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AN INEQUALITY FOR BOUNDED FUNCTIONS

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Abstract. In this note we prove optimal inequalities for bounded functions in terms of their deviation from their mean. These results extend and generalize some known inequalities due to Thong (2011) and Perfetti (2011).

1. Introduction

Let $L^\infty([0,1])$ be the space of essentially bounded measurable real functions on $[0,1]$ equipped with the well-known essential supremum norm $\|\cdot\|_\infty$, and consider two real numbers m and M such that $m<0<M$. Let $\mathscr{F}_{m,M}$ denote the closed subset of $L^\infty([0,1])$ consisting of functions $f:[0,1]\longrightarrow\mathbb{R}$ such that $m\leqslant f\leqslant M$ and $\int_0^1 f(x)\,dx=0$, that is,

$$\mathscr{F}_{m,M}=\left\{f\in L^\infty([0,1]) : m\leqslant f\leqslant M \text{ and } \int_0^1 f(x)\,dx=0\right\}. \tag{1}$$

For $f\in L^\infty([0,1])$ we define the continuous function $J(f):[0,1]\longrightarrow\mathbb{R}$ by

$$\forall x\in[0,1],\qquad J(f)(x)=\int_0^x f(t)\,dt. \tag{2}$$

In [4] it was asked to be shown that for every *continuous* f that belongs to $\mathscr{F}_{m,M}$ one has the following inequality:

$$\left|\int_0^1 xf(x)\,dx\right|\leqslant\frac{1}{2}\cdot\frac{-mM}{M-m} \tag{3}$$

Note that for continuous functions f from $\mathscr{F}_{m,M}$ we have

$$\begin{aligned}\int_0^1 xf(x)\,dx&=\int_0^1 x(J(f))'(x)\,dx\\&=\big[xJ(f)(x)\big]_{x=0}^{x=1}-\int_0^1 J(f)(x)\,dx\\&=-\int_0^1 J(f)(x)\,dx\end{aligned}$$

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Now, we see that (3) would follow from the stronger inequality

$$\int_0^1 |J(f)(x)|\,dx \leqslant \frac{1}{2}\cdot\frac{-mM}{M-m}. \tag{4}$$

Also it was proposed in [3] to prove that for every f in $\mathscr{F}_{m,M}$ one has

$$\int_0^1 \big(J(f)(x)\big)^2\,dx \leqslant \frac{-mM}{6(M-m)^2}(3M^2-8mM+3m^2). \tag{5}$$

but in [2] the following sharper result was proved

$$\left(\int_0^1 \big(J(f)(x)\big)^2\,dx\right)^{1/2} \leqslant \frac{1}{\sqrt{3}}\cdot\frac{-mM}{M-m}, \tag{6}$$

and the cases of equality were characterised.

In this note we will generalize these results to give sharp bounds in terms of m, M and φ for $\int_0^1 \varphi\big(|J(f)(x)|\big)\,dx$, where φ is an increasing function, and we will characterise the cases of equality.

2. The main results

Clearly we have the following simple property:

PROPOSITION 2.1. *For every $f\in\mathscr{F}_{m,M}$ we have*

$$\|J(f)\|_\infty \leqslant \frac{-mM}{M-m}.$$

Proof. Indeed, consider $f\in\mathscr{F}_{m,M}$ and $x\in[0,1]$. We distinguish two cases:

(i) $x\in\left[0,\frac{-m}{M-m}\right]$. Since $f(t)\leqslant M$ for $t\in[0,x]$ we deduce that

$$J(f)(x)=\int_0^x f(t)\,dt \leqslant Mx \leqslant \frac{-mM}{M-m}.$$

(ii) $x\in\left[\frac{-m}{M-m},1\right]$. Here we have $-f(t)\leqslant -m$ for $t\in[x,1]$ so

$$J(f)(x)=\int_x^1 (-f)(t)\,dt \leqslant -m(1-x)\leqslant \frac{-mM}{M-m}.$$

So we have shown that for every $f\in\mathscr{F}_{m,M}$ we have

$$\forall x\in[0,1],\qquad J(f)(x)\leqslant\frac{-mM}{M-m}. \tag{7}$$

Applying (7) to $-f\in\mathscr{F}_{-M,-m}$ we conclude also that

$$\forall x\in[0,1],\qquad -J(f)(x)\leqslant\frac{-mM}{M-m}. \tag{8}$$

Now, from (7) and (8), we arrive to the conclusion that

$$\forall x \in [0,1], \qquad |J(f)(x)| \leqslant \frac{-mM}{M-m},$$

as desired. □

The next lemma is a well-known result on convex functions, (See for example [1, Ch 4].) But, since we are interested in the strict inequality, we will include a proof for the convenience of the reader.

LEMMA 2.2. *Let $\varphi : [0,T] \longrightarrow \mathbb{R}$ be a monotonous increasing function which is not constant on $(0,T)$. For $t \in (0,T]$ we define $K(\varphi,t)$ by*

$$K(\varphi,t) = \frac{1}{t}\int_0^t \varphi(x)\,dx.$$

Then, for all $t \in (0,T)$ we have $\varphi(0^+) \leqslant K(\varphi,t) < K(\varphi,T)$.

Proof. The fact that $\varphi(0^+) \leqslant K(\varphi,t)$ for $t \in (0,T)$ follows since φ is monotonous increasing.

On the other hand, for $\alpha \in (0,1)$, we have

$$K(\varphi,\alpha T) = \frac{1}{\alpha T}\int_0^{\alpha T} \varphi(x)\,dx = \frac{1}{T}\int_0^T \varphi(\alpha u)\,du.$$

So, if $0 < \alpha < 1$ then

$$K(\varphi,T) - K(\varphi,\alpha T) = \frac{1}{T}\int_0^T \big(\varphi(u) - \varphi(\alpha u)\big)\,du \geqslant 0.$$

The last inequality follows from the fact that $u \mapsto \big(\varphi(u) - \varphi(\alpha u)\big)$ is nonnegative on $[0,T]$ because φ is increasing.

Now suppose that we have $K(\varphi,T) = K(\varphi,\alpha T)$ for some $\alpha \in (0,1)$. This implies that the set

$$\mathscr{S} = \{u \in [0,T] : \varphi(u) = \varphi(\alpha u)\}$$

has Lebesgue measure equal to $\lambda([0,T]) = T$. It follows that the set

$$\mathscr{S}' = \bigcap_{n \geqslant 1} \left(\alpha^{-n}\mathscr{S}\right)$$

has also Lebesgue measure equal to T. In particular, $\mathscr{S}'$ is a dense subset of $(0,T)$. Now, consider $u \in \mathscr{S}'$. We have $\varphi(\alpha^k u) = \varphi(\alpha^{k+1}u)$ for every $k \geqslant 0$. Thus, for every $k \geqslant 0$ we have $\varphi(u) = \varphi(\alpha^k u)$, so letting k tend to $+\infty$ we obtain $\varphi(u) = \varphi(0^+)$. Since $\mathscr{S}'$ is a dense subset of $(0,T)$, there is an increasing sequence $(u_n)_{n\geqslant 0}$ in $\mathscr{S}'$ that converges to T, thus $\varphi(0^+) = \lim_{n\to\infty} \varphi(u_n) = \varphi(T^-)$. This means that φ is constant on $(0,T)$ which is contrary to the hypothesis. So we must have $K(\varphi,T) > K(\varphi,\alpha T)$ for every $\alpha \in (0,1)$ and the proof of the Lemma is complete. □

The next theorem is the main result of this note:

THEOREM 2.3. *Let φ be a positive monotonous increasing function on $\left[0, \frac{-mM}{M-m}\right]$. For every $f \in \mathscr{F}_{m,M}$ we have*

$$\int_0^1 \varphi(|J(f)(x)|)dx \leqslant K\left(\varphi, \frac{-mM}{M-m}\right),$$

where $K(.,.)$ is defined in Lemma 2.2. Moreover, if φ is not constant on $\left(0, \frac{-mM}{M-m}\right)$, then equality holds if and only if f coincides for almost every x in $[0,1]$ with one of the functions f_0 or f_1 defined by

$$f_0(x) = \begin{cases} M & \text{if } x \in \left[0, \frac{-m}{M-m}\right), \\ m & \text{if } x \in \left[\frac{-m}{M-m}, 1\right]. \end{cases} \qquad f_1(x) = \begin{cases} m & \text{if } x \in \left[0, \frac{M}{M-m}\right), \\ M & \text{if } x \in \left[\frac{M}{M-m}, 1\right]. \end{cases}$$

Proof. Since f is integrable, $J(f)$ is continuous on $[0,1]$. If $J(f) = 0$, (*i.e.* $f = 0$ *a.e.*,) the claim is obvious. So, in what follows we will suppose that $J(f) \neq 0$.

The continuity of $J(f)$ shows that the set $\mathscr{O} = \{x \in (0,1) : J(f)(x) \neq 0\}$ is an open set. Moreover, since $J(f)(0) = J(f)(1) = 0$, we see that $J(f)(t) = 0$ for every $t \in [0,1] \setminus \mathscr{O}$.

The open set $\mathscr{O}$ is the union of at most countable family of disjoint open intervals. Thus there exist $\mathscr{N} \subset \mathbb{N}$ and a family $(I_n)_{n\in\mathscr{N}}$ of non-empty *disjoint* open sub-intervals of $(0,1)$ such that $\mathscr{O} = \cup_{n\in\mathscr{N}} I_n$.

Suppose that $I_n = (a_n, b_n)$. Since a_n and b_n belong to $[0,1] \setminus \mathscr{O}$, we conclude that $J(f)(a_n) = J(f)(b_n) = 0$, while $J(f)$ keeps a constant sign on I_n. So, let us consider two cases:

a) $J(f)(x) > 0$ for $x \in I_n$. From $m \leqslant f \leqslant M$ we conclude that, for $x \in I_n$, we have

$$J(f)(x) = J(f)(x) - J(f)(a_n) = \int_{a_n}^x f(t)\,dt \leqslant M(x - a_n) \tag{9}$$

and

$$J(f)(x) = -(J(f)(b_n) - J(f)(x)) = \int_x^{b_n} (-f)(t)\,dt \leqslant -m(b_n - x) = m(x - b_n). \tag{10}$$

Combining (9) and (10) we obtain

$$\forall x \in I_n, \qquad 0 < J(f)(x) \leqslant \min\{M(x-a_n), m(x-b_n)\},$$

and consequently, using the definition of $K(.,.)$ from Lemma 2.2, we obtain

$$\begin{aligned}
&\int_{I_n} \varphi(|J(f)(x)|)\,dx \\
&\leqslant \int_{a_n}^{b_n} \varphi\big(\min\{M(x-a_n), m(x-b_n)\}\big)\,dx \\
&= \int_{a_n}^{a_n-m(b_n-a_n)/(M-m)} \varphi(M(x-a_n))dx + \int_{b_n-M(b_n-a_n)/(M-m)}^{b_n} \varphi(m(x-b_n))\,dx \\
&= \frac{1}{M}\int_0^{-mM(b_n-a_n)/(M-m)} \varphi(t)dt + \frac{1}{-m}\int_0^{-mM(b_n-a_n)/(M-m)} \varphi(t)dt \\
&= (b_n-a_n)K\left(\varphi, \frac{-mM(b_n-a_n)}{M-m}\right) \qquad (11)
\end{aligned}$$

with equality if and only if $J(f)(x) = \min\{M(x-a_n), m(x-b_n)\}$ for every $x \in I_n$, that is, if and only if, $f(x) = M$ for almost every $x \in \left[a_n, \frac{Ma_n - mb_n}{M-m}\right)$, and $f(x) = m$ for almost every $x \in \left[\frac{Ma_n - mb_n}{M-m}, b_n\right]$.

b) $J(f)(x) < 0$ for $x \in I_n$. From $m \leqslant f \leqslant M$ we conclude that, for $x \in I_n$, we have

$$J(f)(x) = J(f)(x) - J(f)(a_n) = \int_{a_n}^{x} f(t)\,dt \geqslant m(x-a_n) \qquad (12)$$

and

$$J(f)(x) = -(J(f)(b_n) - J(f)(x)) = \int_x^{b_n} (-f)(t)\,dt \geqslant -M(b_n - x). \qquad (13)$$

Again, combining (12) and (13) we get

$$\forall x \in I_n, \qquad 0 < -J(f)(x) \leqslant \min\{-m(x-a_n), M(b_n-x)\},$$

and consequently

$$\begin{aligned}
&\int_{I_n} \varphi(|J(f)(x)|)\,dx \\
&\leqslant \int_{a_n}^{b_n} \varphi\big(\min\{m(a_n-x), M(b_n-x)\}\big)\,dx \\
&= \int_{a_n}^{a_n+M(b_n-a_n)/(M-m)} \varphi(m(a_n-x))dx + \int_{b_n+m(b_n-a_n)/(M-m)}^{b_n} \varphi(M(b_n-x))\,dx \\
&= \frac{1}{-m}\int_0^{-mM(b_n-a_n)/(M-m)} \varphi(t)dt + \frac{1}{M}\int_0^{-mM(b_n-a_n)/(M-m)} \varphi(t)dt \\
&= (b_n-a_n)K\left(\varphi, \frac{-mM(b_n-a_n)}{M-m}\right), \qquad (14)
\end{aligned}$$

with equality if and only if $J(f)(x) = \max\{m(x-a_n), M(x-b_n)\}$ for every $x \in I_n$, that is, if and only if, $f(x) = m$ for almost every $x \in \left[a_n, \frac{Mb_n - ma_n}{M-m}\right)$, and $f(x) = M$ for almost every $x \in \left[\frac{Mb_n - ma_n}{M-m}, b_n\right]$.

So, comparing (11) and (14) we see that in both cases we have

$$\int_{I_n} \varphi(|J(f)(x)|)\,dx \leqslant |I_n|\cdot K\left(\varphi, \frac{-mM\,|I_n|}{M-m}\right).$$

Therefore, using Lemma 2.2, we can write

$$\begin{aligned}
\int_0^1 \varphi(|J(f)(x)|)\,dx &= \sum_{n\in\mathcal{N}} \int_{I_n} \varphi(|J(f)(x)|)\,dx + \int_{[0,1]\setminus\mathcal{O}} \varphi(|J(f)(x)|)dx \\
&\leqslant \sum_{n\in\mathcal{N}} |I_n|\, K\left(\varphi, \frac{-mM\,|I_n|}{M-m}\right) + \varphi(0)\int_{[0,1]\setminus\mathcal{O}} dx \\
&\leqslant K\left(\varphi, \frac{-mM}{M-m}\right) \sum_{n\in\mathcal{N}} |I_n| + \varphi(0)(1-|\mathcal{O}|) \\
&= K\left(\varphi, \frac{-mM}{M-m}\right)|\mathcal{O}| + \varphi(0)(1-|\mathcal{O}|) \\
&\leqslant K\left(\varphi, \frac{-mM}{M-m}\right)|\mathcal{O}| + K\left(\varphi, \frac{-mM}{M-m}\right)(1-|\mathcal{O}|) \\
&= K\left(\varphi, \frac{-mM}{M-m}\right)
\end{aligned}$$

where the last inequality follows from the increasing property of $t \mapsto K(\varphi,t)$ proved in Lemma 2.2.

Moreover, analyzing the case of equality, and using Lemma 2.2, we see that it can occur if and only if $\mathcal{O} = (0,1)$ and $f(x) = f_0(x)$ *a.e.* or $f(x) = f_1(x)$ *a.e.*, where f_0 and f_1 are the functions defined in the statement of the Theorem. This concludes the proof. □

Let us give some corollaries. For a positive real p and a function $f \in \mathcal{L}^\infty([0,1])$, we recall the notation

$$\|f\|_p = \left(\int_0^1 |f(x)|^p\,dx\right)^{1/p}.$$

The following corollary gives sharp bounds for $\|J(f)\|_p$ when $f \in \mathcal{F}_{m,M}$. This generalizes the inequalities from [2] (corresponding to $p = 2$) and [4] (corresponding to $p = 1$).

COROLLARY 2.4. *Let p be a positive real number. Then, for every $f \in \mathcal{F}_{m,M}$ we have*

$$\|J(f)\|_p \leqslant \frac{1}{(p+1)^{1/p}} \cdot \frac{-mM}{M-m},$$

with equality if and only if f coincides for almost every x in $[0,1]$ with one of the functions f_0 or f_1 defined by

$$f_0(x) = \begin{cases} M & \text{if } x \in \left[0, \frac{-m}{M-m}\right), \\ m & \text{if } x \in \left[\frac{-m}{M-m}, 1\right]. \end{cases} \qquad f_1(x) = \begin{cases} m & \text{if } x \in \left[0, \frac{M}{M-m}\right), \\ M & \text{if } x \in \left[\frac{M}{M-m}, 1\right]. \end{cases}$$

Proof. This follows from Theorem 2.3, by choosing $\varphi(x) = x^p$. □

Applying Theorem 2.3 for the function $\varphi_\varepsilon(x) = \log(\varepsilon + x)$ for $\varepsilon > 0$, and then letting ε tend to 0 we obtain the following corollary:

COROLLARY 2.5. *For every $f \in \mathscr{F}_{m,M}$ we have*

$$\exp\left(\int_0^1 \log|J(f)(x)|\,dx\right) \leqslant \frac{1}{e} \cdot \frac{-mM}{M-m}.$$

REMARK. Note that Corollary 2.5 follows also from Corollary 2.4 by letting p tend to 0.

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