [8] refers to
\[
\frac{\sin x}{x} < \frac{2}{3} + \frac{1}{3} \cos x
\]
holds for $x \in (0, \pi/2)$. It is obvious that our Theorem A and B are improvements of (1.19). Other improvements and refinements for Cusa-Huygens inequality can be found in [2], [10], [13], [14], [15].

A hyperbolic counterpart of the inequality (1.16) is due to Zhu [22, Theorem 1.1].

2. Lemmas

**Lemma 1.** Let $M(a, b)$ be a homogeneous mean of positive arguments $a$ and $b$. Then
\[
M(a, b) = \sqrt{ab} M(e^t, e^{-t}),
\]
where $t = \frac{1}{2} \log (a/b)$. 

LEMMA 2. Let the function \( t \mapsto F_p(t) \) be defined on \((0, \infty)\) by
\[
F_p(t) = \begin{cases} 
\log \frac{2 \sinh t}{4 \arctan e^t - \pi} - \frac{1}{p} \log \left( \frac{2}{3} \cosh^p t + \frac{1}{2} \right) & \text{if } p \neq 0, \\
\log \frac{2 \sinh t}{4 \arctan e^t - \pi} - \cosh^{2/3} t & \text{if } p = 0.
\end{cases}
\]
(2.1)

Then we have
\[
\lim_{t \to 0^+} \frac{F_p(t)}{t^4} = \frac{1}{45} - \frac{1}{36} p
\]
(2.2)
\[
F_p(\infty) = \lim_{t \to \infty} F_p(t) = \begin{cases} 
\frac{1}{p} \log \frac{3}{2} - \log \frac{5}{2} & \text{if } p > 0, \\
\infty & \text{if } p \leq 0.
\end{cases}
\]
(2.3)

Proof. Using power series expansion we have
\[
F_p(t) = -\frac{5p - 4}{180} t^4 + O(t^6),
\]
which yields (2.2).

To obtain (2.3), we write \( F_p(t) \) as
\[
F_p(t) = \log 2 - \log \left( 4 \arctan e^t - \pi \right) - \frac{1}{p} \log \left( \frac{2}{3} \left( \frac{\cosh t}{\sinh t} \right)^p + \frac{1}{3} \left( \frac{1}{\sinh t} \right)^p \right),
\]
from which (2.3) easily follows.

The proof ends. \( \square \)

LEMMA 3. Let the function \( t \mapsto F_p(t) \) be defined on \((0, \infty)\) by (2.1). Then \( F_p \) is strictly increasing on \((0, \infty)\) if \( p \in (0, 4/5] \).

Proof. Differentiation and arrangement yield
\[
F'_p(t) = \frac{2 \cosh^p t + \cosh^2 t}{(\cosh t \sinh t) \left( 2 \cosh^p t + 1 \right) \left( 4 \arctan e^t - \pi \right)} f_1(t),
\]
(2.4)
where
\[
f_1(t) = 4 \arctan e^t - \pi - 2 \sinh t + \frac{2 \sinh^3 t}{\cosh^2 t + 2 \cosh^p t}.
\]
(2.5)

Differentiation again and factoring lead to
\[
f'_1(t) = \frac{4 \sinh^2 t}{(\cosh^3 t) \left( 1 + 2 \cosh^{p-2} t \right)^2} f_2(\cosh t),
\]
(2.6)
here
\[
f_2(x) = (1 - p)x^p - 2x^{2p-2} + px^{p-2} + 1, \quad x \in (1, \infty).
\]
(2.7)
Simple computation reveals that
\[ x^{3-p} f_2' (x) = p (1-p)x^2 + 4(1-p)x^p + p(p-2) := f_3 (x), \]  
(2.8)  
\[ f_3' (x) = 2p (1-p) (x+2x^{p-1}). \]  
(2.9)  
If \( p \in (0, 4/5] \), then  
\[ f_3' (x) = 2p (1-p) (x+2x^{p-1}) > 0 \]  
for all \( x > 1 \), that is, \( f_3 \) is increasing on \((1, \infty)\), it is derived that  
\[ f_3 (x) > f_3 (1) = 4 - 5p > 0, \]  
which together with (2.8) leads to \( f_2' (x) > 0 \), that is, \( f_2 \) is increasing on \((1, \infty)\). Hence, we have  
\[ f_2 (x) > f_2 (1) = 0, \]  
which in conjunction with (2.6) implies that \( f_1' (t) > 0 \) for all \( t > 0 \), and then, \( f_1 (t) > f_1 (0) = 0 \). Thus it is obtained that \( F_p' (t) > 0 \), that is, the desired result.

The proof is completed.  

From the proof of Lemma 3 it is obtained that  
\[ f_1 (t) = 4 \arctan e^t - \pi - 2 \sinh t + \frac{2 \sinh^3 t}{\cosh^2 t + 2 \cosh^p t} > 0, \]  
which can be written as  
\[ \frac{2 \sinh t}{4 \arctan e^t - \pi} < \frac{\cosh^2 t + 2 \cosh^p t}{1 + 2 \cosh^p t}, \]  
(2.10)  
where \( p \in (0, 4/5] \). It is easy to verify that  
\[ \frac{d}{dp} \frac{\cosh^2 t + 2 \cosh^p t}{1 + 2 \cosh^p t} = -\cosh^p t \frac{\log (\cosh t)}{(2 \cosh^p t + 1)^2} (\cosh 2t - 1) < 0, \]  
that is, the function \( p \mapsto \frac{\cosh^2 t + 2 \cosh^p t}{1 + 2 \cosh^p t} \) is decreasing on \( \mathbb{R} \). By Lemma 1 the result can be stated as a corollary of Lemma 3.

**Corollary 1.** We have  
\[ P < \frac{A^2 + 2A^p G^{2-p}}{G^2 + 2A^p G^{2-p}} G, \]  
(2.11)  
where the right hand of (2.11) decreases as \( p \) increases on \((-\infty, 4/5]\). Particularly, putting \( p = 4/5, 0, ..., \rightarrow -\infty \) we have  
\[ P < \frac{A^{4/5} G^{-1/5} A^{6/5} + 2G^{6/5}}{2A^{4/5} + G^{4/5}} < \frac{A^{2} + 2G^{2}}{3G} < \cdots < \frac{A^{2}}{G}. \]  
(2.12)
Lemma 4. Let $p \in (4/5, 1)$ and the function $t \mapsto F_p(t)$ be defined on $(0, \infty)$ by \eqref{2.1}. Then there is a unique number $t_3 \in (0, \infty)$ to satisfy $f_1(t_3) = 0$ such that $F_p$ is decreasing on $(0, t_3)$ and increasing on $(t_3, \infty)$.

Proof. We start with \eqref{2.9} to prove this lemma. If $p \in (4/5, 1)$ then

$$f_3'(x) = 2p(1-p)(x + 2x^{p-1}) > 0,$$

and note that

$$f_3(1) = 4 - 5p < 0 \quad \text{and} \quad f_3(\infty) = \text{sgn}(p(1-p)) > 0,$$

it is seen that there is a unique number $x_1 \in (1, \infty)$ such that $f_3(x) < 0$ for $x \in (1, x_1)$ and $f_3(x) > 0$ for $x \in (x_1, \infty)$. From \eqref{2.8} it is deduced that $f_2$ is decreasing on $(1, x_1)$ and increasing on $(x_1, \infty)$. And then, $f_2(x) < f_2(1) = 0$ for $x \in (1, x_1)$, but $f_2(\infty) = \text{sgn}(1-p)$, it follows that there is a unique number $x_2 \in (x_1, \infty)$ such that $f_2(x) < 0$ for $x \in (1, x_2)$ and $f_2(x) > 0$ for $x \in (x_2, \infty)$. Due to \eqref{2.6} this implies that there exits a unique $t_2 \in (0, \infty)$ to satisfy $\cosh t_2 = x_2$ so that the function $t \mapsto f_1(t)$ is decreasing on $(0, t_2)$ and increasing on $(t_2, \infty)$. Hence, we have

$$f_1(t) < f_1(0) = 0 \text{ if } t \in (0, t_2).$$

However,

$$\lim_{t \to \infty} f_1(t) = \frac{\pi}{4} > 0,$$

thus there is a unique number $t_3 \in (t_2, \infty)$ to satisfy $f_1(t_3) = 0$ such that $f_1(t) < 0$ if $t \in (0, t_3)$ and $f_1(t) > 0$ if $t \in (t_3, \infty)$, which from \eqref{2.4} reveals that the function $t \mapsto F_p(t)$ is decreasing on $(0, t_3)$ and increasing on $(t_3, \infty)$.

This completes the proof. \hfill \square

3. Proofs of Main Results

Proof of Theorem 1. By symmetry, we assume that $b > a > 0$. We have

$$P(e^t, e^{-t}) = \frac{2 \sinh t}{4 \arctan e^t - \pi}, \quad A(e^t, e^{-t}) = \cosh t, \quad G(e^t, e^{-t}) = 1,$$

where $t = \frac{1}{2} \log(b/a) > 0$. From Lemma 1, in order to prove that inequality \eqref{1.13} holds if and only if $p \leq 4/5$, it is enough to show that inequalities

$$\log \frac{2 \sinh t}{4 \arctan e^t - \pi} > \frac{1}{p} \log \left(\frac{2}{3} (\cosh t)^p + \frac{1}{3} \right),$$

that is, $F_p(t) > 0$ holds if and only if $p \leq 4/5$, where $F_p(t)$ is defined by \eqref{2.4}.

Necessity. If $F_p(t) > 0$ holds for all $t > 0$, then by Lemma 2 we have

$$\left\{ \begin{array}{l}
\lim_{t \to +0} \frac{F_p(t)}{t^p} = \frac{1}{45} - \frac{1}{36} p > 0, \\
\lim_{t \to \infty} F_p(t) = \frac{1}{p} \log \frac{3}{2} - \log \frac{4}{3} \
\end{array} \right. \geq 0 \text{ if } p > 0$$
or
\[
\begin{cases}
\lim_{t \to 0^+} \frac{F_p(t)}{t^4} = \frac{1}{45} - \frac{1}{36}p \geq 0, \\
\lim_{t \to \infty} F_p(t) = \infty \text{ if } p \leq 0.
\end{cases}
\]

Solving the inequalities for \( p \) yields \( p \leq 4/5 \).

**Sufficiency.** Suppose that \( p \leq 4/5 \). Since the function

\[
p \mapsto \frac{1}{p} \log \left( \frac{2}{3} (\cosh t)^p + \frac{1}{3} \right)
\]

is clearly increasing, so the function \( p \mapsto F_p(t) \) is decreasing, thus it is suffices to show that

\[ F_p(t) > 0 \text{ for all } t > 0 \text{ if } p = p_1 = 4/5. \]

By Lemma 3, we see that \( F_{p_1} \) is strictly increasing on \((0, \infty)\). It follows that

\[
0 = F_{p_1}(0) < F_{p_1}(t) < F_{p_1}(\infty) = \frac{5}{4} \log \frac{3}{2} - \log \frac{\pi}{2},
\]

which proves the sufficiency and inequalities (1.14). Clearly,

\[
\alpha_1 = \exp(0) = 1 \text{ and } \alpha_2 = \exp \left( \frac{5}{4} \log \frac{3}{2} - \log \frac{\pi}{2} \right) = 3^{\frac{4}{3}} \frac{\sqrt{24}}{(2\pi)}
\]

are the best possible constants.

Thus the proof of Theorem 1 is finished. \( \square \)

**Proof of Theorem 2.** Clearly, the reverse inequality of (1.13) is equivalent to \( F_p(t) < 0 \) for \( t > 0 \). Now we show that \( F_p(t) < 0 \) holds for all \( t > 0 \) if and only if \( p \geq p_2 = (\log 3 - \log 2) / (\log \pi - \log 2) \).

**Necessity.** The condition \( p \geq p_2 \) is necessary. Indeed, if \( F_p(t) < 0 \) holds for all \( t > 0 \), then we have

\[
\lim_{t \to 0^+} \frac{F_p(t)}{t^4} = \frac{1}{45} - \frac{1}{36}p \leq 0,
\]

\[
\lim_{t \to \infty} F_p(t) = \frac{1}{p} \log \frac{3}{2} - \log \frac{\pi}{2} \leq 0 \text{ if } p > 0,
\]

which leads to \( p \geq \log_{\pi/2} (3/2) = p_2 \).

**Sufficiency.** The condition \( p \geq p_2 \) is also sufficient. As mentioned in proof of Theorem 1, the function \( p \mapsto F_p(t) \) is decreasing, thus it is suffices to show that \( F_p(t) < 0 \) for all \( t > 0 \) if \( p = p_2 \).

Lemma 4 reveals that for \( p \in (4/5, 1) \) there is a unique number \( t_3 \in (t_2, \infty) \) to satisfy

\[
f_1(t) = \arctan e^t - \frac{2 \sinh t + 4 \cosh^p t \sinh t + \pi \cosh^2 t + 2 \pi \cosh^p t}{8 \cosh^p t + 4 \cosh^2 t} = 0 \tag{3.1}
\]
such that $F_p$ is decreasing on $(0,t_3)$ and increasing on $(t_3,\infty)$. It is acquired that for $p_2 = \log_{\pi/2} (3/2) \in (4/5, 1)$

$$F_{p_2}(t_3) < F_{p_2}(t) < F_{p_2}(0) = 0 \text{ if } t \in (0,t_3),$$

$$F_{p_2}(t_3) < F_{p_2}(t) < F_{p_2}(\infty) = 0 \text{ if } t \in (t_3,\infty),$$

that is,

$$F_{p_2}(t_3) < \log \frac{2 \sinh t}{4 \arctan \frac{\pi}{2} - \pi} \left( \frac{2}{3} (\cosh t)^{p_2} + \frac{1}{3} \right)^{1/p_2} < 0,$$

which proves the sufficiency and inequalities (1.15).

Furthermore, for $p = p_2 = \log_{\pi/2} (3/2)$, solving the equation (3.1) for $t$ by mathematical computation software yields $t_3 \approx 2.6630245$, and then

$$\beta_1 = \exp (F_{p_2}(t_3)) \approx 0.99237 \text{ and } \beta_2 = \exp (0) = 1.$$

Clearly, $\beta_1 \approx 0.99237$ and $\beta_2 = 1$ are the best possible constants.

This completes the proof of Theorem 2. □

4. Comparisons of certain bounds for $P$

It is mentioned in the Introduction that there has many bounds for $P$, some comparisons of them can refer to [7]. In this section, the bounds in the form of $M_{r_1} (A, G; 2/3)$ will be compared with other ones in the form of $M_{r_2} (a, b; 1/2)$, where $M_r (a, b; w)$ is defined by (1.1).

**Lemma 5.** The inequalities

$$\left( \frac{2}{3} A^p + \frac{1}{3} G^p \right)^{1/p} < \left( \frac{a^{2/3} + b^{2/3}}{2} \right)^{3/2} < \left( \frac{2}{3} A^q + \frac{1}{3} G^q \right)^{1/q} \tag{4.1}$$

hold if and only if $p \leq 10/9$ and $q \geq \log_2 (9/4)$.

**Proof.** By Lemma 1, in order to prove this lemma, it is enough to prove that for $t > 0$ inequalities

$$\frac{1}{p} \log \left( \frac{2}{3} (\cosh t)^p + \frac{1}{3} \right) < \frac{3}{2} \log \cosh \frac{2}{3} t < \frac{1}{q} \log \left( \frac{2}{3} (\cosh t)^q + \frac{1}{3} \right) \tag{4.2}$$

hold if and only if $p \leq 10/9$ and $q \geq \log_2 (9/4)$.

Define that

$$G_p(t) := \frac{1}{p} \log \left( \frac{2}{3} (\cosh t)^p + \frac{1}{3} \right) - \frac{3}{2} \log \cosh \frac{2}{3} t. \tag{4.3}$$
Then we easily get
\[
\lim_{t \to 0^+} \frac{G_p(t)}{t^4} = \frac{1}{36} p - \frac{5}{162},
\]
(4.4)

\[
G_p(\infty) = \begin{cases} 
\frac{1}{2} \log 2 - \frac{1}{p} \log \frac{3}{2} & \text{if } p > 0, \\
-\infty & \text{if } p \leq 0.
\end{cases}
\]
(4.5)

On the other hand, differentiation yields
\[
G_p'(t) = \frac{(2 \sinh \frac{1}{2}t \cosh t) \log \cosh t}{(\cosh \frac{3}{2}t \cosh t) (2 \cosh^p t + 1)} L \left( \cosh^{p-1} t, \cosh \frac{1}{3} t \right) \times g_1(t),
\]
(4.6)

where
\[
g_1(t) = p - 1 - \frac{\log \cosh \frac{1}{2} t}{\log \cosh t}
\]
(4.7)

and \( L(x, y) \) is the logarithmic mean of positive numbers \( x \) and \( y \).

Differentiation again leads to
\[
\frac{\frac{3}{2} (\cosh \frac{1}{2} t \cosh t) \log^2 (\cosh t)}{3 \cosh \frac{1}{2} t \sinh t} g_1'(t) = \log (\cosh \frac{1}{2} t) - \frac{\sinh \frac{1}{2} t \cosh t}{3 \cosh \frac{1}{2} t \sinh t} \log (\cosh t) := g_2'(t),
\]
(4.8)

\[
g_2'(t) = \frac{-2}{9} \frac{\sinh^3 \frac{2}{3} t}{\cosh^2 \frac{1}{2} t \sinh^2 t} \log (\cosh t) < 0
\]
(4.9)

for \( t > 0 \). It is acquired that \( g_2(t) < g_2(0) = 0 \), which implies that \( g_1 \) is decreasing on \((0, \infty)\).

Now we are in a position to prove the desired results.

(i) We prove the first inequality of (4.2) holds if and only if \( p \leq 10/9 \). In fact, if the first inequality of (4.2) holds, that is, \( G_p(t) < 0 \) for all \( t > 0 \), then by (4.4) and (4.5) we have

\[
\begin{align*}
\lim_{t \to 0^+} \frac{G_p(t)}{t^4} &= \frac{1}{36} p - \frac{5}{162} \leq 0, \\
G_p(\infty) &= \frac{1}{2} \log 2 - \frac{1}{p} \log \frac{3}{2} \leq 0 \quad \text{if } p > 0
\end{align*}
\]

or

\[
\begin{align*}
\lim_{t \to 0^+} \frac{G_p(t)}{t^4} &= \frac{1}{36} p - \frac{5}{162} \leq 0, \\
G_p(\infty) &= -\infty \quad \text{if } p \leq 0.
\end{align*}
\]

Solving the above inequalities leads to \( p \leq 10/9 \).

Conversely, if \( p \leq 10/9 \), then since \( g_1 \) is decreasing on \((0, \infty)\), it is obtained that

\[
g_1(t) < g_1(0^+) = \lim_{t \to 0^+} \left( p - 1 - \frac{\log \cosh \frac{1}{2} t}{\log \cosh t} \right) = p - \frac{10}{9} \leq 0,
\]
which in combination with (4.6) reveals that \( G'_p(t) < 0 \). Thus we conclude that \( G_p(t) < G_p(0) = 0 \), that is, the first inequality of (4.2) holds.

(ii) Next we show that the second inequality of (4.2) holds if and only if \( p \geq \log_2(9/4) \).

If the second inequality of (4.2) holds, that is, \( G_p(t) > 0 \) for all \( t > 0 \), then by (4.4) and (4.5) we have

\[
\begin{aligned}
\lim_{t \to 0^+} \frac{G_p(t)}{t^4} &= \frac{1}{36} p - \frac{5}{162} \geq 0, \\
G_p(\infty) &= \frac{1}{2} \log 2 - \frac{1}{p} \log \frac{3}{2} \geq 0 \text{ if } p > 0,
\end{aligned}
\]

which yields \( p \geq \log_2(9/4) \).

Conversely, if \( p \geq \log_2(9/4) \), since the function \( p \to G_p(t) \) is increasing, then it suffices to show that \( G_p(t) > 0 \) for all \( t > 0 \) if \( p = \log_2(9/4) \). By the monotonicity of the function \( g_1 \) and the fact that

\[
g_1(0^+) = \lim_{t \to 0^+} \left( \log_2 \frac{9}{4} - 1 - \frac{\log \cosh \frac{1}{2} t}{\log \cosh t} \right) = \log_2 \frac{9}{4} - \frac{10}{9} > 0,
\]

\[
g_1(\infty) = \lim_{t \to \infty} \left( \log_2 \frac{9}{4} - 1 - \frac{\log \cosh \frac{1}{2} t}{\log \cosh t} \right) = \log_2 \frac{9}{4} - \frac{4}{3} < 0,
\]

it is seen that there is a unique number \( t_0 \in (0, \infty) \) such that \( g_1(t) > 0 \) if \( t \in (0, t_0) \) and \( g_1(t) < 0 \) if \( t \in (t_0, \infty) \), which together with (4.6) indicates that the function \( t \to G_p(t) \) is increasing on \((0, t_0)\) and decreasing on \((t_0, \infty)\). Therefore, we conclude that

\[
G_p(t) > G_p(0) = 0 \text{ for } t \in (0, t_0),
\]

\[
G_p(t) > G_p(\infty) = 0 \text{ for } t \in (t_0, \infty),
\]

which is the desired result. Thus the proof ends. \( \square \)

**Lemma 6.** Let \( r_0 = (\log 2) / \log \pi \). Then inequalities

\[
\left( \frac{3}{4} A^p + \frac{1}{4} G^p \right)^{1/p} > \left( \frac{a^{r_0} + b^{r_0}}{2} \right)^{1/r_0}
\]

hold if and only if \( p \geq \log_{\pi/2}(3/2) \), and the two sides of (4.10) are not comparable for all \( a, b > 0 \) with \( a \neq b \) if \( p < \log_{\pi/2}(3/2) \).

**Proof.** From (1.10) and Theorem 2 it follows that (4.10) holds if \( p \geq \log_{\pi/2}(3/2) \), that is, the condition \( p \geq \log_{\pi/2}(3/2) \) is sufficient to (4.10) holds for all \( a, b > 0 \) with \( a \neq b \).
We now show that the condition \( p \geq \log_{\pi/2}(3/2) \) is necessary. Indeed, by symmetry of \( a \) and \( b \), we assume that \( b > a \) and let \( x = a/b \in (0,1) \). Then inequality (4.10) is equivalent with

\[
U_p(x) := \frac{1}{p} \log \left( \frac{2}{3} \left( \frac{x+1}{2} \right)^{p} + \frac{1}{3} (\sqrt{x})^p \right) - \frac{1}{r_0} \log \left( \frac{x^{r_0} + 1}{2} \right) > 0,
\]

where \( x \in (0,1) \).

If (4.10) holds for all \( a,b > 0 \) with \( a \neq b \), then

\[
U_p(0^+) = \begin{cases} \frac{1}{p} \log \frac{2}{3} - \log 2 + \frac{1}{r_0} \log 2 \text{ if } p > 0 \\ -\infty \text{ if } p \leq 0 \end{cases}
\]

has to be nonnegative, which leads to \( p \geq \log_{\pi/2}(3/2) \). This completes the proof of sufficiency.

Next we show that the two sides of (4.10) are not comparable for all \( a,b > 0 \) with \( a \neq b \) if \( p < \log_{\pi/2}(3/2) \). Clearly, \( U_p(0^+) < 0 \) and

\[
\lim_{x \to 1^-} \frac{U_p(x)}{(x-1)^2} = \frac{1}{12} - \frac{1}{8} r_0 = \frac{2 \log \pi - 3 \log 2}{24 \log \pi} > 0.
\]

From this it is seen that there exits \( x_1,x_2 \in (0,1) \) such that \( U_p(x) < 0 \) for \( x \in (0,x_1) \) and \( U_p(x) > 0 \) for \( x \in (x_2,1) \), that is, the sign of \( U_p(x) \) is not a constant. Thus the proof is completed. \( \square \)

**Lemma 7.** The inequality

\[
\left( \frac{2}{3}A^p + \frac{1}{3}G^p \right)^{1/p} > \left( \frac{a^{1/2} + b^{1/2}}{2} \right)^2
\]  

holds if and only if \( p \geq \log_2 3 - 1 = 0.58496... \), and the two sides of (4.11) are not comparable for all \( a,b > 0 \) with \( a \neq b \) if \( p < \log_2 3 - 1 \).

**Proof.** Since the right hand of (4.11) can be written as \( (A + G)/2 \), then the inequality (4.11) is equivalent with

\[
V_p(x) = \frac{1}{p} \log \left( \frac{2}{3} x^p + \frac{1}{3} \right) - \log \left( \frac{1}{2} x + \frac{1}{2} \right) > 0,
\]

where \( x = A/G > 1 \).

If \( V_p(x) > 0 \) for all \( x > 1 \) then \( p > 0 \). If not, then

\[
\lim_{x \to \infty} V_p(x) = -\infty \text{ if } p \leq 0,
\]

which yields a contradiction. Thus we get

\[
V_p(\infty) = \lim_{x \to \infty} V_p(x) = \frac{1}{p} \log \frac{2}{3} - \log \frac{1}{2} \geq 0 \text{ if } p > 0,
\]
which yields \( p \geq \log_2 3 - 1 \).

Conversely, if \( p \geq \log_2 3 - 1 \), then since the function \( p \mapsto V_p(x) \) is increasing, we need to prove \( V_p(x) > 0 \) for all \( x > 1 \).

Differentiation leads to

\[
V'_p(x) = \frac{-x^p}{x(x+1)(2x^p+1)}(x^{1-p} - 2),
\]

which indicates that there is a unique number \( x_0 = 2^{1/(1-p)} \) such that \( V'_p(x) > 0 \) if \( x \in (1,x_0) \) and \( V'_p(x) < 0 \) if \( x \in (x_0,\infty) \). Thus we conclude that

\[
V_p(x) > V_p(1) = 0 \quad \text{if} \quad x \in (1,x_0) \quad \text{and} \quad V_p(x) > V_p(\infty) \geq 0 \quad \text{if} \quad x \in (x_0,\infty),
\]

that is, the desired result.

We now illustrate that the two sides of (4.11) are not comparable for all \( a,b > 0 \) with \( a \neq b \) if \( p < \log_2 3 - 1 \). In fact, if \( p < \log_2 3 - 1 \), then via (4.12) and (4.13) it is easily seen that \( V_p(\infty) < 0 \). On the other hand, it is easy to derive

\[
\lim_{x \to 1^+} \frac{V_p(x)}{x-1} = \frac{1}{6} > 0.
\]

Consequently, there exits \( x_1,x_2 \in (1,\infty) \) such that \( V_p(x) > 0 \) for \( x \in (1,x_1) \) and \( V_p(x) < 0 \) for \( x \in (x_2,\infty) \), that is, \( \text{sgn}(V_p(x)) \) is not a constant.

This completes the proof. \( \Box \)

Using Theorem 1, 2 and Lemma 5, 6, 7, the following theorem is immediate.

**THEOREM 3.** Let \( q \geq \log_2 (9/4), \log_{\pi/2} (3/2) \leq r \leq 10/9, \log_2 3 - 1 \leq s \leq 4/5. \) Then we have

\[
\left( \frac{2}{3}A^q + \frac{1}{3}G^p \right)^{1/q} > M_{2/3} > \left( \frac{2}{3}A^r + \frac{1}{3}G^r \right)^{1/r} > P > \left( \frac{2}{3}A^s + \frac{1}{3}G^s \right)^{1/s} > M_{1/2}.
\]  

(4.14)

**REMARK 2.** In [11] Kouba proved that inequalities

\[
\left( \frac{2}{3}A^p + \frac{1}{3}G^p \right)^{1/p} < I < \left( \frac{2}{3}A^q + \frac{1}{3}G^q \right)^{1/q}
\]  

(4.15)

hold if and only if \( p \leq 6/5 \) and \( q \geq (\log 3 - \log 2) / (1 - \log 2) \).

Relation (4.14) in combination with (4.15) leads to

\[
\left( \frac{2}{3}A^p + \frac{1}{3}G^p \right)^{1/p} > I > \left( \frac{2}{3}A^q + \frac{1}{3}G^q \right)^{1/q} > M_{2/3} > \left( \frac{2}{3}A^r + \frac{1}{3}G^r \right)^{1/r} > P > \left( \frac{2}{3}A^s + \frac{1}{3}G^s \right)^{1/s} > M_{1/2},
\]

(4.16)

where \( p \geq (\log 3 - \log 2) / (1 - \log 2), \log_2 (9/4) \leq q \leq 6/5, \log_{\pi/2} (3/2) \leq r \leq 10/9, \log_2 3 - 1 \leq s \leq 4/5. \)
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