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ON THE OHLIN LEMMA FOR HERMITE–HADAMARD–FEJÉR TYPE INEQUALITIES

TERESA RAJBA

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Abstract. Using Ohlin's Lemma [21] on convex stochastic ordering, we get a simple proof of known Hermite-Hadamard-Fejér type inequalities. We also prove new inequalities. Using s-convex stochastic ordering [12], we also give some Hermite-Hadamard-Fejér type inequalities in the case of higher order convex functions. The obtained results are useful in proving some inequalities between the quadrature operators [31], [32].

1. Introduction

Let $f\colon I \to \mathbb{R}$ be a convex function defined on a real interval I and $a,b \in I$ with $a < b$. The following double inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a} \cdot \int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known as the Hermite-Hadamard inequality for convex functions (see [13] and [20]). In [14] Fejér gave a generalization of the inequality (1.1):

PROPOSITION 1.1. *Let $f\colon I \to \mathbb{R}$ be a convex function defined on a real interval I, $a,b \in I$ with $a < b$ and let $g\colon [a,b] \to \mathbb{R}$ be nonnegative and symmetric with respect to the point $(a+b)/2$ (the existence of integrals is assumed in all formulas). Then*

$$f\left(\frac{a+b}{2}\right) \cdot \int_a^b g(x)\,dx \leqslant \int_a^b f(x)g(x)\,dx \leqslant \frac{f(a)+f(b)}{2} \cdot \int_a^b g(x)\,dx. \tag{1.2}$$

The double inequality (1.2) is known in the literature as the Hermite-Hadamard-Fejér inequality (see [20], [13] and [22] for the historical background).

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REMARK 1.1.

(i) Note that for $g(x) = w(x)$ such that $\int_a^b w(x)dx = 1$, the inequality (1.2) can be rewritten in the form

$$f\left(\frac{a+b}{2}\right) \leqslant \int_a^b f(x)w(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1.3}$$

(ii) Conversely, from the inequality (1.3) it follows (1.2). Indeed, if $\int_a^b g(x)dx > 0$, it suffices to take $w(x) = \left(\int_a^b g(x)dx\right)^{-1} g(x)$. If $\int_a^b g(x)dx = 0$, then (1.2) is obvious.

For various modifications of (1.1) and (1.2) see e.g. [3], [4], [5], [10], [11], [13], and the references given there.

In a recent paper [17] by M. Klaričić Bakula, J. Pečarić and J. Perić some improvements of various forms of the Hermite-Hadamard inequality can be found; namely, that of Fejér, Lupas, Brenner-Alzer, Beesack-Pečarić. These improvements imply the Hammer-Bullen inequality.

In this paper we offer some useful tools for obtaining and proving of various forms of the Hermite-Hadamard inequality, also for higher-order convex functions.

We describe the inequality (1.3) in terms of convex stochastic ordering. Using the Ohlin lemma [21] we get a simple proof of (1.3).

We obtain a generalization of (1.3), in the case when the function w is not symmetric. We give some generalization of the Brenner and Alzer inequalities [9]. We also consider a generalization of (1.3) in the case of higher order convex functions. The so obtained inequalities are applied to prove some inequalities between the quadrature operators.

2. Some generalizations of the Fejèr inequality

In the sequel we will to make use of the theory of stochastic order relations. Let us review some notations.

As usual, F_X denotes the cumulative distribution function (or the distribution function) of a random variable X and μ_X is the distribution corresponding to X.

For real valued random variables X,Y with a finite expectation we say that X is dominated by Y in *convex ordering* sense if $\mathbb{E}f(X) \leqslant \mathbb{E}f(Y)$ for all convex functions $f\colon \mathbb{R} \to \mathbb{R}$, for which the expectations exist. In that case we write $X \leqslant_{cx} Y$, or $\mu_X \leqslant_{cx} \mu_Y$.

A sufficient condition for convex stochastic ordering is the following Ohlin lemma [21].

LEMMA 2.1. *Let X,Y be two random variables such that $\mathbb{E}X = \mathbb{E}Y$. If the distribution functions F_X,F_Y cross exactly one time, i.e., for some x_0 holds*

$$F_X(x) \leqslant F_Y(x) \text{ if } x < x_0 \text{ and } F_X(x) \geqslant F_Y(x) \text{ if } x > x_0,$$

then $X \leqslant_{cx} Y$.

From Lemma 2.1, a simple proof of (1.2) follows.

Proof of Proposition 1.1. Let f and g satisfy the assumptions of Proposition 1.1. Let X, Y, Z be three random variables such that $\mu_X = \delta_{(a+b)/2}$, $\mu_Y(dx) = (\int_a^b g(x)dx)^{-1} g(x)dx$, $\mu_Z = \frac{1}{2}(\delta_a + \delta_b)$. Then, by Lemma 2.1, we obtain that $X \leqslant_{cx} Y$ and $Y \leqslant_{cx} Z$, which implies (1.2). □

As Fink noted in [15], one wonders what the symmetry has to do with the inequality (1.2) and if such an inequality holds for other functions (cf. [13, p. 53]).

As an immediate consequence of Lemma 2.1, we obtain the following lemma.

LEMMA 2.2. *Let* $0 < p < 1$. *Let* $f\colon I \to \mathbb{R}$ *be a convex function,* $a, b \in I$ *with* $a < b$. *Let* μ *be a finite measure on* $\mathscr{B}([a,b])$ *such that (i)* $\mu([a, pa+qb]) \leqslant pP_0$, *(ii)* $\mu((pa+qb, b]) \leqslant qP_0$, *(iii)* $\int_{[a,b]} x\mu(dx) = (pa+qb)P_0$, *where* $q = 1-p$, $P_0 = \mu([a,b])$. *Then*

$$f(pa+qb)P_0 \leqslant \int_{[a,b]} f(x)\mu(dx) \leqslant [pf(a)+qf(b)]P_0. \tag{2.1}$$

REMARK 2.1. If we choose $\mu(dx) = g(x)dx$ in Lemma 2.2, then the conditions (i), (ii) and the inequality (2.1) reduce to (i') $\mu([a, pa+qb]) = pP_0$, (ii') $\mu((pa+qb]) = qP_0$ and

$$f(pa+qb)P_0 \leqslant \int_a^b f(x)g(x)dx \leqslant [pf(a)+qf(b)]P_0, \tag{2.2}$$

respectively, where $P_0 = \int_a^b g(x)dx$.

For convenience, we will take the measure μ to be the probability measure on $\mathscr{B}([a,b])$. We now apply Lemma 2.2 to obtain the following result.

THEOREM 2.1. *Let* $0 < p < 1$. *Let* $f\colon I \to \mathbb{R}$ *be a convex function,* $a, b \in I$ *with* $a < b$. *If we choose* $w\colon [a,b] \to \mathbb{R}$ *such that* w *is nonnegative,* $\int_a^b w(x)dx = 1$ *and*

$$w(pa+(1-p)b+z) = \frac{(1-p)^2}{p^2} w\Big(pa+(1-p)b - \frac{1-p}{p} z\Big) \tag{2.3}$$

for all $0 \leqslant z \leqslant p(b-a)$, *then*

$$f(pa+(1-p)b) \leqslant \int_a^b f(x)w(x)dx \leqslant pf(a)+(1-p)f(b). \tag{2.4}$$

Proof. Put $A = pa + (1-p)b$. Then we have

$$\begin{aligned}
\int_a^b w(x)dx &= \int_a^A w(x)dx + \int_A^b w(x)dx \\
&= \int_0^{(1-p)(b-a)} w(A-z)dz + \int_0^{p(b-a)} w(A+z)dz \\
&= \int_0^{(1-p)(b-a)} w(A-z)dz + \frac{(1-p)^2}{p^2}\int_0^{p(b-a)} w\Big(A - \frac{1-p}{p}z\Big)dz \\
&= \int_0^{(1-p)(b-a)} w(A-z)dz + \frac{(1-p)}{p}\int_0^{(1-p)(b-a)} w(A-y)dy \\
&= \frac{1}{p}\int_0^{(1-p)(b-a)} w(A-y)dy.
\end{aligned}$$

Since $\int_a^b w(x)dx = 1$, we obtain

$$\int_0^{(1-p)(b-a)} w(A-y)dy = p. \tag{2.5}$$

Taking into account that $\int_a^A w(x)dx = \int_0^{(1-p)(b-a)} w(A-y)dy$, we get

$$\int_a^A w(x)dx = p,$$

which implies that

$$\int_A^b w(x)dx = 1-p,$$

consequently for the measure $\mu(dx) = w(x)dx$, the conditions (i) and (ii) in Lemma 2.2 are satisfied.

Similarly, we obtain that

$$\begin{aligned}
\int_a^b zw(z)dz &= \int_0^{(1-p)(b-a)} (A-y)w(A-y)dy + \int_0^{p(b-a)} (A+z)\frac{(1-p)^2}{p^2} w\Big(A - \frac{1-p}{p}z\Big)dz \\
&= \int_0^{(1-p)(b-a)} (A-y)w(A-y)dy + \frac{1-p}{p}\int_0^{(1-p)(b-a)} \Big(A + \frac{p}{1-p}y\Big)w(A-y)dy \\
&= A\frac{1}{p}\int_0^{(1-p)(b-a)} w(A-y)dy + \int_0^{(1-p)(b-a)} (-y)w(A-y)dy \\
&\quad + \int_0^{(1-p)(b-a)} yw(A-y)dy \\
&= A\frac{1}{p}\int_0^{(1-p)(b-a)} w(A-y)dy.
\end{aligned}$$

By (2.5), we get $\int_a^b zw(z)dz = A$, consequently (iii) in Lemma 2.2 is satisfied. Thus, by Lemma 2.2, we obtain (2.4). □

REMARK 2.2. If we choose $p = \frac{1}{2}$ in Theorem 2.1, then the equality (2.3) means that w is symmetric with respect to $\frac{a+b}{2}$, and the inequalities (2.4) reduce to the Fèjer inequalities (1.3).

Since the set of probability measures μ satisfying the inequalities (2.1) (with $P_0 = 1$) is closed under weak convergence, from Theorem 2.1, we obtain immediately the following result.

THEOREM 2.2. *Let* $0 < p < 1$. *Let* $f\colon I \to \mathbb{R}$ *be a convex function,* $a, b \in I$, *with* $a < b$. *Let* μ *be a probability measure in* $\mathscr{B}([a,b])$ *such that*

$$\mu(pa + (1-p)b + B) = \frac{1-p}{p}\mu\Big(pa + (1-p)b - \frac{1-p}{p}B\Big), \tag{2.6}$$

for any $B \in \mathscr{B}([0, p(b-a)])$. *Then*

$$f(pa + (1-p)b) \leqslant \int_{[a,b]} f(x)\mu(dx) \leqslant pf(a) + (1-p)f(b). \tag{2.7}$$

3. Some results related to the Brenner-Alzer inequality

In 1991, Brenner and Alzer [9] obtained the following result generalizing Fejér's result as well as the result of Vasić and Lacković (1976) [29] and Lupas (1976) [19] (see also [22]).

PROPOSITION 3.1. *Let* p, q *be given positive numbers and* $a_1 \leqslant a < b \leqslant b_1$. *Then the inequalities*

$$f\left(\frac{pa+qb}{p+q}\right) \leqslant \frac{1}{2y}\int_{A-y}^{A+y} f(t)dt \leqslant \frac{pf(a)+qf(b)}{p+q} \tag{3.1}$$

hold for $A = \frac{pa+qb}{p+q}$, $y > 0$, *and all continuous convex functions* $f\colon [a_1, b_1] \to \mathbb{R}$ *iff*

$$y \leqslant \frac{b-a}{p+q}\min\{p, q\}.$$

REMARK 3.1. It is known [22, p. 144] that under the same conditions Hermite-Hadamard's inequality holds, the following refinement of (3.1):

$$f\left(\frac{pa+qb}{p+q}\right) \leqslant \frac{1}{2y}\int_{A-y}^{A+y} f(t)dt \leqslant \frac{1}{2}\{f(A-y)+f(A+y)\} \leqslant \frac{pf(a)+qf(b)}{p+q} \tag{3.2}$$

holds.

In the following theorem we give some generalization of the Brenner and Alzer inequalities (3.2), which we prove by using the Ohlin lemma.

THEOREM 3.1. *Let p,q be given positive numbers, $a_1 \leqslant a < b \leqslant b_1$, $0 < y \leqslant \frac{b-a}{p+q}\min\{p,q\}$ and let $f\colon [a_1,b_1] \to \mathbb{R}$ be a convex function. Then*

$$
\begin{aligned}
f\left(\frac{pa+qb}{p+q}\right) &\leqslant \frac{\alpha}{2}\{f(A-(1-\alpha)y)+f(A+(1-\alpha)y)\}+\frac{1}{2y}\int_{A-(1-\alpha)y}^{A+(1-\alpha)y} f(t)dt \\
&\leqslant \frac{\alpha}{2n}\sum_{k=1}^{n}\left\{f\left(A-y+k\frac{\alpha y}{n}\right)+f\left(A+y-k\frac{\alpha y}{n}\right)\right\}+\frac{1}{2y}\int_{A-(1-\alpha)y}^{A+(1-\alpha)y} f(t)dt \\
&\leqslant \frac{1}{2y}\int_{A-y}^{A+y} f(t)dt, \qquad (3.3)
\end{aligned}
$$

where $0 \leqslant \alpha \leqslant 1$, $n = 1,2,\ldots,$

$$
\begin{aligned}
\frac{1}{2y}\int_{A-y}^{A+y} f(t)dt &\leqslant \frac{\beta}{2}\{f(A-y)+f(A+y)\}+(1-\beta)\frac{1}{2y}\int_{A-y}^{A+y} f(t)dt \\
&\leqslant \frac{1}{2}\{f(A-y)+f(A+y)\}, \qquad (3.4)
\end{aligned}
$$

where $0 \leqslant \beta \leqslant 1$,

$$
\begin{aligned}
\frac{1}{2}\{f(A-y)+f(A+y)\} &\leqslant \left(\frac{1}{2}-\gamma\right)\{f(A-y-c)+f(A+y+c)\}+\gamma\{f(A-y)+f(A+y)\} \\
&\leqslant \frac{pf(a)+qf(b)}{p+q}, \qquad (3.5)
\end{aligned}
$$

where $c = \min\{b-(A+y),(A-y)-a\}$, $\gamma = \left|\frac{1}{2}-p\right|$.

Proof. Let X, Y, W, Z, ξ_n, η and λ be random variables such that:

$$
\begin{aligned}
\mu_X &= \delta_{\frac{pa+qb}{p+q}}, \\
\mu_Y(dx) &= \frac{1}{2y}\chi_{[A-y,A+y]}(x)dx, \\
\mu_Z &= \frac{p}{p+q}\delta_a+\frac{q}{p+q}\delta_b, \\
\mu_W &= \frac{1}{2}\delta_{A-y}+\frac{1}{2}\delta_{A+y}, \\
\mu_{\xi_n}(dx) &= \frac{\alpha}{2n}\sum_{k=1}^{n}\{\delta_{A-y+k\frac{\alpha y}{n}}+\delta_{A+y-k\frac{\alpha y}{n}}\}+\frac{1}{2y}\chi_{[A-(1-\alpha)y,A+(1-\alpha)y]}(x)dx, \\
\mu_\eta(dx) &= \frac{\beta}{2}\{\delta_{A-y}+\delta_{A+y}\}+\frac{1-\beta}{2y}\chi_{[A-y,A+y]}(x)dx, \\
\mu_\lambda &= (\frac{1}{2}-\gamma)\{\delta_{A-y-c}+\delta_{A+y+c}\}+\gamma\{\delta_{A-y}+\delta_{A+y}\}.
\end{aligned}
$$

Then, using the Ohlin lemma, we obtain:

(a) $X \leqslant_{cx} Y$, $Y \leqslant_{cx} W$ and $W \leqslant_{cx} Z$, which implies the inequalities (3.2),

(b) $X \leqslant_{cx} \xi_1$, $\xi_1 \leqslant_{cx} \xi_n$ and $\xi_n \leqslant_{cx} Y$, which implies (3.3),

(c) $Y \leqslant_{cx} \eta$ and $\eta \leqslant_{cx} W$, which implies (3.4),

(d) $W \leqslant_{cx} \lambda$ and $\lambda \leqslant_{cx} Z$, which implies (3.5). □

THEOREM 3.2. *Let p, q be given positive numbers, $0 < \alpha < 1$, $a_1 \leqslant a < b \leqslant b_1$, $0 < y \leqslant \frac{b-a}{p+q}\min\{p,q\}$ and $0 \leqslant \frac{\alpha}{1-\alpha}y \leqslant \frac{b-a}{p+q}\min\{p,q\}$. Let $f\colon [a_1,b_1] \to \mathbb{R}$ be a convex function. Then*

$$\begin{aligned} f(A) &\leqslant \frac{\alpha}{y}\int_{A-y}^{A} f(t)dt + \frac{(1-\alpha)^2}{\alpha y}\int_{A}^{A+\frac{\alpha}{1-\alpha}y} f(t)dt \\ &\leqslant \alpha f(A-y) + (1-\alpha) f\Big(A + \frac{\alpha}{1-\alpha}y\Big) \\ &\leqslant \frac{p}{p+q}f(a) + \frac{q}{p+q}f(b), \end{aligned} \tag{3.6}$$

where $A = \frac{pa+qb}{p+q}$.

Proof. Let X, Y, Z and W be random variables such that:

$$\begin{aligned} \mu_X &= \delta_A, \\ \mu_Y(dx) &= \frac{\alpha}{y}\chi_{[A-y,A]}(x)dx + \frac{(1-\alpha)^2}{\alpha y}\chi_{[A,A+\frac{\alpha}{1-\alpha}y]}(x)dx, \\ \mu_W &= \alpha\delta_{A-y} + (1-\alpha)\delta_{A+\frac{\alpha}{1-\alpha}y}, \\ \mu_Z &= \frac{p}{p+q}\delta_a + \frac{q}{p+q}\delta_b. \end{aligned}$$

Then using the Ohlin lemma we obtain $X \leqslant_{cx} Y$, $Y \leqslant_{cx} W$, $W \leqslant_{cx} Z$, which implies the inequalities (3.6). □

REMARK 3.2. If we choose $\alpha = \frac{1}{2}$ in Theorem 3.2, then the inequalities (3.6) reduce to the inequalities (3.4).

REMARK 3.3. If we choose $\alpha = \frac{p}{p+q}$ and $y = (1-p)z$ in Theorem 3.2, then we have

$$\begin{aligned} f(A) &\leqslant \frac{p}{qz}\int_{A-\frac{q}{p+q}z}^{A} f(t)dt + \frac{q}{pz}\int_{A}^{A+\frac{p}{p+q}z} f(t)dt \\ &\leqslant \frac{p}{p+q}f(A - \frac{q}{p+q}z) + \frac{q}{p+q}f\Big(A + \frac{p}{p+q}z\Big) \\ &\leqslant \frac{p}{p+q}f(a) + \frac{q}{p+q}f(b), \end{aligned}$$

where $A = \frac{pa+qb}{p+q}$, $0 < z \leqslant b-a$.

4. The n-th order case

Convex functions of higher order are very well known and investigated (see e.g. [18], [28], [8], [23], [16], [26]). But up to now there is non common terminology, which sometimes may be confusing.

The notion of nth order convexity (or n-convexity) was defined in terms of divided differences by Popoviciu [24], however, we will not state it have. Instead we list some definitions of nth order convexity which are equivalent to Popoviciu's definition (see [18]).

PROPOSITION 4.1. *A function* $f\colon (a,b) \to \mathbb{R}$ *is* n*-convex on* (a,b) $(n \geqslant 1)$ *if and only if its derivative* $f^{(n-1)}$ *exists and is convex on* (a,b) *(with the convention* $f^{(0)}(x) = f(x)$*).*

PROPOSITION 4.2. *Assume that* $f\colon [a,b] \to \mathbb{R}$ *is* $(n+1)$*-times differentiable on* (a,b) *and continuous on* $[a,b]$ *(*$n \geqslant 1$*). Then* f *is* n*-convex if and only if* $f^{(n+1)}(x) \geqslant 0$*,* $x \in (a,b)$*.*

The Kuczma's monograph [18] devoted to functional equations and inequalities in several variables as well as the classical Roberts and Varberg's book on convex functions [28] use the same terminology (according to which an ordinary convex function is 1-convex).

Some authors (c.f. e.g. [8], [23], [12]) call a function f to be n-convex ($n \geqslant 2$) if $f^{(n-2)}$ is convex (then a convex function is 2-convex).

In this paper we understand the higher order convexity in the sense of Proposition 4.1. Many results on higher-order generalizations of Hermite-Hadamard's inequality one can found, among others, in [2], [30], [13, p. 56].

In the sequel we will to make use of the theory of s-convex stochastic ordering (see Denuit et al. (1998) [12]). Let us review some notations.

For real valued random variables X, Y and any integer $s \geqslant 2$ we say that X is dominated by Y in *s-convex ordering* sense if $\mathbb{E}f(X) \leqslant \mathbb{E}f(Y)$ for all $(s-1)$-convex functions $f\colon \mathbb{R} \to \mathbb{R}$, for which the expectations exist. In that case we write $X \leqslant_{s-cx} Y$, or $\mu_X \leqslant_{s-cx} \mu_Y$. Then the order $\leqslant_{2-cx}$ is just the usual convex order $\leqslant_{cx}$.

A very useful criterion for the verification of the s-convex order is given by Denuit, Lefèvre and Shaked in 1998 [12]. For the statement of this criterion, we need introduce first the following notation. Define the number of sign changes of a function $\varphi\colon \mathbb{R} \to \mathbb{R}$ by

$$S^-(\varphi) = \sup\{S^-[\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n)] \colon x_1 < x_2 < \ldots x_n \in \mathbb{R}, n \in \mathbb{N}\},$$

where $S^-[y_1, y_2, \ldots, y_n]$ denotes the number of sign changes in the sequence $y_1, y_2, \ldots, y_n$ (zero terms are being discarded). Two real functions φ_1, φ_2 are said to have k crossing points (or cross each other k-times) if $S^-(\varphi_1 - \varphi_2) = k$.

PROPOSITION 4.3. ([12]) *Let* X *and* Y *be two random variables such that* $\mathbb{E}(X^j - Y^j) = 0$*,* $j = 1, 2, \ldots, s-1$ *(*$s \geqslant 2$*). If* $S^-(F_X - F_Y) = s-1$ *and the last sign of* $F_X - F_Y$ *is a*$+$*, then* $X \leqslant_{s-cx} Y$*.*

We now apply Proposition 4.3 to obtain the following results.

THEOREM 4.1. *Let* $n \geqslant 1,\ a_1 \leqslant a < b \leqslant b_1$.
Let $a(n) = \left[\frac{n}{2}\right] + 1,\ b(n) = \left[\frac{n+1}{2}\right] + 1$.
Let $\alpha_1, \dots, \alpha_{a(n)},\ x_1, \dots, x_{a(n)},\ \beta_1, \dots, \beta_{b(n)},\ y_1, \dots, y_{b(n)}$ *be real numbers such that*

a) *if* n *is even then*

$$\begin{gathered} 0 < \beta_1 < \alpha_1 < \beta_1 + \beta_2 < \alpha_1 + \alpha_2 < \dots < \alpha_1 + \dots + \alpha_{a(n)} \\ = \beta_1 + \dots + \beta_{b(n)} = 1, \\ a \leqslant y_1 < x_1 < y_2 < x_2 < \dots < x_{a(n)} < y_{b(n)} \leqslant b, \end{gathered} \tag{4.1}$$

b) *if* n *is odd then*

$$\begin{gathered} 0 < \beta_1 < \alpha_1 < \beta_1 + \beta_2 < \alpha_1 + \alpha_2 < \dots < \beta_1 + \dots + \beta_{b(n)} \\ < \alpha_1 + \dots + \alpha_{a(n)} = 1 \\ a \leqslant y_1 < x_1 < y_2 < x_2 < \dots < y_{b(n)} < x_{a(n)} \leqslant b; \end{gathered} \tag{4.2}$$

and

$$\sum_{k=1}^{a(n)} x_i^k \alpha_i = \sum_{j=1}^{b(n)} y_j^k \beta_j,$$

for any $k = 1, 2, \dots, n$.

Let $f\colon [a_1, b_1] \to \mathbb{R}$ *be an* n*-convex function. Then we have the following inequalities:*

i) *if* n *is even then*

$$\sum_{i=1}^{a(n)} \alpha_i f(x_i) \leqslant \sum_{j=1}^{b(n)} \beta_j f(y_j), \tag{4.3}$$

ii) *if* n *is odd then*

$$\sum_{j=1}^{b(n)} \beta_j f(y_j) \leqslant \sum_{i=1}^{a(n)} \alpha_i f(x_i). \tag{4.4}$$

Proof. Let X, Y be random variables such that

$$\mu_X = \sum_{i=1}^{a(n)} \alpha_i \delta_{x_i}, \quad \mu_Y = \sum_{j=1}^{b(n)} \beta_j \delta_{y_j}.$$

Then using Proposition 4.3 we obtain:

a) if n is even then $X \leqslant_{(n+1)-cx} Y$, which implies (4.3),

b) if n is odd then $Y \leqslant_{(n+1)-cx} X$, which implies (4.4). □

EXAMPLE 4.1. Let $n=1$, $a=y_1<x_1=\frac{a+b}{2}<y_2=b$, $\beta_1=\beta_2=\frac{1}{2}$, $\alpha_1=1$. From Theorem 4.1, we obtain the Jensen inequality $f(\frac{a+b}{2})\leqslant\frac{f(a)+f(b)}{2}$, for any convex function f.

REMARK 4.1. The following example shows, that the sharp inequalities in (4.1) and (4.2) are not necessary. Let $n=1$, $a=y_1<x_1=y_2=\frac{a+b}{2}<y_3=b$, $\beta_1=\beta_2=\beta_3=\frac{1}{3}$, $\alpha_1=1$. Then using the Ohlin lemma with $\mu_X=\delta_{\frac{a+b}{2}}$, $\mu_Y=\frac{1}{3}\delta_a+\frac{1}{3}\delta_{\frac{a+b}{2}}+\frac{1}{3}\delta_b$, we obtain the following inequality:

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{3}(f(a)+f(b))+\frac{1}{3}f\left(\frac{a+b}{2}\right),$$

where f is a convex function.

The next theorem is immediate from Proposition 4.3.

THEOREM 4.2. *Let $n\geqslant 1$, $a_1\leqslant a<b\leqslant b_1$. Let $a(n),b(n)\in\mathbb{N}$. Let $\alpha_1,\ldots,\alpha_{a(n)}$, $\beta_1,\ldots,\beta_{b(n)}$ be positive real numbers such that $\alpha_1+\ldots+\alpha_{a(n)}=\beta_1+\ldots+\beta_{b(n)}=1$. Let $x_1,\ldots,x_{a(n)}$, $y_1,\ldots,y_{b(n)}$ be real numbers such that*

a) *$a\leqslant x_1\leqslant x_2\leqslant\ldots\leqslant x_{a(n)}\leqslant b$ and $a\leqslant y_1\leqslant y_2\leqslant\ldots\leqslant y_{b(n)}\leqslant b$,*

b) *$\sum_{k=1}^{a(n)}x_i^k\alpha_i=\sum_{j=1}^{b(n)}y_j^k\beta_j$, for any $k=1,2,\ldots,n$.*

Let $\alpha_0=\beta_0=0$, $x_0=y_0=-\infty$. Let $F_1,F_2\colon\mathbb{R}\to\mathbb{R}$ be two functions given by the following formulas: $F_1(x)=\alpha_0+\alpha_1+\ldots+\alpha_k$ if $x_k<x\leqslant x_{k+1}$ $(k=0,1,\ldots,a(n)-1)$ and $F_1(x)=1$ if $x>x_{a(n)}$; $F_2(x)=\beta_0+\beta_1+\ldots+\beta_k$ if $y_k<x\leqslant y_{k+1}$ $(k=0,1,\ldots,b(n)-1)$ and $F_2(x)=1$ if $x>y_{b(n)}$. If the functions F_1,F_2 have n crossing points and the last sign of F_1-F_2 is a+, then for any n-convex function $f\colon[a_1,b_1]\to\mathbb{R}$ we have the following inequality

$$\sum_{i=1}^{a(n)}\alpha_i f(x_i)\leqslant\sum_{j=1}^{b(n)}\beta_j f(y_j). \tag{4.5}$$

THEOREM 4.3. *Let $n\geqslant 1$, $a_1\leqslant a<b\leqslant b_1$. Let $a(n)=\left[\frac{n}{2}\right]+1$, $b(n)=\left[\frac{n+1}{2}\right]+1$. Let $x_1,\ldots,x_{a(n)},y_1,\ldots,y_{b(n)}$ be real numbers, and $\alpha_1,\ldots,\alpha_{a(n)}$, $\beta_1,\ldots,\beta_{b(n)}$ be positive numbers, such that $\alpha_1+\ldots+\alpha_{a(n)}=1$, $\beta_1+\ldots+\beta_{b(n)}=1$,*

$$\frac{1}{b-a}\int_a^b x^k dx=\sum_{j=1}^{b(n)}y_j^k\beta_j=\sum_{i=1}^{a(n)}x_i^k\alpha_i \quad (k=1,2,\ldots,n),$$

$a\leqslant x_1<x_2<\ldots<x_{a(n)}\leqslant b$, $a\leqslant y_1<y_2<\ldots<y_{b(n)}<b$,

$$\frac{x_1-a}{b-a}<\alpha_1<\frac{x_2-a}{b-a},$$
$$\frac{x_2-a}{b-a}<\alpha_1+\alpha_2<\frac{x_3-a}{b-a},$$
$$\ldots$$
$$\frac{x_{a(n)-1}-a}{b-a}<\alpha_1+\ldots+\alpha_{a(n)-1}<\frac{x_{a(n)}-a}{b-a},$$

$$\frac{y_1-a}{b-a} < \beta_1 < \frac{y_2-a}{b-a},$$
$$\frac{y_2-a}{b-a} < \beta_1+\beta_2 < \frac{y_2-a}{b-a},$$
$$\dots$$
$$\frac{y_{b(n)-1}-a}{b-a} < \beta_1+\ldots+\beta_{b(n)-1} < \frac{y_{b(n)}-a}{b-a};$$

if n is even then $y_1 = a$, $y_{b(n)} = b$, $x_1 > a$, $x_{a(n)} < b$;
if n is odd then $y_1 = a$, $y_{b(n)} < b$, $x_1 > a$, $x_{a(n)} = b$.

Let $f\colon [a_1,b_1] \to \mathbb{R}$ be an n-convex function. Then we have the following inequalities:

i) *if n is even then*

$$\sum_{i=1}^{a(n)} \alpha_i f(x_i) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \sum_{j=1}^{b(n)} \beta_j f(y_j), \tag{4.6}$$

ii) *if n is odd then*

$$\sum_{j=1}^{b(n)} \beta_j f(y_j) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \sum_{i=1}^{a(n)} \alpha_i f(x_i). \tag{4.7}$$

Proof. Let X,Y,Z be random variables such that

$$\mu_X = \sum_{i=1}^{a(n)} \alpha_i \delta_{x_i},\ \mu_Y = \sum_{j=1}^{b(n)} \beta_j \delta_{y_j},\ \mu_Z(dx) = \frac{1}{b-a}\chi_{[a,b]}(x)dx.$$

We now apply Proposition 4.3 to obtain the following $(n+1)$-convex orderings of the random variables X,Y,Z.

If n is even then $X \leqslant_{(n+1)-cx} Z$, $Z \leqslant_{(n+1)-cx} Y$, which implies (4.6).

If n is odd then $Y \leqslant_{(n+1)-cx} Z$, $Z \leqslant_{(n+1)-cx} X$, which implies (4.7). $\square$

5. Inequalities between quadrature operators

In the numerical analysis the inequalities the below type, which are connected with quadrature operators, are called extremalities. Many extremalities are known in the numerical analysis (cf. [1], [7], [6] and the references therein).

The numerical analysts prove them using the suitable differentiability assumptions. As proved Wąsowicz in the papers [31], [32], [34], for convex functions of higher order some extremalities can be obtained without assumptions of this kind, using only the higher order convexity itself. The support-type properties play here the crucial role. As we will show in this paper, some extremalities can be obtained using a probabilistic characterization. The obtained extremalities are also known, however our method using the convex stochastic ordering seems to be quite easy.

For a function $f:[-1,1]\to\mathbb{R}$ we consider six operators approximating the integral mean value

$$\mathscr{I}(f):=\tfrac{1}{2}\int_{-1}^{1}f(x)dx.$$

They are

$$\begin{aligned}
C(f) &:= \tfrac{1}{3}\Big(f\big(-\tfrac{\sqrt{2}}{2}\big)+f(0)+f\big(\tfrac{\sqrt{2}}{2}\big)\Big),\\
\mathscr{G}_2(f) &:= \tfrac{1}{2}\Big(f\big(-\tfrac{\sqrt{3}}{3}\big)+f\big(\tfrac{\sqrt{3}}{3}\big)\Big),\\
\mathscr{G}_3(f) &:= \tfrac{4}{9}f(0)+\tfrac{5}{18}\Big(f\big(-\tfrac{\sqrt{15}}{5}\big)+f\big(\tfrac{\sqrt{15}}{5}\big)\Big),\\
\mathscr{L}_4(f) &:= \tfrac{1}{12}\big(f(-1)+f(1)\big)+\tfrac{5}{12}\Big(f\big(-\tfrac{\sqrt{5}}{5}\big)+f\big(\tfrac{\sqrt{5}}{5}\big)\Big),\\
\mathscr{L}_5(f) &:= \tfrac{16}{45}f(0)+\tfrac{1}{20}\big(f(-1)+f(1)\big)+\tfrac{49}{180}\Big(f\big(-\tfrac{\sqrt{21}}{7}\big)+f\big(\tfrac{\sqrt{21}}{7}\big)\Big),\\
S(f) &:= \tfrac{1}{6}\big(f(-1)+f(1)\big)+\tfrac{2}{3}f(0).
\end{aligned}$$

The operators $\mathscr{G}_2$ and $\mathscr{G}_3$ are connected with Gauss-Legendre rules. The operators $\mathscr{L}_4$ and $\mathscr{L}_5$ are connected with Lobatto quadratures. The operators S and C concern Simpson and Chebyshev quadrature rules, respectively. The operator $\mathscr{I}$ stands for the integral mean value (see e.g. [27], [35], [36], [37], [38]).

We will establish all possible inequalities between these operators in the class of higher order convex functions.

REMARK 5.1. Let X_2, X_3, Y_4, Y_5, U, V and Z be random variables such that

$$\begin{aligned}
\mu_{X_2} &= \frac{1}{2}\left(\delta_{-\frac{\sqrt{3}}{3}}+\delta_{\frac{\sqrt{3}}{3}}\right),\\
\mu_{X_3} &= \frac{4}{9}\delta_0+\frac{5}{18}\left(\delta_{-\frac{\sqrt{15}}{5}}+\delta_{\frac{\sqrt{15}}{5}}\right),\\
\mu_{Y_4} &= \frac{1}{12}(\delta_{-1}+\delta_1)+\frac{5}{12}\left(\delta_{-\frac{\sqrt{5}}{5}}+\delta_{\frac{\sqrt{5}}{5}}\right),\\
\mu_{Y_5} &= \frac{16}{45}\delta_0+\frac{1}{20}(\delta_{-1}+\delta_1)+\frac{49}{180}\left(\delta_{-\frac{\sqrt{21}}{7}}+\delta_{\frac{\sqrt{21}}{7}}\right),\\
\mu_U &= \frac{2}{3}\delta_0+\frac{1}{6}(\delta_{-1}+\delta_1),\\
\mu_V &= \frac{1}{3}\left(\delta_{-\frac{\sqrt{2}}{2}}+\delta_0+\delta_{\frac{\sqrt{2}}{2}}\right),\\
\mu_Z(dx) &= \frac{1}{2}\chi_{[-1,1]}(x)dx.
\end{aligned}$$

Then we have

$$\mathscr{G}_2(f)=E[f(X_2)],\quad \mathscr{G}_3(f)=E[f(X_3)],$$

$$\mathscr{L}_4(f) = E[f(Y_4)], \quad \mathscr{L}_5(f) = E[f(Y_5)],$$
$$S(f) = E[f(U)], \quad C(f) = E[f(V)], \quad \mathscr{I}(f) = E[f(Z)].$$

THEOREM 5.1. *Let $f\colon [-1,1] \to \mathbb{R}$ be 3-convex. Then*

$$\mathscr{G}_2(f) \leqslant \mathscr{I}(f) \leqslant S(f), \tag{5.1}$$

$$\mathscr{G}_2(f) \leqslant C(f) \leqslant T(f) \leqslant S(f), \tag{5.2}$$

where $T \in \{\mathscr{G}_3, \mathscr{L}_5\}$.

Proof. From Theorem 4.3 we obtain $\mathscr{G}_3(f) \leqslant \mathscr{I}(f)$ and $\mathscr{I}(f) \leqslant S(f)$, which implies (5.1). From Theorem 4.1 we obtain $\mathscr{G}_2(f) \leqslant C(f)$. By Theorem 4.2 we get $C(f) \leqslant \mathscr{G}_3(f)$, $C(f) \leqslant \mathscr{L}_5(f)$, $\mathscr{G}_3(f) \leqslant S(f)$, $\mathscr{L}_5(f) \leqslant S(f)$. This completes the proof. □

REMARK 5.2. The inequalities (5.2) can be found in [31]. Wąsowicz in [31] proved that the quadratures $\mathscr{L}_4$, $\mathscr{L}_5$ and $\mathscr{G}_3$ are not comparable in the class of 3-convex functions. Moreover, Wąsowicz [31], [33] proved, that

$$C(f) \leqslant \mathscr{L}_4(f), \tag{5.3}$$

if f is 3-convex. The proof given in [31] is rather complicated. This was done using computer software. In [33] can be found a new easy proof of (5.3), without the use of any computer software, based on the spline approximation of convex functions of higher order. It is worth noting that, Proposition 4.3 does not apply to proving (5.3).

THEOREM 5.2. *Let $f\colon [-1,1] \to \mathbb{R}$ be 5-convex. Then*

$$\mathscr{G}_3(f) \leqslant \mathscr{I}(f) \leqslant \mathscr{L}_4(f), \tag{5.4}$$

$$\mathscr{G}_3(f) \leqslant \mathscr{L}_5(f) \leqslant \mathscr{L}_4(f). \tag{5.5}$$

Proof. The inequalities (5.4) and (5.5) we derive from Theorems 4.3 and 4.2. □

REMARK 5.3. The inequalities (5.5) can be found in [32], [34]. Wąsowicz [32] proved, that in the class of 5-convex functions the operators $\mathscr{G}_2, C, S$ are not comparable both with each other and with $\mathscr{G}_3, \mathscr{L}_4, \mathscr{L}_5$.

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Teresa Rajba
Department of Mathematics and Computer Science
University of Bielsko-Biała
ul. Willowa 2
43–309 Bielsko-Biała
Poland
e-mail: trajba@ath.bielsko.pl