ASYMPTOTIC EXPANSIONS OF THE LOGARITHM OF THE GAMMA FUNCTION IN THE TERMS OF THE POLYGAMMA FUNCTIONS

CHAO-PING CHEN

(*Communicated by N. Elezović*)

Abstract. We present new asymptotic expansions of the logarithm of the gamma function in terms of the polygamma functions. Based on these expansions, we prove new complete monotonicity properties of some functions involving the gamma and polygamma functions. As consequences of them we establish new upper and lower bounds for the gamma function in terms of the polygamma functions.

1. Introduction

A function f is said to be completely monotonic on an interval I if it has derivatives of all orders on I and satisfies the following inequality:

$$(-1)^n f^{(n)}(x) \geqslant 0 \quad (x \in I; n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, \quad \mathbb{N} := \{1,2,3,\dots\}). \tag{1}$$

Dubourdieu [2, p. 98] pointed out that, if a non-constant function f is completely monotonic on $I = (a,\infty)$, then strict inequality holds true in (1). See also [3] for a simpler proof of this result. It is known (Bernstein's Theorem) that f is completely monotonic on $(0,\infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xt} \, d\mu(t),$$

where μ is a nonnegative measure on $[0,\infty)$ such that the integral converges for all $x > 0$. See [7, p. 161].

The familiar gamma function:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt \quad (\Re(z) > 0)$$

is one of the most important functions in mathematical analysis and its applications in various diverse areas are widely scattered. The logarithmic derivative of the gamma function:

$$\psi(z) = \frac{d}{dz}\{\ln\Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \quad \text{or} \quad \ln\Gamma(z) = \int_1^z \psi(t) \, dt$$

is known as the psi (or digamma) function. The successive derivatives of the psi function $\psi(z)$:

$$\psi^{(n)}(z) := \frac{d^n}{dz^n}\{\psi(z)\} \qquad (n \in \mathbb{N})$$

are called the polygamma functions.

The following asymptotic formulas are well-known that

$$\ln\Gamma(z) \sim \left(z - \frac{1}{2}\right)\ln z - z + \ln\sqrt{2\pi} + \sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)z^{2m-1}} \tag{2}$$
$$(z \to \infty \text{ in } |\arg z| < \pi)$$

(see [1, p. 257]), where B_n $(n \in \mathbb{N}_0)$ are the n-th Bernoulli numbers defined by the following generating function (see, for example, [6, Section 1.7]):

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \quad (|z| < 2\pi).$$

In this paper, we present new asymptotic expansions of the logarithm of the gamma function in terms of the polygamma functions. Based on these expansions, we prove new complete monotonicity properties of some functions involving the gamma and polygamma functions. As consequences of them we establish new upper and lower bounds for the gamma function in terms of the polygamma functions.

2. Asymptotic expansions

In this section, we present new asymptotic expansions of the logarithm of the gamma function in terms of the polygamma functions.

THEOREM 2.1. *For $x \to \infty$, we have*

$$\ln\Gamma(x+1) \sim \left(x + \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \sum_{j=1}^{\infty} a_j \psi^{(2j-1)}\left(x + \frac{1}{2}\right), \tag{3}$$

where the coefficients a_j are given by

$$a_j = \frac{4j}{2^{2j+1}(2j+1)!} \qquad (j \in \mathbb{N}). \tag{4}$$

Namely,

$$\ln\Gamma(x+1) \sim \left(x + \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \frac{1}{12}\psi'\left(x + \frac{1}{2}\right)$$
$$+ \frac{1}{480}\psi'''\left(x + \frac{1}{2}\right) + \frac{1}{53760}\psi^{(5)}\left(x + \frac{1}{2}\right) + \cdots \qquad (x \to \infty). \tag{5}$$

Proof. The noted Binet's first formula [5, p. 16] states that

$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,dt \quad (x > 0). \quad (6)$$

Write (6) as

$$\ln\Gamma(x+1) = \left(x + \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \int_0^\infty \frac{p(t)}{1 - e^{-t}}e^{-(x+1/2)t}\,dt, \quad (7)$$

where

$$p(t) = \frac{(t+2)e^{-t/2} + (t-2)e^{t/2}}{2t^2} = \sum_{j=1}^\infty \frac{4j}{2^{2j+1}(2j+1)!}t^{2j-1}$$

$$= \frac{1}{12}t + \frac{1}{480}t^3 + \frac{1}{53760}t^5 + \frac{1}{11612160}t^7 + \frac{1}{4087480320}t^9 + \cdots,$$

or

$$p(t) = \sum_{j=1}^\infty a_j t^{2j-1} \quad (t > 0), \quad (8)$$

where

$$a_j = \frac{4j}{2^{2j+1}(2j+1)!} \quad (j \geqslant 1).$$

It is known (see [1, p. 260]) that

$$\psi^{(j)}(z) = (-1)^{j+1}\int_0^\infty \frac{t^j}{1 - e^{-t}}e^{-zt}\,dt \quad (\Re(z) > 0; \; j \in \mathbb{N}) \quad (9)$$

and

$$\psi^{(j)}(z) = \frac{(-1)^{j-1}(j-1)!}{z^j} + O\left(\frac{1}{z^{j+1}}\right) \quad (z \to \infty; \; |\arg z| < \pi; \; j \in \mathbb{N}). \quad (10)$$

We find that

$$\psi^{(2N+1)}(x) = \frac{(2N)!}{x^{2N+1}} + O\left(\frac{1}{x^{2N+2}}\right) = O\left(\frac{1}{x^{2N+1}}\right)$$

and

$$O\left(\psi^{(2N+1)}(x)\right) = O\left(\frac{1}{x^{2N+1}}\right).$$

Hence, (7) implies that, for $x \to \infty$,

$$\ln\Gamma(x+1) = \left(x + \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \sum_{j=1}^N a_j\psi^{(2j-1)}\left(x + \frac{1}{2}\right) + R_N(x)$$

and

$$R_N(x) = O\left(\frac{1}{x^{2N+1}}\right) = O\left(\psi^{(2N+1)}(x)\right) \qquad (x \to \infty).$$

Therefore, (3) holds. The proof of Theorem 2.1 is complete. $\square$

THEOREM 2.2. *For $x \to \infty$, we have*

$$\ln\Gamma(x+1) \sim \left(x+\frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \sum_{j=1}^{\infty} b_j \psi^{(j)}(x), \tag{11}$$

where the coefficients b_j are given by

$$b_j = \frac{j}{2(j+2)!} \qquad (j \in \mathbb{N}). \tag{12}$$

Namely,

$$\ln\Gamma(x+1) \sim \left(x+\frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \frac{1}{12}\psi'(x) + \frac{1}{24}\psi''(x)$$
$$+ \frac{1}{80}\psi'''(x) + \frac{1}{360}\psi^{(4)}(x) + \cdots \qquad (x \to \infty). \tag{13}$$

Proof. Write (6) as

$$\ln\Gamma(x+1) = \left(x+\frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \int_0^{\infty} \frac{q(t)}{1-e^{-t}} e^{-xt}\, dt, \tag{14}$$

where

$$q(t) = \frac{(t+2)e^{-t}-2+t}{2t^2} = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}j}{2(j+2)!}t^j$$
$$= \frac{1}{12}t - \frac{1}{24}t^2 + \frac{1}{80}t^3 - \frac{1}{360}t^4 + \frac{1}{2016}t^5 - \frac{1}{13440}t^6 + \frac{1}{103680}t^7 - \frac{1}{907200}t^8 + \cdots,$$

or

$$q(t) = \sum_{j=1}^{\infty}(-1)^{j-1}b_j t^j \qquad (t > 0), \tag{15}$$

where

$$b_j = \frac{j}{2(j+2)!} \qquad (j \geqslant 1).$$

It follows from (10) that

$$\psi^{(N+1)}(x) = \frac{(-1)^N N!}{x^{N+1}} + O\left(\frac{1}{x^{N+2}}\right) = O\left(\frac{1}{x^{N+1}}\right)$$

and

$$O\left(\psi^{(N+1)}(x)\right) = O\left(\frac{1}{x^{N+1}}\right).$$

Hence, (14) implies that, for $x \to \infty$,

$$\ln\Gamma(x+1) = \left(x+\frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \sum_{j=1}^{N} b_j \psi^{(j)}(x) + \mathscr{R}_N(x)$$

and

$$\mathscr{R}_N(x) = O\left(\frac{1}{x^{N+1}}\right) = O\left(\psi^{(N+1)}(x)\right) \qquad (x \to \infty).$$

Therefore, (11) holds. The proof of Theorem 2.2 is complete. □

3. Completely monotonic functions

Şevli and Batir [4, Theorem 2.1] have proved that the function

$$S(x) = \ln\Gamma(x+1) - \left(x+\frac{1}{2}\right)\ln x + x - \ln\sqrt{2\pi} - \frac{1}{12}\psi'\left(x+\frac{1}{2}\right)$$

is completely monotonic on $(0, \infty)$.

The formula (5) motivated us to observe the following Theorem 3.1.

THEOREM 3.1. *The functions*

$$f(x) = \ln\Gamma(x+1) - \left[\left(x+\frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \frac{1}{12}\psi'\left(x+\frac{1}{2}\right) + \frac{1}{480}\psi'''\left(x+\frac{1}{2}\right)\right]$$

and

$$g(x) = \ln\Gamma(x+1) - \left[\left(x+\frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \frac{1}{12}\psi'\left(x+\frac{1}{2}\right)\right.$$
$$\left. + \frac{1}{480}\psi'''\left(x+\frac{1}{2}\right) + \frac{1}{53760}\psi^{(5)}\left(x+\frac{1}{2}\right)\right]$$

are completely monotonic on $(0, \infty)$.

Proof. By using (6) and (9), we have

$$f(x) = \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,dt - \frac{1}{12}\int_0^\infty \frac{te^{t/2}}{e^t - 1}e^{-xt}\,dt - \frac{1}{480}\int_0^\infty \frac{t^3 e^{t/2}}{e^t - 1}e^{-xt}\,dt$$
$$= \int_0^\infty \frac{\lambda(t)}{480t^2(e^t - 1)}e^{-xt}\,dt,$$

$$(16)$$

where

$$\lambda(t) = 480 + 240t + (240t - 480)e^t - (t^5 + 40t^3)e^{t/2}$$
$$= \sum_{n=7}^{\infty} \left(15 \cdot 2^{n-1} - (n^2 - 7n + 22)n(n-1)\right) \frac{(n-2)t^n}{2^{n-5} \cdot n!}.$$

By induction on n, it is easy to show that

$$15 \cdot 2^{n-1} - (n^2 - 7n + 22)n(n-1) > 0 \qquad (n \geqslant 7).$$

Hence $\lambda(t) > 0$ for $t > 0$. From (16) we obtain

$$(-1)^n f^{(n)}(x) > 0 \qquad (x > 0; n \in \mathbb{N}_0).$$

By using (6) and (9), we have

$$g(x) = \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2}\right) \frac{e^{-xt}}{t^2} \, dt - \frac{1}{12} \int_0^\infty \frac{te^{t/2}}{e^t - 1} e^{-xt} \, dt$$
$$- \frac{1}{480} \int_0^\infty \frac{t^3 e^{t/2}}{e^t - 1} e^{-xt} \, dt - \frac{1}{53760} \int_0^\infty \frac{t^5 e^{t/2}}{e^t - 1} e^{-xt} \, dt \qquad (17)$$
$$= \int_0^\infty \frac{\mu(t)}{53760 t^2 (e^t - 1)} e^{-xt} \, dt,$$

where

$$\mu(t) = 53760 + 26880t + (26880t - 53760)e^t - (4480t^3 + 112t^5 + t^7)e^{t/2}$$
$$= \sum_{n=9}^{\infty} \left((3360n - 6720)2^n - 16n(n-1)(n-2)\right.$$
$$\left. \times (n^4 - 18n^3 + 147n^2 - 538n + 976)\right) \frac{t^n}{2^{n-3} \cdot n!}.$$

By induction on n, it is not difficult to show that

$$2^n - \frac{16n(n-1)(n-2)(n^4 - 18n^3 + 147n^2 - 538n + 976)}{3360n - 6720} > 0 \qquad (n \geqslant 9).$$

Hence $\mu(t) > 0$ for $t > 0$. From (17) we obtain

$$(-1)^n g^{(n)}(x) > 0 \qquad (x > 0; n \in \mathbb{N}_0).$$

The proof of Theorem 3.1 is complete. $\square$

From the decreasingness of $f(x)$ and $g(x)$, we obtain the following upper and lower bounds for the gamma function in terms of the polygamma functions.

COROLLARY 3.1. *For $x \geqslant 1$,*

$$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\frac{1}{12}\psi'\left(x+\frac{1}{2}\right)+\frac{1}{480}\psi'''\left(x+\frac{1}{2}\right)\right) < \Gamma(x+1)$$

$$\leqslant \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(f(1)+\frac{1}{12}\psi'\left(x+\frac{1}{2}\right)+\frac{1}{480}\psi'''\left(x+\frac{1}{2}\right)\right)$$

and

$$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\frac{1}{12}\psi'\left(x+\frac{1}{2}\right)+\frac{1}{480}\psi'''\left(x+\frac{1}{2}\right)+\frac{1}{53760}\psi^{(5)}\left(x+\frac{1}{2}\right)\right)$$

$$< \Gamma(x+1)$$

$$\leqslant \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(g(1)+\frac{1}{12}\psi'\left(x+\frac{1}{2}\right)+\frac{1}{480}\psi'''\left(x+\frac{1}{2}\right)+\frac{1}{53760}\psi^{(5)}\left(x+\frac{1}{2}\right)\right),$$

where

$$f(1) = -\frac{1}{2}\ln(2\pi) + \frac{23}{15} - \frac{\pi^2}{24} - \frac{\pi^4}{480} = 0.000225676\ldots$$

and

$$g(1) = -\frac{1}{2}\ln(2\pi) + \frac{176}{105} - \frac{\pi^2}{24} - \frac{\pi^4}{480} - \frac{\pi^6}{6720} = 0.000018951\ldots.$$

Şevli and Batir [4, Theorem 2.1] have proved that the function

$$f_1(x) = \left[\left(x+\frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \frac{1}{12}\psi'(x)\right] - \ln\Gamma(x+1)$$

is completely monotonic on $(0,\infty)$.

The formula (13) motivated us to observe the following Theorem 3.2.

THEOREM 3.2. *Let*

$$b_1 = \frac{1}{12}, \quad b_2 = \frac{1}{24}, \quad b_3 = \frac{1}{80}, \quad b_4 = \frac{1}{360}.$$

Then, the functions

$$f_2(x) = \ln\Gamma(x+1) - \left[\left(x+\frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \sum_{k=1}^{2} b_k \psi^{(k)}(x)\right],$$

$$f_4(x) = \ln\Gamma(x+1) - \left[\left(x+\frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \sum_{k=1}^{4} b_k \psi^{(k)}(x)\right]$$

and

$$f_3(x) = \left[\left(x+\frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \sum_{k=1}^{3} b_k \psi^{(k)}(x)\right] - \ln\Gamma(x+1)$$

are completely monotonic on $(0,\infty)$.

   Proof. By using (6) and (9), we have

$$f_2(x) = \int_0^\infty \left(\frac{t}{e^t-1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,dt - \frac{1}{12}\int_0^\infty \frac{te^t}{e^t-1}e^{-xt}\,dt + \frac{1}{24}\int_0^\infty \frac{t^2 e^t}{e^t-1}e^{-xt}\,dt$$

$$= \int_0^\infty \frac{v(t)}{24t^2(e^t-1)}e^{-xt}\,dt,$$

(18)

where

$$v(t) = (t^4 - 2t^3 + 12t - 24)e^t + 12t + 24$$

$$= \sum_{n=5}^\infty \left(36 + 72(n-5) + 47(n-5)^2 + 12(n-5)^3 + (n-5)^4\right)\frac{t^n}{n!} > 0 \qquad (t>0),$$

so that (18) yields

$$(-1)^n f_2^{(n)}(x) > 0 \qquad (x>0; n \in \mathbb{N}_0).$$

   By using (6) and (9), we have

$$f_4(x) = \int_0^\infty \left(\frac{t}{e^t-1} - 1 + \frac{t}{2}\right)\frac{e^{-xt}}{t^2}\,dt - \frac{1}{12}\int_0^\infty \frac{te^t}{e^t-1}e^{-xt}\,dt$$

$$+ \frac{1}{24}\int_0^\infty \frac{t^2 e^t}{e^t-1}e^{-xt}\,dt - \frac{1}{80}\int_0^\infty \frac{t^3 e^t}{e^t-1}e^{-xt}\,dt + \frac{1}{360}\int_0^\infty \frac{t^4 e^t}{e^t-1}e^{-xt}\,dt$$

$$= \int_0^\infty \frac{\omega(t)}{720t^2(e^t-1)}e^{-xt}\,dt,$$

(19)

where

$$\omega(t) = (-720 + 360t - 60t^3 + 30t^4 - 9t^5 + 2t^6)e^t + 360t + 720$$

$$= \sum_{n=7}^\infty \left(1800 + 4350(n-7) + 3923(n-7)^2 + 1725(n-7)^3\right.$$

$$\left. + 395(n-7)^4 + 45(n-7)^5 + 2(n-7)^6\right)\frac{t^n}{n!} > 0 \qquad (t>0),$$

so that (19) yields

$$(-1)^n f_4^{(n)}(x) > 0 \qquad (x>0; n \in \mathbb{N}_0).$$

By using (6) and (9), we have

$$
\begin{aligned}
f_3(x) = & -\int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t}{2} \right) \frac{e^{-xt}}{t^2} \, dt + \frac{1}{12} \int_0^\infty \frac{te^t}{e^t - 1} e^{-xt} \, dt \\
& -\frac{1}{24} \int_0^\infty \frac{t^2 e^t}{e^t - 1} e^{-xt} \, dt + \frac{1}{80} \int_0^\infty \frac{t^3 e^t}{e^t - 1} e^{-xt} \, dt \\
= & \int_0^\infty \frac{\sigma(t)}{240 t^2 (e^t - 1)} e^{-xt} \, dt,
\end{aligned}
\tag{20}
$$

where

$$
\begin{aligned}
\sigma(t) = & -120t - 240 + (240 - 120t + 20t^3 - 10t^4 + 3t^5)e^t \\
= & \sum_{n=6}^\infty \left(480 + 1072(n-6) + 850(n-6)^2 \right. \\
& \left. + 305(n-6)^3 + 50(n-6)^4 + 3(n-6)^5 \right) \frac{t^n}{n!} > 0 \qquad (t > 0),
\end{aligned}
$$

so that (20) yields

$$
(-1)^n f_3^{(n)}(x) > 0 \qquad (x > 0; n \in \mathbb{N}_0).
$$

The proof of Theorem 3.2 is complete. □

Using the facts $f_3(x) > 0$ and $f_4(x) > 0$, we obtain the following upper and lower bounds for the gamma function in terms of the polygamma functions.

COROLLARY 3.2. *For $x > 0$,*

$$
\sqrt{2\pi x} \left(\frac{x}{e} \right)^x \exp\left(\sum_{k=1}^4 b_k \psi^{(k)}(x) \right) < \Gamma(x+1) < \sqrt{2\pi x} \left(\frac{x}{e} \right)^x \exp\left(\sum_{k=1}^3 b_k \psi^{(k)}(x) \right).
$$

In view of Theorems 3.1 and 3.2, we pose two conjectures.

CONJECTURE 3.1. *For all $m \in \mathbb{N}_0$, the functions $R_m(x)$ defined by*

$$
R_m(x) = \ln\Gamma(x+1) - \left[\left(x + \frac{1}{2} \right) \ln x - x + \ln\sqrt{2\pi} + \sum_{j=1}^m a_j \psi^{(2j-1)} \left(x + \frac{1}{2} \right) \right]
$$

are completely monotonic on $(0,\infty)$. Here a_j are given in (4).

CONJECTURE 3.2. *For all $m \in \mathbb{N}_0$, the functions $S_m(x)$ defined by*

$$
S_m(x) = (-1)^m \left[\ln\Gamma(x+1) - \left(x + \frac{1}{2} \right) \ln x + x - \ln\sqrt{2\pi} - \sum_{j=1}^m b_j \psi^{(j)}(x) \right]
$$

are completely monotonic on $(0,\infty)$. Here b_j are given in (12).

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (EDITORS), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series **55**, Ninth printing, National Bureau of Standards, Washington, D.C., 1972.

[2] J. DUBOURDIEU, *Sur un théorème de M. S. Bernstein relatif à la transformation de Laplace-Stieltjes*, Compositio Math. **7** (1939), 96–111 (in French).

[3] H. VAN HAERINGEN, *Completely monotonic and related functions*, J. Math. Anal. Appl. **204** (1996), 389–408.

[4] H. ŞEVLI AND N. BATIR, *Complete monotonicity results for some functions involving the gamma and polygamma functions*, Math. Comput. Modelling **53** (2011), 1771–1775.

[5] H. M. SRIVASTAVA, J. CHOI, *Series associated with the zeta and related functions*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

[6] H. M. SRIVASTAVA AND J. CHOI, *Zeta and q-Zeta functions and fssociated feries and integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[7] D. V. WIDDER, *The Laplace Transform*, Princeton Univ. Press, Princeton, 1941.

Chao-Ping Chen
School of Mathematics and Informatics
Henan Polytechnic University
Jiaozuo City 454000
Henan Province, China
e-mail: chenchaoping@sohu.com

Mathematical Inequalities & Applications
www.ele-math.com
mia@ele-math.com